# IMBEDDING IN LOW DIMENSIONS

BY

L. NEUWIRTH

## I. Introduction

The purpose of this paper is to give some applications of the idea of in-groups and out-groups introduced in [1] and [2].

In particular these groups were designed to study imbedding problems with co-dimension 1. Here we investigate the problem of imbedding a complex of dimension 1 or 2 in a manifold of dimension 2 or 3 respectively.

Among other geometric results, we obtain a necessary and sufficient condition for the planarity of a graph, this condition being quite different from the classical theorem of Kuratowski [3]. In the case of imbedding a 2-complex in a 3-manifold we find necessary and sufficient conditions that a 2 dimensional cw-complex with 1 vertex may be imbedded in some 3-manifold. Some properties that such a 3-manifold must satisfy are also determined.

We show also that any group which has a presentation in which each generator occurs exactly twice in the set of relations has a solvable word problem; in particular such a group is of the form $F *_{i=1}^{n} \pi_1(S_i)$, where $F$ is free, and the $S_i$ are 2-manifolds (not necessarily orientable). This mildly generalizes a result of K. Frederick [4] which was concerned only with groups presented with one relation, and required that $x_i$ and $x_i^{-1}$ each occur once.

To fix ideas we give a number of examples of presentations of the sort described above, first satisfying the hypotheses, and then as they may be factored.

Finally we show how the genus of a graph may be described algebraically.

## II. Standard 2-complexes

Beginning with a finite connected simplicial complex $K$ it is well known that we may deform $K$ over itself so as to contract a maximal tree in the 1-skeleton of $K$ to a single vertex of $K$. If $K$ is 2-dimensional then the resulting space $\hat{K}$ is a cw-complex with 1 0-cell $e^0$, a number of 1-cells $e_1, e_2, \cdots, e_n$ with boundaries attached at $e^0$, and a number of 2-cells $r_1, r_2, \cdots, r_m$ with attaching maps which send the boundary of each $r_i$ into some word in the $e_j$. We denote these attaching maps which completely determine the topological type of the deformed version of $K$ by the word $w_i$ in the $e_j$ which describes them.

Of course the group presented by $(e_1, e_2, \cdots, e_n; w_1, w_2, \cdots, w_m)$ is isomorphic to $\pi_1(K, e^0)$. Conversely corresponding to a presentation $(e_1, \cdots, e_n; w_1, w_2, \cdots w_m)$ there is a connected 2-dimensional cw-complex $\hat{K}$ with one vertex $e^0$, $n$ 1-cells, and $m$ 2-cells such that

$$\pi_1(K, e^0) \approx |(e_1, \cdots, e_n; w_1, \cdots, w_m)|.$$

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Now if $K$ is a 2-dimensional subcomplex of the closed 3-manifold $M$, then the deformation of $K$ over itself described above may be extended to a deformation of $M$ over itself with the resulting space being homeomorphic to $M$. The net result of this is the following; *$K$ is imbeddable in $M$ only if $\hat{K}$ is imbeddable in $M$.*

We study in the next sections the problem of imbedding $\hat{K}$ in some 3-manifold $M$. This is not necessarily the same problem as the imbedding problem for $K$, however notice $\hat{K}$ is of the same homotopy type as $K$. We will henceforth consider $\hat{K}$ to be presented by $(e_1, \cdots, e_n; w_1, \cdots, w_m)$.

In another paper we will study the problem of determining necessary and sufficient conditions that $\hat{K}$ be the spine of a 3-manifold $M$.

All of the above questions are intimately related to the problem of the planarity of a 1-complex $G$. In view of this we commence by considering in the next section the manner in which a 1-dimensional cw-complex $G$ may be imbedded in a 2-manifold $M$.

## III. Graphs

Given a connected graph (1-dimensional cw-complex) $G$ we may construct the out-groups of $G$ [1], [2]. J. Edmunds [6] and J. Youngs [5] have described how an ordering of the edges of a 1-dimensional simplicial complex at each vertex determines an imbedding of $S$ in an orientable 2-manifold $N$ such that $N - S$ consists of a number of disjoint open 2-cells. The construction they describe is easily seen to be valid for a 1-dimensional cw-complex rather than a simplicial complex.

