DIRICHLET SPACES ASSOCIATED WITH INTEGRO-DIFFERENTIAL OPERATORS. PART II

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0. Introduction

In [3], we studied integro-differential operators and semi-groups connected with the stable densities of order α, $0 < \alpha < 2$ in R^n $(n \geq 2)$. Given a bounded domain E, we considered the operator

$$(0.1) \qquad A_\alpha^0 u(P) = \triangle \int_E u(Q) \,|\, PQ \,|^{2-n-\alpha} \, dQ$$

and associated with it a Dirichlet space D_α^0 which was obtained by completing the pre-Hilbert space of infinitely differentiable functions with compact support contained in E using the inner product

$$(0.2) \qquad (u, v)_\alpha^0 = -\int_E v A_\alpha^0 u.$$

Following a valuable suggestion of Beurling we used the theory of Dirichlet spaces (cf., [1] and [2]) to study (i) potentials, i.e. solutions in D_α^0 of $-A_\alpha^0 u = f$ for given f and (ii) positive contraction semi-groups generated by A_α^0 in D_α^0 and other spaces. These semi-groups were associated with the absorbing barrier α-processes on E. Many of the results in that note were known, but the theory of Dirichlet spaces provided a method of unexpected simplicity in deriving them.

This paper is a sequel to that study, and the method of Dirichlet spaces is now used to derive some new results. We consider the compact region $\bar{E}$ in R^n $(n \geq 2)$ and determine extensions of A_α^0 which give rise to Dirichlet spaces containing D_α^0 as a subspace. We start by considering the set $C^2(\bar{E})$ of functions on $\bar{E}$ which can be extended so as to be twice continuously differentiable on some open set containing $\bar{E}$. We then define an operator B_α on $C^2(\bar{E})$ in two parts: if $P \in E$

$$(0.3) \qquad B_\alpha u(P) = \triangle \int_E u(Q) \,|\, PQ \,|^{2-n-\alpha} \, dQ + \int_{\partial E} u(Q')\nu(P, Q') \, dS_{Q'}$$

and if $P \in \partial E$

$$(0.4) \qquad \begin{aligned} B_\alpha u(P) = &\int_E [u(Q) - u(P)]\nu(Q, P) \, dQ \\ &+ 2 \int_{\partial E} [u(Q) - u(P)] b(P, Q) \, dS_P - a(P)u(P). \end{aligned}$$

Received August 24, 1964.

The functions ν, a and b are all ≥ 0 and b is symmetric in P and Q; the precise conditions are given in Section 1. We then define an inner product

$$(0.5)\qquad \begin{aligned}(v, u)_\alpha &= -\int_E vB_\alpha u - \int_{\partial E} vB_\alpha u \\ &= -\int_{\bar{E}} vB_\alpha u \, d\xi,\end{aligned}$$

where ξ is Lebesgue measure in E plus the singular measure of uniform density one concentrated on ∂E, the boundary of E. We show that (0.5) reduces to the symmetric expression (2.7) when u and v are in $C^2(\bar{E})$. The completion, D_α, of this pre-Hilbert space is shown to be a Dirichlet space with underlying set $\bar{E}$ and the measure ξ described above. The method of Dirichlet spaces gives a simple means of deriving the basic properties of B_α and of constructing the positive contraction semi-groups generated by B_α in the spaces $L_\xi^2(\bar{E})$, $L_\xi(\bar{E})$ and $B(\bar{E})$—the space of essentially bounded functions on $\bar{E}$ with respect to the measure ξ.

In Section 3 we study potentials, i.e. solutions of $-B_\alpha u = f$ given suitable f. If, for example, $f \in B(\bar{E})$, then a solution u_f exists in the sense that there is a unique $u_f \in D_\alpha$ with $(u_f, v)_\alpha = \int_{\bar{E}} vf \, d\xi$ for all $v \in D_\alpha$. If the Laplacian $\triangle$ in (0.3) is a distribution derivative on the open set E, then $-B_\alpha u_f(P) = f(P)$ for $P \in E$ in the sense of distributions. The sense in which (0.4) exists at the boundary when the differentiability of u_f is not known is discussed and (3.52) gives a generalized interpretation of B_α at the boundary. This is analogous to interpreting the normal derivative in some generalized sense in the case of differential operators. The need for a generalized interpretation arises only when $\alpha \geq 1$, because then $\nu(P, Q)$ is not integrable over $E \times \partial E$. When $\alpha < 1$ this is not the case, and the integral will exist for any bounded u at least a.e. on ∂E.

The potential equation includes analogues to some classical boundary value problems: namely, if we give a function f which vanishes on ∂E, the potential u_f satisfies $-B_\alpha u_f = f$ in E and (0.4) vanishes on ∂E, at least in some generalized sense. The latter condition is analogous to the classical boundary condition on an elliptic differential operator that a linear combination of the function and its normal derivative vanish on ∂E. We do not speak of boundary conditions here, however, for they are included in the very definition of B_α at the boundary.

We consider also in Section 3 the analogue to the Dirichlet problem for the operator (0.3). We show that given $\phi \in C(\partial E)$, there exists a function f which vanishes in E and whose potential u_f coincides with ϕ on ∂E. Hence $-B_\alpha u_f = 0$ on E and u_f has prescribed boundary values on ∂E.

For each suitable choice of ν, a, and b in (0.3) and (0.4) we get an operator B_α which generates positive contraction semi-groups $\{T_t\,;\, t \geq 0\}$ continuous for $t \geq 0$ and leading to stochastic processes. These processes are interrelated in the same way as the diffusion processes connected with the generalized differential operators of [4] with different definitions of the operators at the boundary. We do not attempt, however, to give a probabilistic discussion of these processes here.

In Section 1 we collect the calculations needed in the sequel; Section 2 is devoted to the construction of the Dirichlet spaces associated with B_α ; in Section 3 we study solutions of $-B_\alpha u = f$, and finally in Section 4 we discuss the positive contraction semi-groups generated by B_α in several spaces.

It is to be noted that in Section 3 we have not treated the cases $\alpha \geq 1$ and $\alpha < 1$ separately, even though the proofs often simplify considerably when $\alpha < 1$; in fact, some proofs are even unnecessary in this case. But this distinction at each step would clutter the exposition to such an extent that we have treated the two cases together whenever possible.

1. Preliminary formulas

In this section we shall establish some formulas needed in the sequel.

We have a bounded domain E in R^n $(n \geq 2)$. We assume that its boundary is regular enough that we can apply the divergence theorem, that is, E is a Greenian domain; in particular, we can define the surface area measure and speak about the unit outer normal n which exists at almost all points of the boundary ∂E. For example, any domain bounded by a finite number of regular surfaces would serve our purposes. In the remainder of this section u and v will denote arbitrary functions which are twice continuously differentiable in some open set containing $\bar{E}$, that is, u and v are elements of $C^2(\bar{E})$. The parameter α always satisfies $0 < \alpha < 2$.

We simply collect here the formulas we shall need without an attempt in this section to interpret their significance.

FORMULA A.

$$\triangle \int_E u(Q) \mid PQ \mid^{2-n-\alpha} dQ = \operatorname{div} \int_E [\nabla u(Q)] \mid PQ \mid^{2-n-\alpha} dQ$$
$$+ \int_{\partial E} u(Q) \partial/\partial \mathbf{n}_Q \mid PQ \mid^{2-n-\alpha} dS_Q ,$$

where $\mathbf{n}_Q$ is the outer unit normal at Q and dS_Q denotes surface area measure on ∂E. Here $\triangle$ and div are taken as distribution derivatives on the open set E.

Proof. Let I_E denote the indicator of E. We know from the theory of distributions that if $\partial/\partial x_i$ is the partial derivative with respect to x_i as a distribution derivative on R^n, then

$$(1.1) \qquad \partial/\partial x_i(uI_E * r^{2-n-\alpha}) = [\partial/\partial x_i(uI_E)] * r^{2-n-\alpha}.$$

But

$$(1.2) \qquad \partial/\partial x_i(uI_E) = I_E \partial u/\partial x_i - u \cos \alpha_i \delta_S ,$$

where $\cos \alpha_i$ is the *i-th* component of the outer unit normal, and δ_S is a singular measure of uniform density one concentrated on the surface $S = \partial E$. Thus, as a distribution on the open set E

$$\nabla \int_E u(Q) \mid PQ \mid^{2-n-\alpha} dQ = \int_E \{\nabla u(Q)\} \mid PQ \mid^{2-n-\alpha} dQ - \int_{\partial E} u(Q)\mathbf{n}_Q \mid PQ \mid^{2-n-\alpha} dS_Q . \tag{1.3}$$

The final result follows from taking the divergence of both sides of (1.3).

