THE STRONG SZEGÖ LIMIT THEOREM

BY

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1. Suppose that f is a complex-valued function belonging to L^1 of the circle group with the usual Haar measure (normalized to 1) and D_n is its n^{th} Toeplitz determinant defined by

$$D_n = \det [\hat{f}(i - j)]_{i,j=0}^{n}. \tag{1}$$

It is of considerable importance to be able to obtain an asymptotic estimate for D_n as $n \to \infty$. The now classical theorem by Szegö [12] states that if $f \geq 0$ and $\log f \in L^1$, then

$$\log D_n = (n+1) \int_0^1 \log f(\theta)\, d\theta + o(n) \quad \text{as} \quad n \to \infty. \tag{2}$$

Considerably later, Szegö [13] obtained a more precise result for a more limited class of functions. For real positive functions enjoying considerable smoothness properties he showed that

$$\log D_n = (n+1) \int_0^1 \log f(\theta)\, d\theta + \sum_{k=1}^{\infty} k \mid (\lambda f)^{\wedge}(k) \mid^2 + o(1), \tag{3}$$

where

$$(\lambda f)^{\wedge}(k) = \int_0^1 e^{-2\pi i k\theta} \log f(\theta)\, d\theta.$$

The problem was subsequently taken up by M. Kac [10] and others. The most recent results, obtained by Banach algebra techniques, were initiated by G. Baxter [1], [2], [3] and continued by I. I. Hirschman, Jr., [8]. The latter's result is the following:

If

α. $\sum_{k=-\infty}^{\infty} |\hat{f}(k)| < \infty$,

β. $\sum_{k=-\infty}^{\infty} | k || \hat{f}(k) |^2 < \infty$,

γ. $f(\theta) \neq 0$,

δ. $\Delta \arg f(\theta) = 0$,

then

$$D_n/\mu^{n+1} \to \exp \textstyle\sum_{k=1}^{\infty} k(\lambda f)^{\wedge}(k)(\lambda f)^{\wedge}(-k), \tag{4}$$

where $\mu = \exp \int_0^1 \log f(\theta)\, d\theta$. He has also obtained analogous results when the functions are defined on the real line [9].

Banach algebra techniques, as beautiful as they are, have for these problems certain inherent limitations in that rather severe smoothness require-

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ments are needed for the functions in question. We have, therefore, gone back to the more classical Hilbert space techniques and have been able to obtain more general results. In particular we obtain as special cases Hirschman's result as well as the following result which seems to be closely related to a result announced by L. Onsager in the 1964 Gibbs lecture:

If the conditions β, γ, *and* δ *are satisfied and if in addition*

α'. f *is continuous and there is a non-negative, continuous, doubly periodic function* $m(\theta, \varphi)$ *with* $m(\theta, \theta) = 0$ *and an* $M > 0$ *so that for every* $\varphi \in [0, 1]$

$$\int_0^1 \left| \frac{f(\theta) - f(\varphi)}{\theta - \varphi} \right| \frac{d\theta}{m(\theta, \varphi)} \leq M,$$

then the relation (4) *is valid.*

2. Our results are based upon results obtained in [7] which we shall briefly review and bring into a form suitable for use in this paper. We shall be working on the circle group with Haar measure normalized to one. Let us begin by supposing that $f \in L^1$ and that for all sufficiently large integers n the Toeplitz determinant $D_n \neq 0$. This means that for all sufficiently large n we can form the numbers

$$\mu_n = D_n/D_{n-1}, \tag{5}$$

and moreover there is a unique polynomial

$$u_n(\theta) = 1 + \textstyle\sum_{k=1}^{n} \hat{u}_n(k) e^{2\pi i k\theta} \tag{6}$$

such that

$$\int_0^1 e^{-2\pi i k\theta} u_n(\theta) f(\theta)\, d\theta = 0, \qquad 1 \leq k \leq n. \tag{7}$$

From this it follows immediately that

$$\mu_n = \int_0^1 | u_n(\theta) |^2 f(\theta)\, d\theta = \int_0^1 u_n(\theta) f(\theta)\, d\theta. \tag{8}$$

