THE PRIMITIVE OPERATORS OF AN ALGEBRA OF SINGULAR INTEGRAL OPERATORS

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In [1] we introduced a C^* algebra $\mathfrak{A}$ of singular integral operators ($\mathfrak{A}$ is a subset of the bounded operators on $L^2(R^n)$) and we extended the σ-symbol of Calderón and Zygmund to a homomorphism σ of $\mathfrak{A}$ onto the bounded continuous functions on $R^n \times S^{n-1}$. Two types of primitive operators are basic in the composition of $\mathfrak{A}$. They are the multiplication operators and operators whose Fourier transforms are multiplication operators. In this note, we give the conditions for such operators to belong to $\mathfrak{A}$. We use the notation introduced in [1]. Note that we freely confuse multiplication by f with f.

THEOREM 1. *Let $f \in L^\infty(R^n)$. Then*

(1) *$f \in \mathfrak{A}$ if and only if f is continuous;*

(2) *If $f \in \mathfrak{A}$ then $\sigma(f)(x, \xi) = f(x)$ for $x \in R^n$, $\xi \in S^{n-1}$.*

THEOREM 2. *Let $g \in L^\infty(S^{n-1})$ and let T be the bounded operator on $L^2(R^n)$ defined by $FTf(x) = g(x/\| x\|)Ff(x)$ where F is the Fourier transform and $\| x \|^2 = \sum_{i=1}^n x_i^2$ for $x = (x_1, \cdots x_n)$. Then*

(1) *$T \in \mathfrak{A}$ if and only if g is continuous;*

(2) *if $T \in \mathfrak{A}$ then $\sigma(T)(x, \xi) = g(x)$ for $x \in R^n$, $\xi \in S^{n-1}$.*

Theorem 1 implies immediately that multiplication by f belongs to the subspace $\mathfrak{C}$ of A ($\mathfrak{C}$ is the set of B^∞-singular integral operators) if and only if $f \in B^\infty(R^n)$ (the set of infinitely differentiable, bounded functions, all of whose derivatives are bounded). Since the operators of $\mathfrak{C}$ leave invariant the Sobolev spaces H_k the following theorem is interesting. (H_k is the set of tempered distributions T on R^n whose Fourier transform T^Λ comes from a function for which $\| T \|_k^2 = \int \mid T^\Lambda \mid^2(1 + \| \quad \|^2)^{k/2} < \infty$.)

THEOREM 3. *Let $f \in L^\infty(R^n)$. Then each H_k (k a non-negative integer) is invariant under multiplication by f if and only if $f \in B^\infty(R^n)$.*

1. The kernel of σ

Recall from [1] that $\sigma : \mathfrak{A} \to BC[R^n \times S^{n-1}]$, that σ is a C^* algebra homomorphism of $\mathfrak{A}$ onto $BC[R^n \times S^{n-1}]$, with kernel

(1.1) $\mathcal{K}^{\text{loc}} = [T : T$ is a bounded operator on $L^2(R^n)$, such that ψT and $T\psi$ are compact for every $\psi \in C_0^\infty(R^n)]$.

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We are interested in the relationship between K^{loc} and two classes of operators; the first are multiplication operators; $\phi \in L^\infty(R^n)$, and the second are operators of the form $F^{-1}\phi F$ where F is the Fourier transform.

For the multiplication operators we have the following well known fact.

LEMMA 1. $L^\infty(R^n) \cap \mathcal{K}^{\text{loc}} = (0)$.

Proof. It is sufficient to show that $L^\infty(R^n) \cap \mathcal{K} = (0)$. Let $f \in L^\infty(R^n) \cap \mathcal{K}$ and assume $f \neq 0$. Then there is a set $E \subset R^n$ of positive Lebesgue measure, and an $\varepsilon > 0$, such that $|f| > \varepsilon$ on E. Then $f \mid E$ is a compact, invertible operator on the infinite-dimensional Hilbert space, $L^2(E)$. This is impossible, QED.

In the case of the second class of operators, the situation is not as simple. For instance if T is convolution by any $C_0^\infty(R^n)$ function ϕ, then T is in this class and also in $\mathcal{K}^{\text{loc}}$. For if $\psi \in C_0^\infty(R^n)$ we have

$$(\psi T)f(x) = \int \psi(x)\phi(x - y)f(y)\, dy$$

and

$$(T\psi)f(x) = \int \phi(x - y)\psi(y)f(y)\, dy.$$

Both ψT and $T\psi$ are integral operators whose kernels are in $C_0^\infty(R^n \times R^n)$ and hence are compact operators. In addition $T = F^{-1}(F\phi)F$.