FORMULA B.

$$-\int_E v(P) \operatorname{div}_P \int_E \{\nabla_Q u(Q)\} \mid PQ|^{2-n-\alpha} dQ\, dP$$

$$= -\int_{\partial E} v(P)\mathbf{n}_P \cdot \int_E \{\nabla_Q u(Q)\} \mid PQ \mid^{2-n-\alpha} dQ\, dS_P$$

$$+ \int_E \int_E [\nabla_P v(P) \cdot \nabla_Q u(Q)] \mid PQ \mid^{2-n-\alpha} dP\, dQ,$$

where the subscripts on the operators ∇ and div indicate the variable with respect to which the differentiation is performed.

Proof. This formula is an application of the identity,

$$\int_E v \operatorname{div} \mathbf{w} = \int_{\partial E} v(\mathbf{w}\cdot\mathbf{n}) - \int_E \nabla v \cdot \mathbf{w}, \tag{1.4}$$

where $\mathbf{n}$ is the outer unit normal.

FORMULA C.

$$\int_E \int_E [\nabla_P v(P) \cdot \nabla_Q u(Q)] \mid PQ \mid^{2-n-\alpha} dP\, dQ$$

$$= \tfrac{1}{2} \int_E \nabla_P v(P) \cdot \left\{\int_{\partial E} \mathbf{n}_Q [u(Q) - u(P)] \mid PQ \mid^{2-n-\alpha} dS_Q\right\} dP$$

$$+ \tfrac{1}{2} \int_E \nabla_P u(P) \cdot \left\{\int_{\partial E} \mathbf{n}_Q [v(Q) - v(P)] \mid PQ \mid^{2-n-\alpha} dS_Q\right\} dP$$

$$- \tfrac{1}{2} \int_E \int_E \nabla_P \{[u(Q) - u(P)][v(Q) - v(P)]\} \cdot \nabla_P \mid PQ \mid^{2-n-\alpha} dP\, dQ$$

Proof *For* simplicity we denote the expression on the left by $\Phi(v, u)$.

Then

$$\Phi(v, u) = \int_E \nabla_P v(P) \cdot \int_E \{\nabla_Q u(Q)\} \mid PQ \mid^{2-n-\alpha} dQ\, dP$$

$$(1.5) \qquad = \int_E \nabla_P v(P) \cdot \left\{ \int_{\partial E} [u(Q) - u(P)]\mathbf{n}_Q \mid PQ \mid^{2-n-\alpha} dS_Q \right\} dP$$

$$- \int_E \nabla_P v(P) \cdot \left\{ \int_E [u(Q) - u(P)] \nabla_Q \mid PQ \mid^{2-n-\alpha} dQ \right\} dP.$$

Here we have applied the identity

$$(1.6) \qquad \int_E \psi \nabla \phi = \int_{\partial E} (\psi\phi)\mathbf{n} - \int_E \phi \nabla \psi$$

to the inner integral in the middle term of (1.5). Since $\Phi(u, v) = \Phi(v, u)$, we have $\Phi(v, u) = \frac{1}{2}\{\Phi(v, u) + \Phi(u, v)\}$; this gives the result if we use in addition the fact that

$$(1.7) \qquad \nabla_Q \mid PQ \mid^{2-n-\alpha} = -\nabla_P \mid PQ \mid^{2-n-\alpha}.$$

FORMULA D.

$$\int_E \int_E \nabla_P\{[u(P) - u(Q)][v(P) - v(Q)]\} \cdot \nabla_P \mid PQ \mid^{2-n-\alpha} dP\, dQ$$

$$= \int_E \int_{\partial E} [u(P) - u(Q)][v(P) - v(Q)]\partial/\partial \mathbf{n}_P \mid PQ \mid^{2-n-\alpha} dS_P\, dQ$$

$$- C_\alpha \int_E \int_E [u(P) - u(Q)][v(P) - v(Q)] \mid PQ \mid^{-n-\alpha} dP\, dQ,$$

where

$$(1.8) \qquad C_\alpha = \alpha(n + \alpha - 2).$$

Proof. This is an application of (1.4) with v replaced by

$$[u(P) - u(Q)][v(P) - v(Q)]$$

as a function of P for fixed Q, and $\mathbf{w}$ replaced by $\nabla_P \mid PQ \mid^{2-n-\alpha}$. The vector function $\mathbf{w}$ in this case has a singularity at $P = Q$, but the formula is still valid since u and v are continuously differentiable in $\bar{E}$. This can be verified by cutting out a sphere of radius r and center Q. The formula is valid in E minus the sphere and the result follows on letting $r \to 0$.

FORMULA E.

$$\int_E \nabla_P v(P) \left\{ \int_{\partial E} \mathbf{n}_Q [u(Q) - u(P)] \mid PQ \mid^{2-n-\alpha} dS_Q \right\} dP$$

$$+ \int_E \nabla_P u(P) \cdot \left\{ \int_{\partial E} \mathbf{n}_Q [v(Q) - v(P)] \mid PQ \mid^{2-n-\alpha} dS_Q \right\} dP$$

$$= -\int_{\partial E} \int_{\partial E} (\mathbf{n}_P \cdot \mathbf{n}_Q)[u(P) - u(Q)][v(P) - v(Q)] \mid PQ \mid^{2-n-\alpha} dS_P \, dS_Q$$

$$- \int_{\partial E} \int_E [u(P) - u(Q)][v(P) - v(Q)] \partial / \partial \mathbf{n}_Q \mid PQ \mid^{2-n-\alpha} dS_Q \, dP.$$

Proof. The left side can be written:

$$-\int_{\partial E} \mathbf{n}_Q \cdot \left\{ \int_E \mid PQ \mid^{2-n-\alpha} \nabla_P([u(P) - u(Q)][v(P) - v(Q)]) \, dP \right\} dS_Q .$$

In the inner integral we use (1.6). Again the singularities do not cause trouble here because of the differentiability properties of u and v. In addition to (1.6) we also make use of (1.7).

Now combining (C), (D), and (E) we get

FORMULA F.

$$\int_E \int_E \{ \nabla_P v(P) \cdot \nabla_Q u(Q) \} \mid PQ \mid^{2-n-\alpha} dP \, dQ$$

$$= -\int_{\partial E} \int_E [u(P) - u(Q)][v(P) - v(Q)] \partial / \partial \mathbf{n}_Q \mid PQ \mid^{2-n-\alpha} dP \, dS_Q$$

$$+ \frac{C_\alpha}{2} \int_E \int_E [u(P) - u(Q)] \mid PQ \mid^{-n-\alpha} dP \, dQ$$

$$- \tfrac{1}{2} \int_{\partial E} \int_{\partial E} (\mathbf{n}_P \cdot \mathbf{n}_Q) \mid PQ \mid^{2-n-\alpha} [u(P) - u(Q)][v(P) - v(Q)] \, dS_P \, dS_Q .$$

The left side of (F) is the second member of the right side of (B). We now derive another form for the first term on the right of (B).