For any complex number z let Az designate the principal argument of z. In order to proceed we shall suppose that in addition to the condition $f \in L^1$ that

$$\log | f | \in L^1 \tag{9}$$

and there exists a $\gamma \in H^\infty$ such that $1/\gamma \in H^\infty$ and

$$\| A\gamma f \|_\infty < \pi/2, \tag{10}$$

where $\|\cdot\|_\infty$ designates the usual supremum norm. From the facts that $| f |$ and $\log | f |$ are summable it follows from the well-known Szegö factorization theorem that

$$(11) \qquad | f | = | \Phi |^2,$$

where Φ is an outer factor in H^2. Further, from the condition (10) it follows from the considerations of [5] that we may write

$$(12) \qquad e^{iAf} = \Psi/\Psi^* \qquad (z^* = \text{complex conjugate of } z)$$

where Ψ and $1/\Psi$ are outer factors in H^q for some $q > 2$. Hence, using (11) and (12) we may write

$$(13) \qquad f = gh^*,$$

where $g = \Phi\Psi$ and $h = \Phi/\Psi$, g and h are outer factors in H^p, for some $p > 1$.

Let us denote the conjugation operator by C; i.e., for any $u \in L^1$,

$$Cu(\varphi) = \int_0^1 u(\theta) \cot \pi(\varphi - \theta)\, d\theta.$$

It is well known [4] that any $g \in H^1$ is an outer factor if and only if

$$g = \exp \{\log | g | + iC \log | g | + i\alpha\},$$

where α is any value of the argument of $\hat{g}(0)$. By Jensen's inequality it is always true that $\log | g | \in L^1$. However, in general, $C \log | g | \in L^p$, $0 < p < 1$. We can take

$$\log g = \log | g | + iC \log | g | + i\alpha,$$

and in case $C \log | g | \in L^1$ we can integrate to get

$$\int_0^1 \log g(\theta)\, d\theta = \log | \hat{g}(0) | + i\alpha.$$

We have used the fact that g is an outer factor and hence $\log | \hat{g}(0) |$ is the integral of $\log | g |$. Hence we may write

$$\hat{g}(0) = \exp \int_0^1 \log g(\theta)\, d\theta.$$

In case $C \log | h | \in L^1$ we get the same formula for $\hat{h}(0)$. Hence, if we define

$$(14) \qquad \log f = \log g + \log h^*$$

we arrive at the fact that

$$(15) \qquad \hat{g}(0)\hat{h}(0)^* = \exp \int_0^1 \log f(\theta)\, d\theta.$$

Notice that $\log f$ depends on the values chosen for the arguments of $\hat{g}(0)$ and $\hat{h}(0)$ but that $\exp \int_0^1 \log f(\theta)\, d\theta$ is independent of this choice.

In the case where one of the functions $C \log | g |$ or $C \log | h |$ is not summable a similar analysis (carried out in [7] shows that we may take

$$(16) \qquad \log f = \log | f | + 2iC \log | \Psi | + i\beta,$$

where now β is any argument of $\hat{\Psi}(0)$, and in this formulation the formula (15) remains valid. Indeed, it is easy to see that the values of $\log f$ given by (14), coincide with the values given by (16) up to an additive constant $2k\pi i$, where k is an integer. Roughly speaking, the additive constant determines the "branch of $\log f$", and does not affect the formula (15). *For convenience we shall designate the left side, and hence the right side of* (15) *by* μ.

We have broken the main thread of our development in order to get an expression for $\hat{g}(0)\hat{h}(0)^*$. Let us return to it. For *any* n^{th} degree trigonometric polynomial of the form $q_n(\theta) = 1 + \sum_1^n \hat{g}_n(k)e^{2\pi ik\theta}$, it follows from (7) and (8) that

$$\mu^u = \int_0^1 u_n\, g(q_n\, h)^*\, d\theta,$$

and hence we can write

$$\begin{aligned}\mu_n - \mu &= \int_0^1 u_n\, g(q_n\, h)^*\, d\theta - \hat{h}(0)^* \int_0^1 u_n\, g\, d\theta \\ &= \int_0^1 u_n\, \Phi \left\{ \frac{\Psi}{\Psi^*} [q_n\, \Phi - \hat{h}(0)\Psi]^* \right\} d\theta.\end{aligned}$$