Since each local ordering of the edges of $G$ at a vertex leads on the one hand to an out-group, and on the other to an embedding of $G$ on a closed orientable 2-manifold we are able to connect these two notions. Let us denote by $N_G(\mathfrak{o})$ the 2-manifold defined by the ordering $\mathfrak{o}$ at each vertex of $G$. We now orient each 1-cell of $G$ and obtain from the ordering $\mathfrak{o}$ an out-group presentation which we denote $P_\mathfrak{o}(G)$ and an out-group which we denote $\pi_\mathfrak{o}(G)$.

THEOREM III.1. *$\pi_\mathfrak{o}(G) \approx F * \pi_1(N_G(\mathfrak{o}))$, where $F$ is a free group.*

*Proof.* Suppose $N_G(\mathfrak{o}) - G$ has $t$ components $C_1, \cdots, C_t$; select in each $C_i$ a point $p_i$. Let $N$ denote a $t - ad$, ($t$ closed arcs, disjoint except at one end-point of each, the latter point being common to all the arcs), with end-points $q_1, \cdots, q_t$. Form the space $N_G(\mathfrak{o}) \cup N$ and identify $p_i$ with $q_i$, denote the resulting space $F_G(\mathfrak{o})$. Now $F_G(\mathfrak{o}) - G$ is connected and simply connected. Adjoining the 1-cells of $G$ to $F_G(\mathfrak{o}) - G$ we obtain a space $Y$ which by virtue of the van Kampen theorem has $\pi_1$ free, with free generators $g_1, g_2, \cdots, g_l$ in 1-1 correspondence with the oriented 1-cells of $G$. Now finally adjoining to $Y$ the vertices of $G$ we see that the relations we must adjoin are exactly those of that out-group determined by the ordering $\mathfrak{o}$. Thus

$$\pi_\mathfrak{o}(G) \approx \pi_1(N_G(\mathfrak{o}) \cup N),$$

and $\pi_1(N_G(\mathfrak{o}) \cup N)$ is clearly $F * \pi_1(N_G(\mathfrak{o}))$ where the rank of $F$ is $t - 1$.

COROLLARY III.1. *$G$ is planar $\Leftrightarrow$ $\pi_{\mathfrak{o}}(G)$ is free for some $\mathfrak{o}$.*

*Proof.* If $G$ is planar then some $N_G(\mathfrak{o})$ is a sphere, so $\pi_1(N_G(\mathfrak{o}) \cup N)$ is free; conversely if $\pi_1(N_G(\mathfrak{o}) \cup N)$ is free $N_G(\mathfrak{o})$ must be simply connected, so $N_G(\mathfrak{o})$ is a 2-sphere.

Clearly the previous corollary is not stated in its strongest form. If we make use of Young's observation [5] that an imbedding of $G$ on an orientable surface of minimal genus has the property that each component of the complement is a 2-cell, then we see that the following stronger version of the preceding corollary is true.

COROLLARY III.2. *$G$ is imbeddable on an orientable surface of genus $g$ and not on one of genus $g - 1$ if and only if*

$$\pi_G(\mathfrak{o}) \approx F * |(x_1, x_2, \cdots, x_{2g} : \textstyle\prod_{i=1}^{g} [x_i, x_{i+g}])|$$

*for some $\mathfrak{o}$, and*

$$\pi_G(\mathfrak{o}) \not\approx F * |(x_1, x_2, \cdots, x_{2g-2} : \textstyle\prod_{i=1}^{g-1} [x_i, x_{i+g-1}]$$

*for any other $\mathfrak{o}$.*

This gives a purely algebraic determination of the genus of a graph in terms of the properties of the out-groups of the graph.

In the next section we will make use of Theorem III.1. in the form: $\pi_0(G)$ is free if and only if $N_G(\mathfrak{o})$ is a sphere.

The graph of a 2-complex is described below.[1]

We assume as indicated earlier that $K$ is a 2-complex with a single vertex $e^0$, and 1-cells $e_1, \cdots, e_n$, 2-cells $r_1, \cdots, r_m$ attached to $K^{(1)}$ by $w_1, w_2, \cdots, w_m$. To avoid trivialities we suppose that each $e_i$ actually occurs in at least one of the words $w_j$.