FORMULA G.

$$-\int_{\partial E} v(P) \mathbf{n}_P \cdot \left\{ \int_E (\nabla_Q u) \mid PQ \mid^{2-n-\alpha} dQ \right\} dS_P$$

$$= \tfrac{1}{2} \int_{\partial E} \int_{\partial E} [u(P) - u(Q)][v(P) - v(Q)](\mathbf{n}_P \cdot \mathbf{n}_Q) \mid PQ \mid^{2-n-\alpha} dS_P \, dS_Q$$

$$- \int_{\partial E} v(P) \left\{ \int_E [u(Q) - u(P)] \partial / \partial \mathbf{n}_P \mid PQ \mid^{2-n-\alpha} dQ \right\} dS_P .$$

Proof. We have

$$\int_E \{ \nabla_Q u(Q) \} \mid PQ \mid^{2-n-\alpha} dQ = \int_{\partial E} [u(Q) - u(P)] \mid PQ \mid^{2-n-\alpha} \mathbf{n}_Q \, dS_Q$$

$$- \int_E [u(Q) - u(P)] \nabla_Q \mid PQ \mid^{2-n-\alpha} dQ.$$

Now multiply each side by $-\mathbf{n}_P v(P)$ (in the sense of the inner product) and integrate over ∂E using (1.7) plus the fact that

$$-\int_{\partial E} v(P)\mathbf{n}_P \cdot \left\{\int_{\partial E} [u(Q) - u(P)] \mid PQ \mid^{2-n-\alpha} \mathbf{n}_Q \, dS_Q\right\} dS_P$$
$$= \tfrac{1}{2} \int_{\partial E} \int_{\partial E} (\mathbf{n}_P \cdot \mathbf{n}_Q)[u(P) - u(Q)][v(P) - v(Q)] \mid PQ \mid^{2-n-\alpha} dS_P \, dS_Q \,.$$

Combining (B), (F), and (G):

FORMULA H.

$$-\int_E v(P) \operatorname{div}_P \left\{\int_E [\nabla_Q u(Q)] \mid PQ \mid^{2-n-\alpha} dQ\right\} dP$$
$$= -\int_{\partial E} v(P) \left\{\int_E [u(Q) - u(P)] \partial/\partial \mathbf{n}_P \mid PQ \mid^{2-n-\alpha} dQ\right\} dS_P$$
$$- \int_{\partial E} \int_E [u(P) - u(Q)][v(P) - v(Q)] \partial/\partial \mathbf{n}_Q \mid PQ \mid^{2-n-\alpha} dS_Q \, dP$$
$$+ \frac{C_\alpha}{2} \int_E \int_E [u(P) - u(Q)][v(P) - v(Q)] \mid PQ \mid^{-n-\alpha} dP \, dQ.$$

In what follows we use the notation:

$$\begin{aligned} m(P, Q) &= -\partial/\partial \mathbf{n}_Q \mid PQ \mid^{2-n-\alpha} \\ &= (n + \alpha - 2)(\overrightarrow{PQ} \cdot \mathbf{n}_Q)/\mid PQ \mid^{n+\alpha}, \end{aligned} \tag{1.9}$$

where $\overrightarrow{PQ}$ denotes the vector from P to Q. Note that

$$\int_{\partial E} m(P, Q) \, dS_Q = m(P), \tag{1.10}$$

where $m(P)$ was the function appearing in [3, formula (3.2)].

Let $\nu(P, Q)$ be a function measurable on $E \times \partial E$ with respect to the product of Lebesgue measure on E and the surface area measure on ∂E; further suppose

$$\mu(P) = m(P) - \int_{\partial E} \nu(P, Q) \, dS_Q \geq 0; \qquad \mu \in L(E), \tag{1.11}$$

$$\int_E \int_{\partial E} \mid PQ \mid \nu(P, Q) \, dS_Q \, dP < \infty, \tag{1.12}$$

$$\nu(P, Q) \geq c > 0 \tag{1.13}$$

for some positive constant c. As we have remarked in [3], if $\alpha < 1$, then $m \in L(E)$; thus (1.13) and (1.11) imply (1.12) in that case, since E is a bounded domain. If E is convex, an example is provided by $\mu \equiv 0$, $\nu(P, Q) = m(P, Q)$.[1]

We shall consider an operator B_α for each fixed $(0 < \alpha < 2)$ defined at least on our basic set $C^2(\bar{E})$ and given by

$$(1.14) \quad B_\alpha u = \triangle \int_E u(Q) \mid PQ \mid^{2-n-\alpha} dQ + \int_{\partial E} u(Q)\nu(P, Q) \, dS_Q \, .$$

In fact, for $u \in C^2(\bar{E})$, this reduces to

$$B_\alpha u(P) = \operatorname{div} \int_E [\nabla \, u(Q)] \mid PQ \mid^{2-n-\alpha} dQ + \int_{\partial E} u(Q)\mu(P, Q) \, dS_Q$$

$$(1.14a) \quad = \int_E [\triangle \, u(Q)] \mid PQ \mid^{2-n-\alpha} dQ + \int_{\partial E} \partial u/\partial \mathbf{n} \mid PQ \mid^{2-n-\alpha} dS_Q - \int_{\partial E} u(Q)\mu(P, Q) \, dS_Q \, ,$$

where

$$(1.15) \quad \mu(P, Q) = m(P, Q) - \nu(P, Q).$$

Note that the function on the right of (1.14a) is in $L(E)$; see the proof of Lemma 3.2 for the details of the verification.

We then have

FORMULA I.

$$-\int_E vB_\alpha u = \int_{\partial E} v(Q) \int_E [u(P) - u(Q)]\nu(P, Q) \, dP \, dS_Q$$
$$+ \int_{\partial E} \int_E [u(P) - u(Q)][v(P) - v(Q)]\nu(P, Q) \, dP \, dS_Q$$
$$+ \frac{C_\alpha}{2} \int_E \int_E [u(P) - u(Q)][v(P) - v(Q)] \mid PQ \mid^{-n-\alpha} dP \, dQ$$
$$+ \int_E u(P)v(P)\mu(P) \, dP,$$

where $\mu(P)$ is defined in (1.11).

Proof. Using the definition of B_α along with the formula (H) and (1.15), we get

$$-\int_E vB_\alpha u = -\int_E v \operatorname{div} \int_E [\nabla_Q u] \mid PQ \mid^{2-n-\alpha} dQ + \int_E v(P) \int_{\partial E} u(Q)\mu(P, Q) \, dS_Q \, dP.$$

[1] We sketched a preliminary version for $\alpha < 1$ of this special case in *Une application des espaces de Dirichlet*, Faculté des Sciences de Paris, Sém. Potentiel, 1961/62, Fascicule 1.

But

$$\int_E v(P)\,dP \int_{\partial E} u(Q)\mu(P,Q)\,dS_Q$$

$$= -\int_E \int_{\partial E} [u(P) - u(Q)][v(P) - v(Q)]\mu(P,Q)\,dP\,dS_Q$$
$$+ \int_E u(P)v(P)\mu(P)\,dP$$
$$- \int_{\partial E} v(P)\,dS_Q \int_E [u(P) - u(Q)]\mu(P,Q)\,dP,$$

which gives the result.

2. The integro-differential operators and associated Dirichlet spaces

We introduce the measure ξ which is the Lebesgue measure in E plus the singular measure of uniform density one concentrated on the surface ∂E. Hence if u is continuous on $\bar{E}$

$$\int_{\bar{E}} u\,d\xi = \int_E u(Q)\,dQ + \int_{\partial E} u(Q)\,dS_Q. \tag{2.1}$$

As in Section 1, we shall consider the class $C^2(\bar{E})$ of restrictions to $\bar{E}$ of functions which are twice continuously differentiable in some open set containing $\bar{E}$. We define an operator B_α on $C^2(\bar{E})$ as follows: for $P \in E$, $B_\alpha u(P)$ is defined by (1.14) and for $Q \in \partial E$

$$\begin{aligned} B_\alpha u(Q) &= \int_E [u(P) - u(Q)]\nu(P,Q)\,dP \\ &\quad + 2\int_{\partial E} [u(P) - u(Q)]b(P,Q)\,dS_P - a(Q)u(Q), \end{aligned} \tag{2.2}$$

where a and b are measurable functions on ∂E and $\partial E \times \partial E$ respectively and further satisfy

$$a(Q) \geq 0, \qquad b(P,Q) \geq 0, \qquad b(P,Q) = b(Q,P); \tag{2.3}$$

$$a \in L(\partial E), \qquad |PQ|\, b(P,Q) \in L(\partial E \times \partial E). \tag{2.4}$$

We also assume that either

$$a(Q) > d > 0 \tag{2.5}$$

for some constant d and $Q \in \partial E$, or

$$\mu(P) > k > 0 \tag{2.6}$$

for some constant k and $P \in E$.

We shall discuss later the case where neither (2.5) nor (2.6) is satisfied. It is to be noted that under any circumstances $\mu(P) \geq 0$.