Further since $\hat{h}(0) = \hat{\Phi}(0)/\hat{\Psi}(0)$, it follows that $(q_n\, \Phi)\hat{}\,(0) = \hat{h}(0)\hat{\Psi}(0)$ and hence

$$\int_0^1 \frac{1}{\Psi} \left\{ \frac{\Psi}{\Psi^*} [q_n\, \Phi - \hat{h}(0)\Psi]^* \right\} d\theta = 0.$$

Therefore, we get

$$\mu_n - \mu = \int_0^1 \{\mu_n\, \Phi - \hat{g}(0)/\Psi\} \left\{ \frac{\Psi}{\Psi^*} [q_n\, \Phi - \hat{h}(0)\Psi]^* \right\} d\theta.$$

Taking absolute values, and applying the Schwarz inequality we arrive at the estimate

$$|\mu_n - \mu| \le \| u_n\, \Phi - \hat{g}(0)/\Psi \| \; \| q_n\, \Phi - \hat{h}(0)\Psi \|, \tag{17}$$

where $\|\cdot\|$ indicates the usual L^2 norm.

In order to proceed further it is necessary to obtain an estimate for the quantity $\| u_n\, \Phi - \hat{g}(0)/\Psi \|$. To do this it is necessary to use a theorem proved in [7]. Let us set $\varphi = \Psi/\Psi^*$ and T_φ the corresponding Toeplitz operator. Let P_n be the projection of L^2 onto the $(n + 1)$-dimensional subspace of H^2 generated by the set $\{e^{ik\theta}\Phi : 0 \le k \le n\}$. If we denote the latter subspace by $H_n^2(\Phi)$, the following is true:

THEOREM A. (a) *If* $|f|$ *and* $\log |f|$ *are in* L^1 *and* $\| Af \|_\infty < \pi/2$, *then there exists an* $m > 0$ *so that for all* n *and all* $u \in H_n^2(\Phi)$

$$m \| u \| \le \| P_n\, T_\varphi\, u \|. \tag{18}$$

(b) *Suppose*

$$| f | = e^{u+Cv},$$

where $u \in L^\infty$, $\| v \|_\infty < \pi/2$ *and there is a* $\gamma \in H^\infty$ *with* $1/\gamma \in H^\infty$ *so that*

$$\| A\gamma f \|_\infty < \pi/2.$$

If in addition g/g^* *or* h/h^* *is continuous, then there is an* $m > 0$ *and an* N *so that* (18) *is valid for all* $n > N$ *and all* $u \in H_n^2(\Phi)$.

Although we have stated the theorem here and in [7] only for the special situation for which it is needed, the proof given in [7] makes it quite clear that it can be formulated in a way so that it will constitute a generalization of a theorem of Reich [11].

The inequality (18) implies that $D_n \neq 0$ (see [7]). Hence the considerations prior to Theorem A are valid. If $p(\theta)$ is any trigonometric polynomial of the form $p(\theta) = \sum_{k=0}^{n} \hat{p}(k) e^{2\pi i k\theta}$, then we may write

$$\hat{p}(0)^* \{\mu_n - \mu\} = \int_0^1 \frac{\Psi}{\Psi^*} \{\mu_n \Phi - \hat{g}(0)/\Psi\} (p\Phi)^* \, d\theta.$$

Hence,

$$\begin{aligned} \| P_n T_\varphi(u_n \Phi - \hat{g}(0)/\Psi) \| &= \sup \{ | (\varphi[u_n \Phi - \hat{g}(0)/\Psi] \mid p\Phi) : \| p\Phi \| = 1 \} \\ &= \sup \{ | \hat{p}(0) | \, | \mu_n - \mu | : \| p\Phi \| = 1 \} \\ &\leq | \mu_n - \mu | / | \hat{\Phi}(0) |. \end{aligned}$$

The last inequality follows from the fact that $\| p\Phi \| = 1$ implies that

$$| \hat{p}(0) | \, | \hat{\Phi}(0) | \leq 1.$$

Therefore, if $p_n \Phi$ is any element of $H_n^2(\Phi)$ and if f satisfies either of the hypotheses of Theorem A we have from (18), for all sufficiently large n,

$$\begin{aligned} m \| u_n \Phi - p_n \Phi \| &\leq \| P_n T_\varphi(u_n \Phi - p_m \Phi) \| \\ &\leq | \mu_n - \mu | / | \hat{\Phi}(0) | + \| P_n T_\varphi(p_n \Phi - \hat{g}(0)/\Psi \|. \end{aligned}$$