Any nice small neighborhood $L$ of $e^0$ is the join of a graph $G(K)$ with $e^0$, and it is this graph we wish to characterize. This may be done as follows. Let $f_i^{+1}$, $f_i^{-1}$ denote 0-cells (vertices) corresponding to the intersection of the neighborhood $L$ with $e_i$. We construct the set of 1-cells of $G(K)$ by letting each 1-cell correspond to a pair of adjacent letters in $w_j$ ($j = 1, 2, \cdots, m$). (We consider the first letter of $w_j$ to immediately succeed the last letter of $w_j$.) In particular when a word $w_j$ contains the successive letters $e_{j_i}^{\varepsilon_i} e_{j_{i+1}}^{\varepsilon_{i+1}}$ then we map the boundary of a 1-cell ${}_jt_i$ onto the pair (not necessarily distinct) of 0-cells $f_{j_i}^{-\varepsilon_i}, f_{j_{i+1}}^{\varepsilon_{i+1}}$. In this way we associate with a single vertexed 2-dimensional cw-complex $K$, a graph $G(K)$. A little reflection will reveal the fact that $G(K)$ is in fact homeomorphic to the boundary of a small nice neighborhood (for example a star neighborhood in a sufficiently fine subdivision of a triangulation of $K$) of $e^0$.

[1] T. R. Brahana has results closely related to these in a paper which appeared in the Duke Math. J., vol. 30 (1963), pp. 215–220.

Clearly a necessary condition for the imbeddability of $K$ in some 3-manifold is the imbeddability of $G(K)$ on a 2-sphere.

Since a graph is planar if and only if each component is planar we assume without loss of generality that $G(K)$ is connected.

In the next section we shall refine the idea of imbedding $K$ by imbedding $G(K)$. For the present, notice when $K$ is imbedded in a 3-manifold, the boundary of a small ball about the image of $e^0$ will intersect the image of $K$ in a 1-complex homeomorphic to $G(K)$.

## IV. Restrictions on the imbedding of $G(K)$ in $S^2$

As indicated earlier we may not imbed $K$ in a 3-manifold unless $G(K)$ may be embedded on a 2-sphere. On the other hand not every imbedding of $G(K)$ on a 2-sphere is induced by imbedding $K$ in a 3-manifold. For example if the 3-manifold is orientable, then a neighborhood of each $e_i \cup e^0$ is a solid torus, and an ordering of edges of $G(K)$ at $f_i^{+1}$ must induce an ordering of edges at $f_i^{-1}$, since the edges at each correspond to the intersection of 2-cells along $e_i$. On the other hand if the 3-manifold is non-orientable, then a neighborhood of $e_i \cup e^0$ is a generalized moebius band for some $i$, so again cyclic ordering of the edges at $f_i^{+1}$ determines a cyclic ordering of the edges at $f_i^{-1}$.

Let us restrict our attention to an orientable 3-manifold which purports to contain $K$. This implies the existence of an ordering of the edges at each $f_i^{+1}$ so that the induced ordering on the edges at $f_i^{-1}$ yields an ordering, $\mathfrak{o}$, of all edges of $G(K)$ at each vertex, and this ordering enjoys the property that $N_G(\mathfrak{o})$ is a sphere, hence, by Corollary III.1, $\pi_{\mathfrak{o}}(G(K))$ is free. Notice that by the definition of $G(K)$ a cyclic ordering at $f_i^{+1}$ amounts to a cyclic ordering of the collection of successors of $e_i^{-1}$ and predecessors of $e_i$ in the words $w_j$. On the other hand this is equivalent to cyclically ordering all the occurrences of both $e_i$ and $e_i^{-1}$ in the $w_j$. The latter process induces the ordering of the successors of $e_i^{-1}$, and the predecessors of $e_i$ on the one hand and the ordering of the predecessors of $e_i^{-1}$ and the succesors of $e_i$ on the other. The former gives an ordering to the 1-cells at $f_i^{+}$, and the latter gives the induced ordering to the 1-cells at $f_i^{-1}$. Let us denote by $\theta$ those orderings $\mathfrak{o}$ of the edges at vertices of $G(K)$ which arise from an ordering of the occurrences of a generator in a presentation of $K$. Then we have shown

PROPOSITION IV.1. *A necessary condition for the imbeddability of $K$ in an orientable 3-manifold is that $\pi_{\mathfrak{o}}(G(K))$ is free for some $\mathfrak{o} \in \theta$.*

We now show the converse;

PROPOSITION IV.2. *A sufficient condition for the imbeddability of $K$ in an orientable 3-manifold is that $\pi_{\mathfrak{o}}(G(K))$ is free for some $\mathfrak{o} \in \theta$.*