With this definition of B_α, it then follows from formula (I) of Section 1 that

$$
\begin{aligned}
-\int_{\bar{E}} vB_\alpha u \, d\xi = & \int_{\partial E} u(Q)v(Q)a(Q) \, dS_Q \\
& + \int_{\partial E}\int_{\partial E} [u(P) - u(Q)][v(P) - v(Q)]b(P, Q) \, dS_P \, dS_Q \\
& + \int_{\partial E}\int_{E} [u(P) - u(Q)][v(P) - v(Q)]\nu(P, Q) \, dP \, dS_Q \\
& + \frac{C_\alpha}{2}\int_{E}\int_{E} [u(P) - u(Q)][v(P) - v(Q)] \, |PQ|^{-n-\alpha} \, dP \, dQ \\
& + \int_{E} u(P)v(P)\mu(P) \, dP \\
= & \, (u, v)_\alpha ,
\end{aligned} \tag{2.7}
$$

where μ is defined in (1.11). To get the second term on the right, we note that the symmetry of b implies

$$
\begin{aligned}
-2\int_{\partial E} v(Q)\left\{\int_{\partial E} [u(Q) - u(P)]b(P, Q) \, dS_P\right\} dS_Q & \\
= \int_{\partial E}\int_{\partial E} [u(P) - u(Q)][v(P) - v(Q)]b(P, Q) \, dS_P \, dS_Q .&
\end{aligned} \tag{2.8}
$$

Let us now consider the set $C^1(\bar{E})$ of restrictions to $\bar{E}$ of functions continuously differentiable on some open set containing $\bar{E}$. For the class $C^1(\bar{E})$, the right side of (2.7) is well defined and using $(u, v)_\alpha$ as inner product, we make $C^1(\bar{E})$ into a pre-Hilbert space. Our aim is now to show that the completion D_α of this space is a Dirichlet space. To this end, we now show that the three postulates for a Dirichlet space are satisfied.

$$
\int_{\bar{E}} |u| \, d\xi \leq A \, \| u \|_\alpha , \tag{i}
$$

where A is a constant and $\| u \|_\alpha^2 = (u, u)_\alpha$.

Proof. Case 1. $a(Q) > d > 0$. First, we have

$$
\begin{aligned}
\int_{\partial E} |u| \, dS_Q & \leq \left\{\int_{\partial E} |u|^2 a\right\}^{1/2} d^{-1/2}\{S(E)\}^{1/2} \\
& \leq \text{const.} \, \| u \|_\alpha ,
\end{aligned} \tag{2.9}
$$

where $S(E) = \int_{\partial E} dS_Q$. Now

$$
\int_{E} |u(P)| \, dP \leq \int_{E} |u(P) - u(Q)| \, dP + |u(Q)| \, V(E), \tag{2.10}
$$

where $V(E)$ is the n-dimensional volume of E and Q is an arbitrary point on ∂E. Applying Schwarz's inequality to the right side of (2.10) we get

$$(2.11)\quad \int_E |u(P)|\,dP \le [V(E)]^{1/2}\left\{\int_E [u(P)-u(Q)]^2\,dP\right\}^{1/2} + |u(Q)|\,V(E).$$

Now integrate over ∂E to obtain

$$(2.12)\quad \begin{aligned} S(E)\int_E |u(P)|\,dP &\le [V(E)]^{1/2}\int_{\partial E}\left\{\int_E |u(P)-u(Q)|\,dP\right\}^{1/2} dS_Q \\ &\quad + V(E)\int_{\partial E} |u(Q)|\,dS_Q \\ &\le [S(E)V(E)]^{1/2}\left\{\int_{\partial E}\int_E |u(P)-u(Q)|^2\,dP\,dS_Q\right\}^{1/2} \\ &\quad + V(E)\int_{\partial E} |u(Q)|\,dS_Q\,. \end{aligned}$$

Now using (1.13) and (2.9) we get

$$\begin{aligned} \int_{\bar{E}} |u|\,d\xi &\le \text{const.}\left\{\int_{\partial E}\int_E |u(P)-u(Q)|^2\nu(P,Q)\,dP\,dS_Q\right\}^{1/2} \\ &\quad + V(E)\int_{\partial E} |u(Q)|\,dS_Q \\ &\le \text{const.}\ \|u\|_\alpha\,. \end{aligned}$$

Case 2. $\mu(P) > k > 0$. The argument is similar to that in Case 1. We have first

$$(2.13)\quad \int_E |u(P)|\,dP \le k^{-1/2}[V(E)]^{1/2}\|u\|_\alpha\,.$$

Now let P be any point in E. We have

$$(2.14)\quad \begin{aligned} \int_{\partial E} |u(Q)|\,dS_Q &\le \int_{\partial E} |u(Q)-u(P)|\,dS_Q + u(P)S(E) \\ &\le [S(E)]^{1/2}\left\{\int_{\partial E} |u(Q)-u(P)|^2\,dS_Q\right\}^{1/2} \\ &\quad + |u(P)|\,S(E). \end{aligned}$$

Now integrate over E to obtain

$$\begin{aligned} V(E)\int_{\partial E} |u(Q)|\,dS_Q &\le [S(E)]^{1/2}\int_E\left\{\int_{\partial E} |u(Q)-u(P)|^2\,dS_Q\right\}^{1/2} dP \\ &\quad + S(E)\int_E |u(P)|\,dP \end{aligned}$$

$$\leq [S(E)V(E)]^{1/2} \int_E \int_{\partial E} |u(Q) - u(P)|^2 \, dS_Q \, dP + S(E) \int_E |u(P)| \, dP.$$

Now using (2.13) and (1.13) we get the result again in this case.

(ii) $C \cap D_\alpha$ is dense in C and in D_α.

Here C denotes the functions continuous on $\bar{E}$. The statement is a direct consequence of the definition of D_α as the completion of $C^1(\bar{E})$ in the norm $\| u \|_\alpha$.

(iii) If T is a normalized contraction, then $u \in D_\alpha$ implies $Tu \in D_\alpha$ and $\| Tu \|_\alpha \leq \| u \|_\alpha$.

Proof. If $Tu \in D_\alpha$, then clearly $\| Tu \|_\alpha \leq \| u \|_\alpha$, since

$$| Tz_1 - Tz_2 | \leq | z_1 - z_2 |$$

for any normalized contraction. The only problem is in checking that $u \in D_\alpha$ implies $Tu \in D_\alpha$.[2] We prove this in three steps.

Step 1. If $\exists$ a constant M such that $| u(P) - u(Q) | \leq M \, | PQ |$ for all P and Q in $\bar{E}$, then $u \in D_\alpha$.

Proof. This condition certainly assures that $(u, u)_\alpha < \infty$. One has only to check that u can be approximated in the norm by functions in the pre-Hilbert space. To achieve this, we can first extend u to a function $\bar{u}$ on R^n satisfying the same Lipschitz condition as u for all P and Q in R^n. Then take β_n to be an infinitely differentiable function with support B_n, the open ball of radius $1/n$ and center at the origin, and satisfying $\int_{B_n} \beta_n = 1$. The function $\beta_n * \bar{u} = u_n$ satisfies the same uniform Lipschitz condition as $\bar{u}$ and is infinitely differentiable on R^n. Furthermore $u_n \to u$ uniformly on $\bar{E}$ as $n \to \infty$. By dominated convergence it is easily seen that

$$(u, u_n)_\alpha \to (u, u)_\alpha \quad \text{and also} \quad \| u_n \|_\alpha^2 \to (u, u)_\alpha .$$

But this implies that $\| u - u_n \|_\alpha \to 0$. Since the restriction of u_n to $\bar{E}$ is in D_α, the result is proven.

Step 2. If u satisfies the condition in Step 1, then $Tu \in D_\alpha$ and $\| Tu \|_\alpha \leq \| u \|_\alpha$.

Proof. Clearly $| u(P) - u(Q) | \leq M \, | PQ |$ implies

$$| Tu(P) - Tu(Q) | \leq M \, | PQ |,$$

so the assertion follows directly from Step 1.

[2] It is not clear that all functions u for which the expression $(u, u)_\alpha$ of (2.7) is finite are in D_α. It may be that D_α is only a subspace of this larger space.

Step 3. If $u_n \to u$ in the norm of D_α and $u_n \in C^1(\bar{E})$, then for any normalized contraction T, $\exists$ a subsequence $\{u_{m_k}\}$ of $\{u_n\}$ such that $Tu_{m_k} \to Tu$ in the norm of D_α.