Using the estimate (17) and noting that $\| P_n T_\varphi \| \leq 1$ we get

$$\begin{aligned} m \| u_n \Phi - p_n \Phi \| \leq \{ \| u_n \Phi - p_n \Phi \| + \| p_n \Phi - \hat{g}(0)/\Psi \| \} \| q_n \Phi \\ - \hat{h}(0)\Psi \| / | \hat{\Phi}(0) | + \| p_n \Phi - \hat{g}(0)/\Psi \|. \end{aligned}$$

Now Φ is outer and $\Psi \in H^2$ and hence we can find a sequence q_n so that $q_n \Phi \to \hat{h}(0)\Psi$ in H^2. Hence for all n sufficiently large we can find a q_n so that

$$\| q_n \Phi - \hat{h}(0)\Psi \| / | \hat{\Phi}(0) | < m/2.$$

Using this in the previous inequality we have arrived at the following:

THEOREM 1. *If* f *satisfies either of the hypotheses of Theorem* A, *then there is an* $M > 0$ *and an* N *so that for all* $n > N$ *and all* $p_n \Phi \in H_n^2(\Phi)$,

$$\| u_n \Phi - p_n \Phi \| \leq M \| p_n \Phi - \hat{g}(0)/\Psi \|. \tag{19}$$

COROLLARY 1. *Under the hypotheses of Theorem* 1, *there is an* $M > 0$ *and an* N *so that for all* $n > N$ *and all* $p_n \Phi \in H^2_n(\Phi)$,

$$\| u_n \Phi - \hat{g}(0)/\Psi \| \leq M \| p_n \Phi - \hat{g}(0)/\Psi \|. \tag{20}$$

3. In order to discuss the strong Szegö limit theorem it is necessary for us to make some observations about a certain Hilbert space of functions which we will label $H_{1/2}$. This will be the Hilbert space of all functions on the circle group for which

$$\| c \|^2_{1/2} = | \hat{c}(0) |^2 + \textstyle\sum_{k=-\infty}^{\infty} | k | \, | \hat{c}(k) |^2 < \infty. \tag{21}$$

We shall usually be working with a pseudo-norm on this space and we shall designate this by

$$\| c \|'_{1/2} = \{ \textstyle\sum_{k=-\infty}^{\infty} | k | \, | \hat{c}(k) |^2 \}^{1/2}. \tag{22}$$

There is an alternate expression for $\| c \|'_{1/2}$ which is often very useful. This is given by

$$\| c \|'^2_{1/2} = \int_0^1 \int_0^1 \left| \frac{c(\theta) - c(\varphi)}{2 \sin \pi(\theta - \varphi)} \right|^2 d\theta \, d\varphi. \tag{22'}$$

In case $c \in H^2 \cap H_{1/2}$, then c may be extended analytically inside the unit disk D and we have the representation

$$\| c \|'^2_{1/2} = \frac{1}{\pi} \iint_D | c'(z) |^2 \, dx \, dy, \qquad z = x + iy, \tag{22''}$$

where $c'(z)$ is the derivative with respect to z of the analytic extension $c(z)$ of c into the unit disk. As is well known, for almost all θ, $c(re^{2\pi i\theta}) \to c(\theta)$ as $r \to 1$. (We are somewhat abusing the notation here since quite properly the boundary function should be denoted by $c(e^{2\pi i\theta})$ instead of $c(\theta)$. However we think no confusion will result.) Both of these formulas are essentially an application of the Plancherel theorem and we leave their verification to the reader. Of course the formulas (22′) and (22″) are valid even if $c \notin H_{1/2}$ in which case both sides are infinite.