*Proof.* Since $\pi_{\mathfrak{o}}(G(K))$ is free $N_G(\mathfrak{o})$ is a 2-sphere. A neighborhood of $e^0$ may be imbedded in a 3-cell $B$ by taking $B$ as the join of $N_G(\mathfrak{o})$ with a point; then the join of $G(K)$ with a point of homeomorphic to a neighborhood, $N$, of

$e^0$ and lies in $B$. Join $f_i^+$ to $f_i^{-1}$ by disjoint arcs $\alpha_i$ intersecting $B$ only in their endpoints. Thicken the $\alpha_i$ to handles $C_i$ and attach annuli, $A_j$, along the 1-cells of $G(K)$ and $\alpha_i$ according to the words $w_j$. This may clearly be done so that the $A_j$ lie in the $C_i$ except for "fins" where the $A_j$ are attached to the 1-cells of $G(K)$, no difficulty being encountered in making the annuli non-singular since $\theta$ was designed for this contingency. Now by thickening $B$ slightly to accommodate the aforementioned fins we find ourselves with a handlebody, in the boundary of which are to be found one component of the boundary of each $A_i$. These being simple closed curves we complete the proof by attaching discs along the curves, (thus completing the reconstruction of $K$) and then thickening the discs and doubling the resulting manifold with boundary.

*Remark.* The last two propositions may easily be modified to accommodate non-orientable manifolds by altering the class $\theta$ to take account of non-orientable handles.

Summarizing the preceding two propositions we have:

THEOREM IV.1. *$K$ is imbeddable in an orientable 3-manifold if and only if $\pi_{\mathfrak{o}}(G(K))$ is free for some $\mathfrak{o} \in \theta$.*

Notice that in applications the determination of the imbeddability of $K$ now becomes a totally algebraic question. In practice one may deal entirely with a finite presentation of a group, and investigate orderings of the occurrence of each generator. We illustrate this in a later section.

## V. Imbedding $K$ in special manifolds

Suppose we wish to imbed $K$ in a closed simply connected 3-manifold; S. Alexander's duality theorem gives considerable information about $S - K$ in terms of the cohomology of $K$. Furthermore we have noted in an earlier paper [2] that the in-group of $K$ in $S$ is a homomorph of $\pi_1(S)$ if $S - K$ is connected. In case $S - K$ has $h$ components we noted in [2] that the in-Group of $K$ is a homomorph of

$$\pi_1(S) * F_{h-1} \approx F_{h-1},$$

where $F_{h-1}$ is the free group of rank $h - 1$. From these remarks we see that we have for example;

PROPOSITION V.1. *If $H^2(K) = 0$, then $K$ is imbeddable in a simply connected 3-manifold $S$ only if $\pi_{\mathfrak{o}}(K) = 0$ for $\pi_{\mathfrak{o}}(K)$ some out-group of $K$.*

We see in this proposition the first use of the out-groups of $K$, rather than the out-groups of $G(K)$. The relation between these groups is the following.

An ordering of the 2-cells $r_i$ having a fixed 1-cell $e_j$ on the image of their boundary is equivalent to an ordering in the presentation for $\pi_1(K)$ of the occurrences of the generator $e_j$ in the words $w_i$. But as we have seen, an ordering of this latter sort is precisely one leading to an ordering of the edges in $G(K)$ from the class of orderings $\theta$. It thus follows that an ordering of the occurrences of each generator in the set of relators leads to an out-group of $K$, and an out-group of $G(K)$ of the sort required in the preceding theorems. To illustrate, consider the following presentation of a 2-complex $K$,

$$(e_1, e_2; e_1 e_2 e_2 e_1 e_2^{-1}, e_2 e_1 e_1 e_2 e_1^{-1}).$$

Let us number the letters of the relators 1 2 3 4 5, 6 7 8 9 10. Select an ordering of the occurrences of $e_1$, say 1 8 4 7 10, and an ordering of the occurrences of $e_2$, say 2 3 6 5 9. Then this gives rise to the out-group of $K$ presented as follows;

$$(X, Y; X\ Y\ X\ Y\ \bar{Y}, X\ X\ Y\ \bar{X}\ Y).$$

Here $X$ is the 2-cell attached according to the word $e_1 e_2 e_2 e_1 e_2^{-1}$, and $Y$ is the 2-cell attached according to the word $e_2 e_1 e_1 e_2 e_1^{-1}$. The ordering about the 1-cell corresponding to $e_1$ is $XYXY\bar{Y}$ and that about $e_2$ is $XXY\bar{X}Y$.