However we are really interested in $F^{-1}gF$ where g is a homogeneous function of degree zero.

Let $g \in L^\infty(S^{n-1})$, the bounded measurable functions on S^{n-1}, measurability with respect to the usual measure ν on S^{n-1}, defined say by using spherical coordinates. Then g extends to a function in $L^\infty(R^n)$ via the formula $g(x) = g(x/\| x \|)$. The extended function is called a bounded homogeneous function of degree zero.

In the following we use $\int f$ for the Lebesgue integral on R^n, and $\| f \|_0$ for the $L^2(R^n)$ norm of f.

LEMMA 2. *Let g be a bounded homogeneous function of degree zero. Suppose the operator $F^{-1}gF \in \mathcal{K}^{\text{loc}}$; then $g = 0$.*

Proof. Suppose $g \neq 0$. *Let*

$$P = [\xi \in S^{n-1} : | g(\xi) | \geq \| g \|_\infty/2 > 0]$$

where $\| g \|_\infty = \sup_{S^{n-1}} | g |$; then $\nu(P) > 0$.

Let $E = [x \in R^n : 1 \leq \| x \| \leq 2 \text{ and } x/\| x \| \in P]$ and let $E_k = kE = [kk : x \in E]$ where $k = 1, 2 \cdots$. If μ denotes the Lebesgue measure on R^n, then it is easily shown by using spherical coordinates that

$$\mu(E_k) = \nu(P)(2^n - 1)k^n.$$

Let $c = \nu(P)(2^n - 1) > 0$. If g_k is the characteristic function of E_k and

$h_k = (1/\sqrt{c}k^{n/2})g_k$ then $\| h_k \|_0 = 1$ and since support $(h_k) \subset [x \in R^n : \| x \| \geq k]$, we have that $h_k \to 0$ weakly as $k \to \infty$. Note also that $h_k(x) = k^{-n/2}h_1(x/k)$.

We now show that for some $m \geq 0$,

$$(1.2) \qquad \int_{[x \in R^n : \| x \| \leq m]} | F^{-1}h_k |^2 \geq \tfrac{1}{2} \qquad \text{for every } k.$$

For $(F^{-1}h_k)(x) = k^{n/2}(F^{-1}h_1)(kx)$ so that

$$\int_{A_m} | F^{-1}h_k |^2 = k^n \int_{A_m} |(F^{-1}h_1) \circ T_k |^2$$

where $A_m = [y \in R^n : \| y \| \leq m]$ and $T_k(x) = kx$ for $x \in R^n$. By the change of variables theorem, we have that

$$\int_{A_m} | F^{-1}h_k |^2 = \int_{kA_m} | F^{-1}h_1|^2 \geq \int_{A_m} | F^{-1}h_1 |^2 \geq \tfrac{1}{2}$$

for large m, since $\| F^{-1}h_1 \|_0 = \| h_1 \|_0 = 1$. This proves (1.2).

There is a $\psi \in C_0^\infty(R^n)$ such that $\psi = 1$ on A_m with m large enough for (1.2) to hold. Let

$$(1.3) \qquad \begin{aligned} h_k' &= h(x)/g(x) \quad \text{if } x \in E_k \\ &= 0 \qquad\qquad\quad \text{if } x \notin E_k . \end{aligned}$$

Then $\| h_k' \|_0 \leq (2/\| g \|_\infty)\| h_k \|_0 = 2/\| g \|_\infty$ so that $h_k' \to 0$ weakly. If $f_k = F^{-1}h_k'$ then $f_k \to 0$ weakly also. But

$$\| \psi F^{-1}gFf_k \|_0^2 \geq \tfrac{1}{2} \qquad \text{by (1.2)}.$$

Therefore $\psi F^{-1}gFf_k$ does not converge to zero in the norm so that $\psi F^{-1}gF$ is not compact. This means that $F^{-1}gF \notin \mathcal{K}^{\text{loc}}$, QED.