Proof. Case 1. $a(Q) > d > 0$. If $u_n \to u$ in the norm of D_α, then on the boundary ∂E of E, $u_n \to u$ in the L^2 sense with respect to the measure $a(Q)\, dS_Q$. Thus we have a subsequence $\{n_k\}$ such that $\{u_{n_k}\}$ converges pointwise a.e. (dS_Q) on ∂E. But $u_n \to u$ in the norm of D_α also implies that $| u_n(P) - u_n(Q) |$ converges in the L^2 sense on $E \times \partial E$ with respect to the measure $\nu(P, Q)\, dP\, dS_Q$. Since $\nu(P, Q) > c > 0$, there exists a subsequence $\{m_k\}$ of $\{n_k\}$ such that $\{u_{m_k}(P) - u_{m_k}(Q)\}$ converges pointwise a.e. $(dP \times dS_Q)$ on $E \times \partial E$. Combining these two statements, we conclude that $\{u_{m_k}\}$ converges a.e. (dP) on E and a.e. (dS_Q) on ∂E.

Now $| Tu_{m_k} - Tu | \leq | u_{m_k} - u |$ so $Tu_{m_k} \to Tu$ pointwise wherever $u_{m_k} \to u$ pointwise. Since

$$\| Tu_{m_k} \|_\alpha \leq \| u_{m_k} \|_\alpha \leq M,$$

we have $(Tu, Tu_{m_k} - Tu)_\alpha \to 0$. (Here we use the fact that on any space X which is the union of a countable family of sets of finite measure, the measure being denoted by γ, if a sequence $\{f_n\}$ in $L_\gamma^2(X)$ with $\| f_n \|_2 \leq M$ for all n converges pointwise to 0 γ-a.e., then $g \in L_\gamma^2(X)$ implies $\int_X g f_n\, d\gamma \to 0$.)

By dominated convergence, we have also $\| Tu_{m_k} \|_\alpha \to \| Tu \|_\alpha$. The last two statements combined prove that $Tu_{m_k} \to Tu$ in the norm of D_α. Since $u_n \in C^1(\bar{E})$, Step 2 implies that Tu_n is in D_α; thus Tu is also in D_α.

Case 2. $\mu(P) > k > 0$. In this case $u_n \to u$ in D_α implies that $u_n \to u$ in the L_μ^2 sense and we can choose a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u$ a.e. in E. By an argument similar to that in Case 1, we can then choose a subsequence $\{u_{m_k}\}$ converging a.e. with respect to Lebesgue measure in E and a.e. (dS_Q) on ∂E. The rest of the argument is then the same as in Case 1.

We have now verified that the completion of the differentiable functions on $\bar{E}$ with inner product $(u, v)_\alpha$ given by the right side of (2.7) is a Dirichlet space. The proof of (iii) would be simpler if one started with the larger pre-Hilbert space of Lipschitz functions instead of C^1. In that case Steps 1 and 2 would be unnecessary, and in Step 3, $C^1(\bar{E})$ would be replaced by the class of Lipschitz functions.

We mention briefly the case in which neither (2.5) nor (2.6) is satisfied. We can then replace $-B_\alpha u$ by $\lambda_0 u - B_\alpha u$ with λ_0 any positive real number. Then the right side of (2.7) will have μ replaced by $\mu + \lambda_0 = \mu'$ and a replaced by $a + \lambda_0 = a'$. The conditions (2.5) and (2.6) are then both satisfied for μ' and a'.

Note that the space D_α^0 considered in [3] is a subspace of D_α. Namely, if we consider the subspace of D_α for which $u(Q) = 0$ on ∂E we get D_α^0.

3. Potentials in D_α

We recall that if $f \in B(\bar{E})$ there exists a unique element $u_f \in D_\alpha$ such that

$$\int_{\bar{E}} vf \, d\xi = (u_f, v)_\alpha. \tag{3.1}$$

This is true in any Dirichlet space and u_f is called the potential of f. There are other functions giving rise to potentials in D_α, for example, if (2.5) holds and

$$\int_{\partial E} f^2 a^{-1} \, dS_Q < \infty, \tag{3.2}$$

then there exists a unique $u_f \in D_\alpha$ satisfying (3.1) for all $v \in D_\alpha$; if (2.6) holds, it suffices to assume

$$\int_E f^2 \mu^{-1} \, dQ < \infty. \tag{3.3}$$

In all cases $f \geq 0$ implies $u_f \geq 0$.

Before going into additional properties of potentials we recall that

$$\int_{\partial E} a(Q) \, dS_Q < \infty \tag{3.4}$$

and

$$\int_E \mu(P) \, dP < \infty. \tag{3.5}$$

It is to be noted that (3.5) is automatically satisfied if $\alpha < 1$; it must be assumed as hypothesis if $\alpha \geq 1$.

The first question we ask is: under what conditions does $f \in B(\bar{E})$, the space of essentially bounded functions on $\bar{E}$ with respect to the measure ξ, imply that $u_f \in B(\bar{E})$? The simplest result in this direction is

THEOREM 3.1. *If there exists a constant q such that*

$$|f(P)| \leq q \cdot a(P) \text{ on } \partial E \quad \text{and} \quad |f(P)| \leq q \cdot \mu(P) \text{ on } E,$$

then u_f exists and is in $B(\bar{E})$; in fact

$$\| u_f \|_B \leq q. \tag{3.6}$$

In particular, if both (2.5) and (2.6) are satisfied and $f \in B(\bar{E})$, then

$$\| u_f \|_B \leq [\min(k, d)]^{-1} \| f \|_B. \tag{3.7}$$

Proof. It is enough to prove the result when $f \geq 0$. That u_f exists follows from the fact that the function $w \equiv 1 \in D_\alpha$ and is the potential of the function $\omega = u + aI_{\partial E}$ (I_A will always denote the indicator of the set A). Hence,

$$\left| \int_{\bar{E}} vf \right| \leq \int_E |v| \mu + \int_{\partial E} |v| a = (|v|, w)_\alpha \leq \| v \|_\alpha \cdot \| w \|_\alpha. \tag{3.8}$$

Here we have used the fact that in a Dirichlet space $|v|$ has norm not exceeding that of v. Also $f \leq q\omega$ implies that $u_f \leq u_{q\omega} = qu_\omega = q$. The last statement is then an immediate consequence of what we have just proved.

In [3] potentials in D_α^0 were expressed in terms of a Green's function G_α so that

$$u_f^0(P) = \int_E G_\alpha(P, Q) f(Q)\, dQ. \tag{3.9}$$

The potentials u_f^0 are also potentials in D_α, but we must define f properly on the boundary; in D_α we assign u_f^0 the value zero on ∂E. For $Q \in \partial E$, we have

$$-B_\alpha u_f^0(Q) = -\int_E u_f^0(P)\nu(P, Q)\, dP = \bar{f}(Q). \tag{3.10}$$

The function $\bar{f}$ is integrable over ∂E; in fact, using (1.11),

$$\begin{aligned} \int_{\partial E} |\bar{f}(Q)|\, dS_Q &= \int_E u_f^0(P)\nu(P)\, dP \\ &\leq \int_E u_f^0(P) m(P) = \int_E f(Q)\, dQ. \end{aligned} \tag{3.11}$$

Here

$$\nu(P) = \int_{\partial E} \nu(P, Q)\, dS_Q. \tag{3.11a}$$

Thus u_f^0 is the potential of the function which agrees with f on E and with $\bar{f}$ on ∂E. Note, however, that $\bar{f} < 0$ on ∂E, so that u_f^0 is no longer the potential of a positive function when considered in the larger space D_α.

The following theorem expresses potentials in D_α in terms of the kernel G_α.

THEOREM 3.2. *If $f \in B(\bar{E})$ and $u_f \in B(\bar{E})$, then for $P \in E$*

$$u_f(P) = u_f^0(P) + \int_E G_\alpha(P, Q)\left\{\int_{\partial E} u_f(Q')\nu(Q, Q')\, dS_{Q'}\right\} dQ. \tag{3.12}$$

Proof. Again we may suppose $f \geq 0$. The formula (3.12) is meaningful, since

$$\int_{\partial E} u_f(Q')\nu(Q, Q')\, dS_{Q'} \leq \| u_f \|_B\, m(Q) \tag{3.13}$$

for $Q \in E$ and

$$\int_E G_\alpha(P, Q) m(Q)\, dQ = 1 \tag{3.14}$$

almost everywhere in E.