In case c is an *outer factor* it has a non-vanishing extension into D and moreover $\log c$ can be defined as in Section 2 and this belongs certainly to H^p for $0 < p < 1$. If we write $\lambda c = \log c$, then the formula (22″) takes the form

$$\| \lambda c \|'^2_{1/2} = \frac{1}{2\pi} \iint_D \left| \frac{c'(z)}{c(z)} \right|^2 dx \, dy = \frac{1}{2\pi} \iint_D | \{\log c(z)\}' |^2 \, dx \, dy.$$

If $c \in L^1$, let $\sigma_n(c)$ be the n^{th} Fejer mean of c and $s_n(c)$ the n^{th} partial sum of the Fourier series of c; i.e.,

$$\sigma_n(c)(\theta) = \textstyle\sum_{k=-n}^{n} (1 - | k |/(n + 1)) \hat{c}(k) e^{2\pi i k\theta},$$
$$s_n(c)(\theta) = \textstyle\sum_{k=-n}^{n} \hat{c}(k) e^{2\pi i k\theta}.$$

PROPOSITION 1. *If $c \in H_{1/2}$, then*

$$\| s_n(c) - \sigma_n(c) \|_\infty \to 0 \quad \textit{as } n \to \infty.$$

Proof. We have

$$\begin{aligned} | s_n(c)(\theta) - \sigma_n(c)(\theta) |^2 &= | \textstyle\sum_{k=-n}^{n} | k |/(n+1)\hat{c}(k)e^{2\pi ik\theta} |^2 \\ &\leq 2(n+1) \textstyle\sum_{k=-n}^{n} | k/(n+1) |^2 | \hat{c}(k) |^2 \\ &\leq 2 \textstyle\sum_{k=-n}^{n} | k/(n+1)^{1/2} |^2 | \hat{c}(k) |^2. \end{aligned}$$

Break up the sum on the right into two parts. The first part shall consist of those terms for which $| k |/(n+1)^{1/2} \leq 1/(n+1)^{1/4}$ and the second part will consist of the remaining terms. Noting the fact that $| k |/(n+1) < 1$ for $| k | \leq n$, we have

$$\begin{aligned} &| s_n(c)(\theta) - \sigma_n(c)(\theta) |^2 \\ &\qquad \leq 2/(n+1)^{1/2} \textstyle\sum_{|k| \leq (n+1)^{1/4}} | \hat{c}(k) |^2 + 2 \textstyle\sum_{|k| > (n+1)^{1/4}} | k | | \hat{c}(k) |^2. \end{aligned}$$

Since $c \in L^2$ the first term goes to zero as $n \to \infty$, and since $c \in H_{1/2}$ the second term goes to zero. This completes the proof.

PROPOSITION 2. *If $0 < m \leq | f | \leq M < \infty$ and f satisfies the other hypotheses of part* (a) *or* (b) *of Theorem* A, *then for all sufficiently large n,*

$$\left.\begin{aligned} &\| u_n - s_n(\hat{g}(0)/g) \|_\infty \\ &\| u_n - s_n(\hat{g}(0)/g) \|_{1/2} \end{aligned}\right\} = O\{n^{1/2} \| s_n(1/g) - 1/g \|\}$$

$$= O\{\| s_n(1/g) - 1/g \|_{1/2}\}.$$

In particular if $1/g \in H_{1/2}$ both quantities on the left go to zero.

Proof. Since f satisfies the hypotheses of Theorem A we may factor it as $f = gh^*$ and (18) and (19) are valid. Hence, u_n exists for all sufficiently large n and we may write

$$\begin{aligned} | u_n(\theta) - s_n(\hat{g}(0)/g)(\theta) |^2 &= | \textstyle\sum_{k=0}^{n} \{\hat{u}_n(k) - (\hat{g}(0)/g)^\wedge(k)\} e^{2\pi ik\theta} |^2 \\ &\leq n \textstyle\sum_{k=1}^{n} | \hat{u}_n(k) - (\hat{g}(0)/g)^\wedge(k) |^2. \end{aligned}$$

In (19) take $p_n = s_n(\hat{g}(0)/g)$ and use the fact that $0 < m \leq | \Phi |^2 \leq M$. Then for all sufficiently large n,

$$m \| u_n - s_n(\hat{g}(0)/g) \| \leq \| u_n \Phi - s_n \Phi \| \leq M_1 \| s_n(1/g) - 1/g \|,$$

where M_1 is a fixed positive constant. Putting this into the previous inequality gives the first inequality of our proposition.