2. Proofs of theorems

Proof of Theorem 1. Let $f \in L^\infty(R^n)$. We first note that if f is continuous, then $f \in \mathfrak{A}$ and (2) holds. This follows for $f \in B^\infty(R^n)$ from the definition of σ in [1]. For $f \in UC(R^n)$ (i.e. the uniformly continuous functions) the assertion is obtained by using uniform convergence and Lemma 10 of [1]. Finally, if f is continuous, and $\psi \in C_0^\infty(R^n)$, then $\psi f \in UC(R^n)$ so that $f \in \mathfrak{A}$ by the definition of $\mathfrak{A}$. If $\psi(x) = 1$ then also by definition, $\sigma(f)(x, \xi) = f(x)$; hence (2) holds.

To complete the proof, we must show that $f \in \mathfrak{A}$ implies that f is continuous (i.e. that there is a continuous function agreeing with f almost everywhere). We first show that if $\xi_1, \xi_2 \in S^{n-1}$ and $x \in R^n$ then $\sigma(f)(x, \xi_1) = \sigma(f)(x, \xi_2)$.

Let ϕ_m and δ_m be the $C_0^\infty(R^n)$ functions and real numbers of Theorem 2 of [1]. Let $\psi_{mj} = \phi_m(\cdot - x)\varepsilon^{i\langle \cdot - x, \delta_m \xi_j \rangle}$ for $j = 1, 2$. We have $\| \psi_{m1}\|_0 = \| \psi_{m2} \|_0 = 1$. Therefore

$$\begin{aligned}|\sigma(f)(x, \xi_1) - \sigma(f)(x, \xi_2)| &= \|(\sigma(f)(x, \xi_1) - \sigma(f)(x, \xi_2))\psi_{m1}\|_0 \\ &\leq \|(\sigma(f)(x, \xi_1) - f)\psi_{m1}\|_0 + \|(f - \sigma(f)(x, \xi_2))\psi_{m1}\|_0 .\end{aligned}$$

But from the definition of ψ_{mj} it follows that $|\psi_{m1}| = |\psi_{m2}|$, so that

$$\|(f - \sigma(f)(x, \xi_2))\psi_{m1}\|_0 = \|(f - \sigma(f)(x, \xi_2))\psi_{m2}\|_0 .$$

Now by Theorem 2 of [1] both terms of the sum tend to zero as $m \rightarrow \infty$ which means that

$$\sigma(f)(x, \xi_1) = \sigma(f)(x, \xi_2).$$

Let $h(x) = \sigma(f)(x, \xi)$. Then h is well defined; it is a bounded continuous function on R^n. By the first part of the proof, $\sigma(h)(x, \xi) = h(x) = \sigma(f)(x, \xi)$. Therefore $f - h \in$ kernel $\sigma = \mathcal{K}^{\text{loc}}$; hence $f = h$ by Lemma 1, QED.

Proof of Theorem 2. We note that if g is continuous, then $T \in \mathfrak{A}$ and (2) holds. This follows for $g \in B^\infty(S^{n-1})$ from the definition of σ and for $g \in C(S^{n-1})$ by the Stone-Weierstrass theorem.

To complete the proof, we must show that $T \in \mathfrak{A}$ implies that g is continuous (i.e.—that there is a continuous function agreeing with g almost everywhere on S^{n-1}). We first show that if $x_1, x_2 \in R^n$ and $\xi \in S^{n-1}$, then $\sigma(T)(x_1, \xi) = \sigma(T)(x_2, \xi)$.

With ϕ_m and δ_m as in the proof of Theorem 1, this time let

$$\psi_{mj} = \phi_m(\cdot - x_j)e^{i\langle \cdot - x_j, \delta_m \xi\rangle}$$

for $j = 1, 2$. Note that $\|\psi_{m1}\| = \|\psi_{m2}\| = 1$ and $|F\psi_{m1}| = |F\psi_{m2}|$. Now using also the fact F is an isometry of $L^2(R^n)$ the proof proceeds exactly as in Theorem 1 with T replacing f. Having shown $\sigma(T)$ is independent of x, we define $h(\xi) = \sigma(T)(x, \xi)$ as before; it is a continuous function on S^{n-1}. Let $FSf(y) = h(y/\|y\|)Ff(y)$; then $S \in \mathfrak{A}$ and $\sigma(S) = \sigma(T)$. Therefore $S - T \in \mathcal{K}^{\text{loc}}$; hence $g = h$ by Lemma 2.