On the other hand the out-group of $G(K)$ induced by ordering is presented as follows; $(X_{1,2}, X_{2,3}, X_{3,4}, X_{4,5}, X_{5,1}, X_{6,7}, X_{8,9}, X_{9,10}, X_{10,6}; X_{5,1} X_{7,8} X_{3,4} \cdot X_{6,7} X_{10,1}^{-1}, X_{1,2}^{-1} X_{8,9}^{-1} X_{4,5}^{-1} X_{7,8} X_{9,10}, X_{1,2} X_{2,3} X_{10,6} X_{5,1}^{-1} X_{8,9}, X_{2,3}^{-1} X_{3,4}^{-1} X_{6,7}^{-1} \cdot X_{4,5} X_{9,10}^{-1})$. Here $X_{7,8}$ for example corresponds to an oriented 1-cell of $G(K)$ joining two vertices $f_1^{-1}$ to $f_1^{+1}$ and $X_{10,6}$ to an oriented 1-cell of $G(K)$ joining $f_1^{+1}$ to $f_2^{+1}$. The relation $X_{1,2} X_{2,3} X_{10,6} X_{5,1}^{-1} X_{8,9}$ for example corresponds to a little loop on $N_G(\mathfrak{o})$ about the vertex $f_2^{+1}$.

Since Alexander's duality is valid for any 3-manifold which is a homology sphere we may state a stronger version of Proposition V.1 as

THEOREM V.1. *Suppose $H^2(K)$ is free of rank $h$; then if $K$ can be imbedded in the homology 3-sphere $H$, $\pi_0(K)$ must be a homomorph of $\pi_1(H) * F_h$ for some out-group $\pi_0(K)$.*

## VI. Sufficient conditions for the imbeddability of a simply connected $K$ in a simply connected 3-manifold

We consider in this section some conditions which are sufficient to insure the imbeddability of $K$ in a simply connected manifold.

Given the fact that $\pi_0(G(K))$ is free for some $\mathfrak{o} \in \theta$, there is by Theorem IV.1 an imbedding of $K$ in some orientable 3-manifold; however if $K$ is simply connected the construction of this manifold entailed first the construction of a regular neighborhood of $K$ and then the doubling of this manifold. Notice that this neighborhood is simply connected, as it collapses to $K$. It follows from Seifert-Threlfall p. 223, Satz IV, that no component of the boundary of the regular neighborhood in question has genus $>0$, and this is sufficient for imbedding $K$ in an orientable simply connected manifold, since we may simply cap the 2-sphere boundary components of the boundary of the neighborhood of $K$ by 3-balls. Putting this argument together with Theorem IV.1, we have proved

THEOREM VI.1. *If $K$ is simply connected, then $K$ is imbeddable in a closed simply connected 3-manifold iff $\pi_0(G(K))$ is free for some $\mathfrak{o} \in \theta$.*

Of course this theorem says very little more than Theorem IV.1, and in the case $K$ is not simply connected the situation is much more interesting. We have attacked this problem with some small success, but the results are a bit complicated and we do not present them here but choose to present them in another paper.

## VII. Applications to group theory

Suppose we are given a presentation

$$P = (x_1, \cdots, x_n : r_1, \cdots, r_m)$$

with the property that each generator occurs twice in the set of words $r_j$. Then from such a presentation we may construct a graph $G[P]$ as follows. As vertices of $G[P]$ we take $m$ points which we denote $r_1, \cdots, r_m$. As edges of $G[P]$ we take $n$ edges denoted $x_1, \cdots, x_n$ and as endpoints of $x_i$ we select that pair $r_j$ and $r_k$ in which $x_i$ appears (we allow $x_i$ to begin and end at the same vertex if necessary). Now we construct a 2-manifold containing $G[P]$ as follows. At the vertex $r_i$ the edges corresponding to those $x_j$ occurring in $r_i$ are incident, and these edges are cyclically ordered by the word $r_i$ in the $x_j$. Preserving this cyclic ordering we imbed a nice small neighborhood of each $r_i$ in $G[P]$ on disjoint discs $D_i$. The resulting space $x$ is the union of $m$ discs and $n$ arcs $a_j$, each $a_j$ being a subarc of $x_j$. We will imbed $x$ in a 2-manifold with boundary by thickening each arc $a_i$ to a 2-disc $A_i$. Before doing this we orient each $D_i$ so that the induced orientation on the boundary intersects the $x_j$ in a manner agreeing with the word $r_i$ (at this point we ignore the exponents of the $x_j$). Now in thickening the $a_j$ to a 2-disc each $A_j$ will meet two $D_i$ (not necessarily distinct) along two arcs on $\partial A_j$. We orient $A_j$ and this orients the two arcs meeting the $D_i$. In thickening $a_j$ we may have the orientation of these arcs agree or disagree with that of each of the $D_i$ meeting $\partial A_j$. If the two occurrences of $x_j$ have different exponents in the $r_i$ then we thicken $a_j$ so the arcs' orientations agree with those of the appropriate $\partial D_i$, if they have the same exponents we let one arc's orientation agree with the orientation of a $\partial D_i$ and the other disagree. This puts a "twist" in some of the $A_j$ relative to others. This construction is illustrated in Figure 1.