Similarly, using (19),

$$\begin{aligned} \textstyle\sum_{k=1}^{n} k | \hat{u}_n(k) - (\hat{g}(0)/g)^\wedge(k) |^2 &\leq n \textstyle\sum_{k=1}^{n} | \hat{u}_n(k) - (\hat{g}(0)/g)^\wedge(k) |^2 \\ &= O\{n \| s_n(1/g) - 1/g \|^2\}. \end{aligned}$$

The fact that

$$n \| s_n(1/g) - 1/g \|^2 = O\{\|s_n(1/g) - 1/g\|_{1/2}^2\}$$

is clear.

PROPOSITION 3. *Suppose* $0 < m \leq |f| \leq M < \infty$ *and there is a* $\gamma \in H^\infty$ *with* $1/\gamma \in H^\infty$ *so that* $\| A\gamma f \|_\infty < \pi/2$ *and moreover that* γ *and* $\log \gamma$ *belong to* $H_{1/2}$. *Then* $f \in H_{1/2}$ *if and only if* $\log f \in H_{1/2}$ *where* $\log f$ *is defined by* (14).

Proof. Let us set

$$f_1 = \gamma f.$$

If $\operatorname{Log} z$ represents the principal value of the logarithm then we can clearly choose the "branches" of $\log \gamma$ and $\log f$ so that

$$\operatorname{Log} f_1 = \log \gamma + \log f.$$

Let us suppose that $f \in H_{1/2}$. Since $\gamma \in H_{1/2}$ and both γ and f are bounded it follows from (22′) that $f_1 = \gamma f$ belongs to $H_{1/2}$. As we shall now show this will imply that $\operatorname{Log} f_1 \in H_{1/2}$, and since $\log \gamma \in H_{1/2}$ it follows that $\log f \in H_{1/2}$. Clearly the "branch" of $\log f$ that we choose does not affect $\| \log f \|'_{1/2}$ and hence if one "branch" of $\log f$ belongs to $H_{1/2}$, any "branch" will also.

To show that $\operatorname{Log} f_1$ belongs to $H_{1/2}$ we simply notice that f_1 is bounded and bounded away from zero and hence its range lies in a compact set in the open right complex plane (remember that $\| Af_1 \|_\infty < \pi/2$!). Hence, there is a constant K so that

$$|\operatorname{Log} f_1(\theta) - \operatorname{Log} f_1(\varphi)| \leq K |f_1(\theta) - f_1(\varphi)|$$

for all θ and φ. If we use this inequality in the right hand side of (22′) we see that indeed $\operatorname{Log} f_1 \in H_{1/2}$.

Conversely, suppose that $\log f$ is in $H_{1/2}$; then $\operatorname{Log} f_1 \in H_{1/2}$. Since f_1 is bounded and bounded away from zero, $\operatorname{Log} f_1$ lies in a compact set in the complex plane. Hence, since

$$f_1 = e^{\operatorname{Log} f_1}$$

there is a constant K so that

$$|f_1(\theta) - f_1(\theta)| \leq K |\operatorname{Log} f_1(\theta) - \operatorname{Log} f_1(\varphi)|$$

for all θ and φ. Hence $f_1 \in H_{1/2}$ and since $f = f_1/\gamma$, it follows that $f \in H_{1/2}$. We have of course used the fact that since γ and $1/\gamma$ are in L^∞, $\gamma \in H_{1/2}$ if and only if $1/\gamma \in H_{1/2}$.

COROLLARY 2. *If* $0 < m \leq |f| \leq M < \infty$ *and the argument of* f *can be chosen as a continuous function on the circle group, then the conclusions of Proposition 3 are valid.*

Proof. Since $\arg f(\theta)$ is continuous there is a real trigonometric polynomial $p(\theta) = \sum_{p=-n}^{n} \hat{p}(k) e^{2\pi i k\theta}$ so that $\| p(\theta) - \arg f(\theta) \|_\infty < \pi/2$. Choose

$$\gamma(\theta) = \exp - 2i\{\hat{p}(0)/2 + \textstyle\sum_{k=1}^{n} \hat{p}(k) e^{2\pi i k\theta}\}.$$

Clearly γ and $1/\gamma$ are in H^∞, $\| A\gamma f \|_\infty < \pi/2$ and $\log \gamma \in H_{1/2}$. Further, since γ is continuously differentiable as a function of θ, it also belongs to $H_{1/2}$. The result now follows from Proposition 3.