Let g be an arbitrary element of $C(\bar{E})$; we have

$$\begin{aligned} (u_f, u_g^0)_\alpha &= \int_E u_g^0 f \\ &= \int_E u_f g - \int_{\partial E} u_f(Q)\left\{\int_E u_g^0(P)\nu(P, Q)\, dP\right\} dS_Q \end{aligned} \tag{3.15}$$

using (3.1) and (3.12). Thus

$$(3.16)\qquad \int_E u_f g = \int_E u_g^0 f + \int_{\partial E} u_f(Q)\left\{\int_E u_g^0(P)\nu(P, Q)\, dP\right\} dS_Q .$$

Let the right side of (3.10) be denoted by w; then

$$(3.17)\qquad \int_E wg = \int_E g u_f^0 + \int_{\partial E} u_f(Q)\left\{\int_E u_g^0(P)\nu(P, Q)\, dP\right\} dS_Q .$$

Since

$$(3.18)\qquad \int_E g u_f^0 = \int_E f u_g^0 ,$$

we conclude that

$$(3.19)\qquad \int_E u_f g = \int_E wg.$$

Since g was arbitrary in $C(\bar{E})$, we must have $u_f = w$ a.e.

It is to be noted that (3.12) gives the decomposition of u_f into a sum $u_f^0 + u_f^1$ with u_f^0 in D_α^0 and u_f^1 in the orthogonal complement.

Lemma 3.1. *If $\phi \in B(E)$, then the function ψ defined by*

$$(3.20)\qquad \int_E G_\alpha(P, Q)\phi(Q)m(Q)\, dQ = \psi(P)$$

is bounded and continuous in E.[3]

Proof. The kernel $G_\alpha(P, Q)$ can be chosen so that for any bounded f the function defined everywhere by (3.9) is in $C(E)$, and furthermore (3.14) holds everywhere in E. We shall postpone the proof of this statement until the end of Section 4. From now on we assume that G_α has been chosen in this way. If $\phi \in B(E)$, $\phi \geq 0$, put $\phi_n = \phi I_{K_n}$ where I_{K_n} is the indicator of a compact subset K_n of E. We also assume that as $n \to \infty$, we have $K_n \uparrow E$. Now let $\psi_n = G_\alpha \phi_n m$, and note that

$$(3.21)\qquad 0 \leq \psi(P) - \psi_n(P) \leq \| \phi \|_B \left\{\int_{E-K_n} G_\alpha(P, Q)m(Q)\, dQ \right\}.$$

But the monotone convergence of the continuous functions $G_\alpha I_{K_n} m$ to 1 is uniform on every compact subset of E by Dini's theorem. Thus since $\psi_n \in C(E)$, we conclude $\psi \in C(E)$.

Corollary 3.1. *If $f \in B(\bar{E})$ and $u_f \in B(\bar{E})$, then $u_f \in C(E)$.*

Proof. This follows from Lemma 3.1, putting $\phi = m^{-1}(Q)g(Q)$ where

[3] We shall take the usual liberty throughout this paper of referring to elements of D_α as functions rather than equivalence classes of functions.

$$g(Q) = \int_{\partial E} u_f(Q')\nu(Q, Q')\, dS_{Q'}\, .$$

By (3.13), $\phi \in B(E)$.

Before stating the next lemma we introduce some notations. Let

$$c(P', Q') = \tfrac{1}{2} \int_E \nu(P, P') \left\{ \int_E G_\alpha(P, Q)\nu(Q, Q')\, dQ \right\} dP \tag{3.22}$$

for P' and Q' on ∂E, $P' \neq Q'$. We have

$$\int_{\partial E} \int_{\partial E} |\, PQ \,|\, c(P', Q')\, dS_{P'}\, dS_{Q'} < \infty, \tag{3.23}$$

since
$$G_\alpha(P, Q) \leq \text{const.}\, |\, PQ \,|^{-n+\alpha}$$
(cf. [3 formula (4.31)]) and
$$\nu(Q, Q') \leq \text{const.}\, |\, QQ' \,|^{1-n-\alpha}.$$
We also define

$$d(P') = \int_E u_\mu^0(P)\nu(P, P')\, dP \tag{3.24}$$

for $P' \in \partial E$. The existence of u_μ^0 in D_α^0 is assured by the condition

$$\int_E \mu^2 m^{-1} < \int_E \mu < \infty.$$

We then have

$$\int_E d(P')\, dS_{P'} = \int_E u_\mu^0(P)\nu(P)\, dP \leq \int_E \mu(P)\, dP. \tag{3.25}$$

LEMMA 3.2. *Let $\eta \in C^2(\bar{E})$; then η is the potential in D_α of a function $f \in L(\bar{E})$. On ∂E, f is given by*

$$\begin{aligned} f(P') = -2 \int_{\partial E} [\eta(Q') - \eta(P')][c(P', Q') + b(P', Q')]\, dS_P\, dS_Q \\ + \eta(P')\{a(P') + d(P')\}. \end{aligned} \tag{3.26}$$

Proof. We know that $\eta \in D_\alpha$. Also for $P \in E$, cf. (1.14a),

$$\begin{aligned} -B_\alpha \eta &= \int_{\partial E} \eta(Q')\mu(P, Q')\, dS_{Q'} - \int_E (\Delta\eta)\, |\, PQ \,|^{2-n-\alpha}\, dQ \\ &\quad - \int_{\partial E} \partial\eta/\partial \mathbf{n}\, |\, PQ \,|^{2-n-\alpha}\, dS_Q \\ &= f(P). \end{aligned} \tag{3.27}$$

To see that $f \in L(E)$ we write the first term as

$$\int_{\partial E} [\eta(Q') - \eta(P)]\mu(P, Q')\, dS_{Q'} + \eta(P)\mu(P). \tag{3.28}$$

Both $m(P, Q)$ and $\nu(P, Q)$ satisfy (1.12). Thus since $\eta \in C^2(\bar{E})$ and $\mu \in L(E)$ the right side of (3.27) is in $L(E)$. By direct computation we find that $-B_\alpha \eta$ coincides with (3.26) on ∂E. That $f \in L(\partial E)$ follows from (3.23) and the fact that $d \in L(\partial E)$.

COROLLARY 3.2. *Suppose $\eta \in C^2(\bar{E})$ and $\partial\eta/\partial\mathbf{n} = 0$ a.e. on ∂E. If, in addition,*

$$(3.29) \qquad \int_E G_\alpha(P, Q)\left\{\int_{\partial E} \eta(Q')\mu(Q, Q')\, dS_{Q'}\right\} dQ \in D_\alpha^0 ,$$

then the function u defined by

$$(3.30) \qquad u(P) = \int_E G_\alpha(P, Q) \int_{\partial E} \eta(Q')\nu(Q, Q')\, dS_{Q'}\Big\} dQ$$

when $P \in E$ and by $u(P) = \eta(P)$ when $P \in \partial E$, is in D_α and is the potential of the function f equal to zero in E and to (3.26) on ∂E. Furthermore,

$$(3.31) \qquad \begin{aligned} \| u \|_\alpha^2 &= \int_{\partial E} \eta^2(P)\{a(P)\} + d(P)\}\, dS_P \\ &+ \int_{\partial E}\int_{\partial E} [\eta(P) - \eta(Q)]^2[c(P, Q) + b(P, Q)]\, dS_P\, dS_Q . \end{aligned}$$

Proof. By Theorem 3.2 and Lemma 3.2

$$(3.32) \qquad \eta(P) = u(P) + u_f^0(P),$$

with f given by (3.27). Since $\eta \in C^2(\bar{E})$ we have $(\triangle\eta)*r^{2-n-\alpha} \in C(\bar{E})$ and so $u_f^0 \in D_\alpha$. Thus $u \in D_\alpha$ and is the potential of the function described in the statement of the corollary. Equation (3.31) simply states that

$$\| u_f \|_\alpha^2 = \int_{\bar{E}} u_f f = \int_{\partial E} \eta\bar{f}.$$

The result of Corollary 3.2 can be viewed as the solution to the "Dirichlet Problem" for B_α. That is, given η, the function u in (3.30) satisfies

$$(3.33) \qquad -B_\alpha u(P) = 0 \qquad (P \in E)$$

(in the sense of distributions) and $u(P) = \eta(P)$ for $P \in \partial E$. The kernel

$$(3.34) \qquad K_\alpha(P, Q') = \int_E G_\alpha(P, Q)\nu(Q, Q')\, dQ$$

defined on $E \times \partial E$ plays the role of the Poisson kernel. The result will be extended somewhat in the course of this section.