Proof of Theorem 3. (a) Suppose $f \in B^\infty(R^n)$ (bounded functions in $C^\infty(R^n)$ whose derivatives are in $L^\infty(R^n)$). Then by the Leibniz rule for distributions, we have that if $g \in H_k$ and $|\alpha| \leq k$ then

$$D_\alpha(fg) = \textstyle\sum_{\beta \leq \alpha} C_\beta(D_\beta f)(D_{\alpha-\beta}\, g).$$

Here differentiation is in the sense of Schwartz and C_β is a constant for each β. Since $D_\beta(f) \in L^\infty(R^n)$ and $D_{\alpha-\beta}\, g \in L^2(R^n)$, we have that $D_\alpha(fg) \in L^2(R^n)$. Therefore $fg \in H_k$. This proves the "if" part of the assertion.

(b) We will show that if multiplication by f maps H_{k_j} into H_{k_j} for a sequence $k_j \rightarrow \infty$ (k_j is a non-negative integer) then $f \in B^\infty(R^n)$.

For any compact set G, there is a $\psi \in C_0^\infty(R^n)$ such that $\psi = 1$ on G. Then since $\bigcap H_{k_j} \subset B^\infty(R^n)$ we have $f\psi \in B^\infty(R^n)$, which shows that $f \in C^\infty(R^n)$.

We will first complete the proof under the additional assumption that multiplication by f is a bounded operator on each of the Hilbert Spaces H_{k_j}. Let $\phi \in C_0^\infty$; $\phi = 1$ on $[x \in R^n : \| x \| \leq 1]$. Then if $x_0 \in R^n$, $\| \phi(\cdot - x_0) \|_{k_j} = \| \phi \|_{k_j}$ because $F(\phi(\cdot - x_0)) = e^{i\langle x_0, \cdot \rangle}(F\phi)$. Therefore, by Sobolev's lemma, we have for $| \alpha | < k_j - n/2$ that

$$\sup_{R^n} | D_\alpha(f(\phi(\cdot - x_0))) | \leq C(k_j) \| f(\phi(\cdot - x_0)) \|_{k_j} \leq C'(k_j) \| \phi \|_{k_j}.$$

But this means that $| D_\alpha f(x) | \leq C'(k_j) \| \phi \|_{k_j}$ if $\| x - x_0 \| \leq 1$ where $C'(k_j)$ does not depend on x_0. Since x_0 is arbitrary, this shows that $f \in B^\infty(R^n)$.

We will now remove the added assumption. Let $\phi_m(x) = \phi(x/m)$. We wish to show that if $s = k_i$, then for any $g \in H_s$, $\phi_m fg$ converges to fg in the H_s norm as $m \to \infty$. Then since $\phi_m f \in B^\infty(R^n)$, it is easily seen from the method of part (a) that multiplication by $\phi_m f$ is a bounded operator from H_s into H_s. Therefore, by the uniform boundedness theorem, multiplication by f is also a bounded operator from H_s into H_s.

It is sufficient to show that if $| \alpha | \leq s$ and $g \in H_s$ then

$$\| D_\alpha(\phi_m fg - fg) \|_0 \to 0 \quad \text{as} \quad m \to \infty.$$

For this let $\varepsilon > 0$. Since $fg \in H_s$, we have $D_\beta(fg) \in L^2(R^n)$ for $| \beta | \leq s$. Therefore there is a number N such that $\int_{E_N} | D_\beta fg |^2 < \varepsilon$ for $| \beta | \leq s$ where $E_N = [x \in R^n : \| x \| \geq N]$. Then if $m \geq N$,

$$\int | D_\alpha((\phi_m - 1)(fg)) |^2 = \int_{E_N} | \textstyle\sum_{\beta < \alpha} C_\beta D_\beta(\phi_m - 1) D_{\alpha-\beta}(fg) |^2.$$

But $D_\beta \phi_m = (1/m^{|\beta|})(D_\beta \phi)_m$ so that $| D_\beta(\phi_m - 1) | \leq \sup_{R^n} | D_\beta \phi |$. Therefore, if $m \geq N$, $\| D_\alpha(\phi_m - 1) fg \|_0^2 \leq M\varepsilon$, where M is independent of ε, QED.

Reference

1. S. M. Newberger, *The σ-symbol of the singular integral operators of Calderón and Zygmund*, Illinois J. Math., vol. 9 (1965), pp. 428–443.

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