The (possibly disconnected) manifold, $T$, with boundary components $C_1, \cdots, C_r$ we have constructed may be nonorientable, but this is no obstacle. We add discs $J_1, \cdots, J_r$ to $C_1, \cdots, C_r$ and obtain a closed 2-manifold with components $U_1, \cdots, U_t$. As in the proof of Theorem III.1 we adjoin to $\bigcup_l U_l$ an $r - ad\ \rho$ with endpoints $p_1, p_2 \cdots p_r$ by identifying each $p_i$ with an interior point of $J_i$. Letting $F_q$ denote a free group of rank $q$ the resulting space $V$ has fundamental group isomorphic to $F_q *_{\alpha=1}^{t} \pi_1(U_\alpha)$ on the one hand

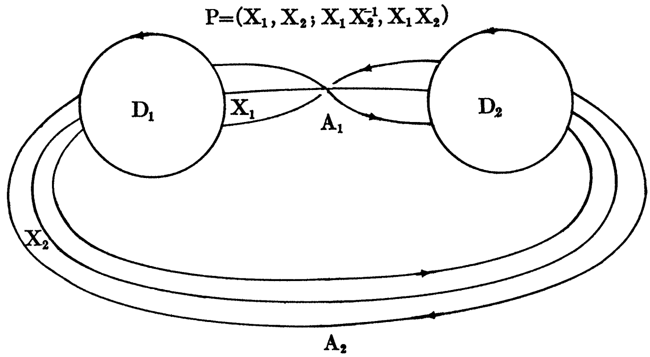
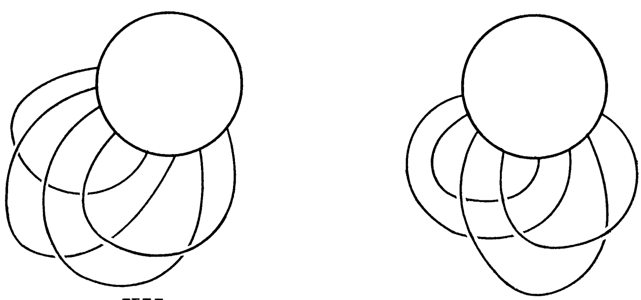



FIGURE 1



$abcd\bar{a}\bar{b}\bar{c}\bar{d}$ $abcd\bar{a}\bar{b}\bar{d}\bar{c}$

FIGURE 2

and to the group $(x_1, x_2, \cdots, x_n; r_1, \cdots, r_m)$ on the other. That the latter is true is easily seen by first adjoining to $\rho \cup_i J_i$, (a contractible space) the open arcs $x_i$ yielding a space with $\pi_1 \approx F_n$; then adjoining the vertices $r_1, \cdots, r_m$ each of which gives rise to the corresponding relation. We have thus proved:

THEOREM VII.1. *If the group G has a quadratic presentation, that is, one in which each generator occurs twice, then it is isomorphic to the free product of a free group with the fundamental groups of a finite number of surfaces.*

Not only is the above theorem true, it is also useful, for one may easily compute the Euler characteristics of the surfaces constructed in the proof of the theorem, and so determine in a very efficient manner exactly the group being presented. For example, Figure 2 shows immediately that

$$(abcd,\ abcd\bar{a}\bar{b}\bar{c}\bar{d}) \approx (X, Y, Z, W; [X, Y][Z, W])$$

and

$$(abcd;\ abcd\bar{a}\bar{b}\bar{d}\bar{c}) \approx (X, Y, Z, W; [X, Y]).$$

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Institute for Defense Analyses
Princeton, New Jersey