LEMMA 3.3. *Suppose $P_0 \in \partial E$ is such that $\exists$ an open neighborhood of P_0 on ∂E where the surface is represented by the n equations $x_i = f_i(u_1, \cdots u_{n-1})$ with f_i twice continuously differentiable and the Jacobians*

$$J_k = | \partial f_i/\partial u_j | \qquad (i \neq k, j = 1, \cdots, n - 1)$$

are not all 0 *at* P_0. *Then a neighborhood* N *of* P_0 *can be chosen so that whenever* N' *and* N'' *are neighborhoods of* P_0 *on* ∂E *such that* $\bar{N}' \subset N'' \subset N$, *there exists a function* η *satisfying the condition of Lemma* 3.2 *with* $\eta = 1$ *in* N', $\eta = 0$ *on* $C\bar{N}''$, *and* $0 \leq \eta \leq 1$ *everywhere.*

Proof. Let $P_0 = f_i(u_1^0, \cdots, u_{n-1}^0)$. Our hypothesis assures that for some neighborhood U of $(u_1^0, \cdots, u_{n-1}^0)$ and some interval $|t| < \delta$ the equations

$$x_i' = x_i(u_1, \cdots, u_{n-1}) + t \cos \alpha_i(u_1, \cdots, u_{n-1}) \tag{3.35}$$

$(i = 1, \cdots, n)$ with $\cos \alpha_i$ the *i-th* direction cosine of the outer normal, defines one and only one point in a neighborhood of P_0. Let S denote the set of points

$$(x_1', \cdots, x_n')$$

with $(u_1, \cdots, u_{n-1}) \in U$ and $|t| < \delta$. Now let N be a neighborhood of P_0, contained in $S \cap \partial E$, and let N' and N'' be neighborhoods of P_0 such that $\bar{N}' \subset N'' \subset N$. If A and B are open subsets of R^n such that $A \cap \partial E = C\bar{N}''$ and $B \cap \partial E = N'$, we can find a function ϕ which is infinitely differentiable on R^n, $0 \leq \phi \leq 1$, $\phi = 0$ on A, and $\phi = 1$ on B.

To construct η, we let g be a function on R^1 satisfying the conditions: $0 \leq g \leq 1$, g is twice continuously differentiable vanishing for $|t| > \delta'$ for some $\delta' < \delta$; also $g \equiv 1$ in some neighborhood of $t = 0$. Now define

$$\eta(x_1', \cdots, x_n') = \phi(x_1, \cdots, x_n) g(t) \tag{3.36}$$

when $(x_1', \cdots, x_n') \in S$, and

$$\eta(P) = 0 \qquad (P \notin S). \tag{3.37}$$

Our assumptions assure that η is twice continuously differentiable on R^n; furthermore, $\partial\eta/\partial\mathbf{n} = \phi g'(0) = 0$ on ∂E, and since η agrees with ϕ on ∂E, it assumes the desired values in N.

LEMMA 3.4. *Suppose that* P_0 *is a point on* ∂E *satisfying the hypothesis of Lemma* 3.3. *If there exists a constant* M *such that* $|\mu(P, Q)| \leq M$, *then for each sufficiently small neighborhood* $N(P_0)$ *of* P_0 *on* ∂E, *the function* u *defined by*

$$u(P) = \int_E G_\alpha(P, Q) \left\{ \int_{N(P_0)} \nu(Q, Q')\, dS_{Q'} \right\} dQ \tag{3.38}$$

satisfies $-B_\alpha u = 0$ *in* E *and*

$$\lim_{P \to Q_0} u(P) = 1 \qquad (Q_0 \in N(P_0)) \tag{3.39}$$

and whenever $Q_0 \notin \bar{N}(P_0)$ *satisfies the condition of Lemma* 3.3,

$$\lim_{P \to Q_0} u(P) = 0. \tag{3.40}$$

Proof. For sufficiently small N, we can construct an η as in Lemma 3.3 so that $\eta = 1$ on a neighborhood $N' \subset N$ and $\eta = 0$ outside $\bar{N}$. We then have

$$\eta(P) = \int_E G_\alpha(P, Q) \int_{N(P_0)} \eta(Q')\nu(Q, Q')\, dS_{Q'}\, dQ \tag{3.41}$$
$$\leq u(P) + u_f^0 .$$

But in a sufficiently small neighborhood $S(P_0)$ of P_0 in E, we have $\eta(P) \equiv 1$ by the construction in Lemma 3.3. Therefore $P \in S(P_0)$ implies

$$0 \leq 1 - u(P) \leq u_f^0 .$$

Our hypothesis implies that $f \in B(E)$ and that every point $Q_0 \in N(P_0)$ is a regular point in the sense of [3], Lemma 4.8. Hence by that Lemma, $\lim u_f^0 = 0$ as $P \to Q_0$. This proves (3.39).

Similarly, if $Q_0 \notin \bar{N}(P_0)$, then $\exists$ a neighborhood $S(Q_0)$ in which $\eta \equiv 0$. Then (3.41) is replaced by

$$0 \geq \int_E G_\alpha(P, Q) \int_{N'(P_0)} \eta(Q')\nu(Q, Q')\, dS_{Q'}\, dQ + u_f^0 . \tag{3.42}$$

Since N' is an arbitrary open neighborhood $\subset N$, we also have

$$0 \geq u(P) + u_f^0$$

in $S(Q_0)$. Again, since $u_f^0(P) \to 0$ as $P \to Q_0$ and $u(P) \geq 0$, we have (3.40).

LEMMA 3.5. *Let the conditions of Lemma* 3.4 *hold and suppose that* $\phi \in B(\partial E)$ *is continuous at* P_0 ; *then*

$$\lim_{P \to P_0} \int_E G_\alpha(P, Q) \left\{ \int_{\partial E} \phi(Q')\nu(Q, Q')\, dS_{Q'} \right\} dQ = \phi(P_0). \tag{3.43}$$

Proof. Let

$$u(P) = \int_E G_\alpha(P, Q) \left\{ \int_{\partial E} \phi(Q')\nu(Q, Q')\, dS_{Q'} \right\} dQ. \tag{3.44}$$

We have

$$u(P) - \phi(P_0) = \phi(P_0) u_\mu^0(P) \tag{3.45}$$
$$+ \int_E G_\alpha(P, Q) \left\{ \int_{\partial E} [\phi(Q') - \phi(P_0)]\nu(Q, Q')\, dS_{Q'} \right\} dQ,$$

since

$$1 = \int_E G_\alpha(P, Q) \left\{ \int_{\partial E} \nu(Q, Q')\, dS_{Q'} \right\} dQ + u_\mu^0(P). \tag{3.46}$$

From $\mu \in B(E)$ it follows that $\lim u_\mu^0(P) = 0$ as $P \to P_0$. If $N(P_0)$ is a neighborhood of P_0 on ∂E in which $| \phi(Q') - \phi(P_0) | < \varepsilon$, then

$$\int_E G_\alpha(P, Q) \left\{ \int_{N'} | \phi(Q') - \phi(P_0) | \nu(Q, Q')\, dS_{Q'} \right\} dQ < \varepsilon, \tag{3.47}$$

and by Lemma 3.4 since $\phi \in B(\partial E)$

$$(3.48) \quad \lim_{P \to P_0} \int_E G_\alpha(P, Q) \left\{ \int_{\partial E - N} | \phi(Q') - \phi(P_0) | \nu(Q, Q') \, dS_{Q'} \right\} dQ = 0.$$

This completes the proof.

Lemma 3.5 gives a slightly more general version of the Dirichlet problem for B_α. The function u defined by (3.44) satisfies $-B_\alpha u = 0$ in E and at points P_0 of the boundary which are regular in the sense of Lemma 3.4 as well as points of continuity of ϕ, we have $\lim u(P) = \phi(P_0)$ as $P \to P_0$. Without further assumptions on ϕ, however, it is not clear that $u \in D_\alpha$. On the other hand, u is certainly bounded by $\| \phi \|_B$.

Finally, we discuss the sense in which $B_\alpha u_f$ exists at the boundary of E. If u_f is not differentiable, it is not clear that the integrals in (2.2) will exist. Suppose P_0 is a point satisfying the conditions of Lemma 3.3. Let $S_r = N_r'(P_0)$ be the intersection of an open ball of radius r and center P_0 with ∂E and suppose r small enough that we can find a differentiable function η_r^ε which is 1 on S_r, 0 outside $S_{r+\varepsilon}$ for some ε, with $0 \leq \eta_r^\varepsilon \leq 1$ and $\partial \eta_r^\varepsilon / \partial \mathbf{n} = 0$ on ∂E. Let

$$(3.49) \qquad u_r^\varepsilon = \int_E G_\alpha(P, Q) \left\{ \int_{\partial E} \eta_r^\varepsilon(Q') \nu(Q, Q') \, dS_{Q'} \right\} dQ.$$

when $P \in E$, and $u_r^\varepsilon(P) = \eta_r^\varepsilon(P)$ when $P \in \partial E$. We have

$$(3.50) \quad \begin{aligned} (u_r^\varepsilon, u_f)_\alpha &= \int_{\partial E} \eta_r^\varepsilon \left[f + \int_E u_f^0 \nu(Q, Q') \, dQ \right] dS_{Q'} \\ &= \int_{S_r} \left\{ f(Q') + \int_E u_f^0(Q) \nu(Q, Q') \, dQ \right\} dS_{Q'} \\ &\quad + \int_{S_{r+\varepsilon} - S_r} \eta_r^\varepsilon \left\{ f(Q') + \int_E u_f^0(Q) \nu(Q, Q') \, dQ \right\} dS_{Q'}. \end{aligned}$$

Thus

$$(3.51) \quad \lim_{\varepsilon \downarrow 0} (u_r^\varepsilon, u_f)_\alpha - \int_{S_r} \int_E u_f^0(Q) \nu(Q, Q') \, dQ \, dS_{Q'} = \int_{S_r} f(Q') \, dS_{Q'},$$

and for each $r > 0$, we can write

$$(3.52) \quad -\int_{S_r(P_0)} B_\alpha u_f = \lim_{\varepsilon \downarrow 0} (u_r^\varepsilon, u_f)_\alpha - \int_{S_r} \int_E u_f^0(Q) \nu(Q, Q') \, dQ \, dS_{Q'}.$$

This gives a generalized version of $B_\alpha u_f$ on ∂E.

The conditions under which a potential $u_f \in C(\bar{E})$ are not so easy to formulate for $\alpha \geq 1$, but if $\alpha < 1$ we have

LEMMA 3.6. *If $\alpha < 1$ and $| \mu(P, Q) | \leq M$, suppose that the transformation T_α defined by*

$$(3.53) \quad T_\alpha u(Q) = \int_E u(P) \nu(P, Q) \, dP + 2 \int_{\partial E} u(P) b(P, Q) \, dS_P$$

takes $u \in B(\bar{E})$ *into* $T_\alpha u \in C(\partial E)$; *then if* $a \in C(\partial E)$, *we have*

$$f \in C(\partial E), \qquad u_f \in C(\bar{E}), \tag{3.54}$$

whenever all $P_0 \in \partial E$ *satisfy the condition of Lemma* 3.3.

Proof. When $\alpha < 1$, we have $m \in L(E)$ and also $\nu \in L(E)$. Hence, the integral in (3.53) is defined a.e. for $Q \in \partial E$ whenever $u \in B(\bar{E})$. We then have for any $f \in B(\partial E)$

$$u_f(P) = [T_\alpha u_f(P) + f(P)][T_\alpha 1 + a(P)]^{-1} \tag{3.55}$$

when $P \in \partial E$ a.e. Our further assumptions on T_α assure that u_f is actually continuous when restricted to ∂E. But then Lemma 3.1 and Lemma 3.5 imply that $u_f \in C(\bar{E})$.

4. Semi-groups generated by B_α

Let us recall (cf., [2, Lemma 3]) that if f is given in $L^2_\xi(\bar{E})$ or in D_α, then for each $\lambda > 0$, $\exists$ a unique element $S_\lambda f$ minimizing the quadratic functional

$$F(u) = \lambda \| u \|_\alpha^2 + \int_{\bar{E}} | u - f |^2 \, d\xi. \tag{4.1}$$

Also, $u = S_\lambda f$ is the only element in D_α such that

$$(u, v)_\alpha + \int_{\bar{E}} (u - f) v \, d\xi = 0 \tag{4.2}$$

whenever $v \in L^2_\xi \cap D_\alpha$. We then consider the operator $R_\lambda = \lambda^{-1} S_{1/\lambda}$; this may be considered as an operator on any one of the spaces D_α, $L^2_\xi(\bar{E})$, or $B(\bar{E})$. In each case, R_λ is the resolvent of a positive contraction semi-group $\{T_t \,;\, t \geq 0\}$, strongly continuous for $t \geq 0$, cf., [2, Section 3]. In addition, (cf. [3, Corollary 5.1]) we can extend R_λ to the space $L_\xi(\bar{E})$ and the result again holds in this space.

We can parallel the results of Section 3 from Lemma 3.2 on, replacing the operator B_α by $B_\alpha - \lambda I$ and D_α by the Dirichlet space D_α^λ with norm

$$(\| u \|_\alpha^\lambda)^2 = \| u \|_\alpha^2 + \lambda \int_E u^2 \, d\xi. \tag{4.3}$$

We can regard $R_\lambda f$ as the potential of f in D_α^λ with respect to the new operator. The kernel $G_\alpha(P, Q)$ is replaced by the resolvent kernel $G_\alpha(P, Q; \lambda)$ which satisfies

$$G_\alpha(P, Q; \lambda) - G_\alpha(P, Q) = -\lambda \int_E G_\alpha(P, \bar{Q}; \lambda) G_\alpha(\bar{Q}, Q) \, d\bar{Q}, \tag{4.4}$$

and the function m is replaced by the function $m' = m + \lambda$. Thus (3.14) becomes

$$\int_E G_\alpha(P, Q; \lambda)[m(Q) + \lambda] \, dQ = 1. \tag{4.5}$$

The kernel $G_\alpha(P, Q; \lambda)$ has the property that in D_α^0, the transformation

$$R_\lambda^0 f(P) = \int_E G_\alpha(P, Q; \lambda) f(Q)\, dQ \tag{4.6}$$

is the resolvent transformation of the semi-groups in [3]. For R_λ we have

$$\begin{aligned} R_\lambda f(P) &= \int_E G_\alpha(P, Q; \lambda) f(Q)\, dQ \\ &+ \int_E G_\alpha(P, Q; \lambda) \left\{ \int_{\partial E} R_\lambda f(Q') \nu(Q, Q')\, dS_{Q'} \right\} dQ. \end{aligned} \tag{4.7}$$

If $\alpha < 1$ and the condition of Lemma 3.6 holds, then R_λ maps $C(\bar{E})$ into $C(\bar{E})$ since

$$R_\lambda f(P) = [T_\alpha R_\lambda f(P) + f(P)][T_\alpha 1 + a(P) + \lambda]^{-1} \tag{4.8}$$

for $P \in \partial E$. In this case, the semi-group would operate in the subspace $C(\bar{E})$ of $B(\bar{E})$.

Finally we note that it was shown in [3] that if $G_\alpha(P, Q)$ is defined so as to satisfy [3, Formula (4.31)], then the function defined by (3.9) everywhere is the continuous representative of the equivalence class u_f^0. We assume from now on that $G_\alpha(P, Q)$ has been so determined. A similar argument can be carried out for $G_\alpha(P, Q; \lambda)$. Then [3, Formula (4.31)] is replaced by

$$\begin{aligned} G_\alpha(P, Q; \lambda) &= K_\alpha(P, Q; \lambda) \\ &- \int_E G_\alpha(Q, R; \lambda) \left[\int_{CE} K_\alpha(P, T; \lambda) |\, TR\, |^{-n-\alpha} dT \right] dR \end{aligned} \tag{4.9}$$

for $P, Q \in E$ and

$$0 = K_\alpha(P, Q; \lambda) - \int_E G_\alpha(Q, R; \lambda) \left[\int_{CE} K_\alpha(P, T; \lambda) |\, TR\, |^{-n-\alpha} dT \right] dR \tag{4.10}$$

for $P \in R^n - E$, $Q \in E$, where $K_\alpha(P, Q; \lambda)$ is the Laplace transform of the symmetric stable density function of order α on R^n. Thus

$$\lambda \int_{R^n} K_\alpha(P, Q; \lambda)\, dQ \equiv 1 \tag{4.11}$$

for all $P \in R^n$. If $G_\alpha(P, Q; \lambda)$ is now chosen to satisfy (4.9) and (4.10) everywhere, then we have (4.5) holding everywhere, as is shown by integrating (4.9) and (4.10) over R^n with respect to P. In addition (4.4) holds for all P and Q in E, so (3.14) actually holds everywhere for this determination of $G_\alpha(P, Q)$.

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