PROPOSITION 4. *If $f = gh^*$ with $g, h, 1/g, 1/h$ in H^∞, then* $\log f$ (as defined by (14)) *belongs to $H_{1/2}$ if and only if g and h are in $H_{1/2}$* (or either of the equivalent conditions: $1/g, 1/h \in H_{1/2}$; $\log g, \log h \in H_{1/2}$).

Proof. Since $\log f = \log g + \log h^*$, it follows that $(\lambda f)^\wedge(k) = (\lambda g)^\wedge(k)$ for $k > 0$ and $(\lambda f)^\wedge(k) = (\lambda h)^\wedge(k)^*$ for $k < 0$, where $\lambda f = \log f$, etc. Hence, it is enough to show that $g \in H_{1/2}$ if and only if $\lambda g \in H_{1/2}$ and the same for h. However, since g and h are bounded and bounded away from zero we have

$$\| \lambda g \|_{1/2}'^2 = \frac{1}{\pi} \iint_D \left| \frac{g'(z)}{g(z)} \right|^2 dx\, dy = O\left\{ \frac{1}{\pi} \iint_D | g'(z) |^2 \, dx\, dz \right\}$$

$$= O\, \{ \| g \|_{1/2}'^2 \},$$

and vice versa. Of course, the same is true for h. This completes the proof.

4. We are now in a position to obtain some results about the limit of D_n/μ^{n+1}. Our first result gives conditions under which this limit exists, but does not specify its value.

THEOREM 2. *If $f \in L^\infty$ and satisfies the hypotheses of either* (a) *or* (b) *of theorem* A *and if in addition $1/g$ and $1/h$ are in $H_{1/2}$, then*

$$\lim_{n\to\infty} D_n/\mu^{n+1}$$

exists as a finite non-zero number.

Proof. Suppose that n_0 is an integer so that for $n \geq n_0$, $D_n \neq 0$. Then we may write

$$\frac{D_n}{\mu^{n+1}} = \frac{\mu_n}{\mu} \cdot \frac{\mu_{n-1}}{\mu} \cdots \frac{\mu_{n_0+1}}{\mu} \cdot \frac{D_{n_0}}{\mu^{n_0+1}}.$$

We should, of course, note that under the conditions of theorem A, μ exists and is a non-zero number (see [7]).

A sufficient condition that the product on the right have a finite non-zero limit is of course that

$$\sum_{k>n_0} | 1 - \mu_k/\mu | < \infty.$$

From the estimates (17) and (20) and the fact that $| \Phi |^2 = | f |$ is bounded we get

$$| 1 - \mu_n/\mu | = O\{ \| p_n - \hat{g}(0)/g \| \cdot \| q_n - \hat{h}(0)/h \| \}$$

where p_n and q_n can be taken to be any n^{th} degree "analytic" trigonometric polynomials with constant coefficient 1. We choose

$$p_n = s_n(\hat{g}(0)/g), \qquad q_n = s_n(\hat{h}(0)/h).$$

Then

$$\sum_{k>n_0} | 1 - \mu_k/\mu | = O\{ \sum_{k>n_0} [\sum_{j>k} |(1/g)\,\hat{}\,(j)|^2]^{1/2} [\sum_{j>k} |(1/h)\,\hat{}\,(j)|^2]^{1/2} \}.$$

Applying the Schwarz inequality on the right we finally get

$$\sum_{k>n_0} | 1 - \mu_k/\mu | = O\{ \| 1/h \|_{1/2}' \, \| 1/h \|_{1/2}' \}.$$

COROLLARY 3. *If the hypotheses of Theorem 2 are satisfied, then*

$$(\mu_n/\mu)^n \to 1 \quad as \quad n \to \infty.$$

Proof. This corollary is really a corollary to the proof of Theorem 2. As we have shown in the proof

$$1 - \mu_n/\mu = O\{ [\sum_{k>n} |(1/g)\,\hat{}\,(k)|^2]^{1/2} [\sum_{k>n} |(1/h)\,\hat{}\,(k)|^2]^{1/2} \}.$$

Since $1/g$ and $1/h$ are in $H_{1/2}$, it follows that

$$1 - \mu_n/\mu = o(1/n).$$

This says that

$$(\mu_n/\mu)^n - 1 = (1 + \varepsilon_n/n)^n - 1,$$

where $\varepsilon_n \to 0$. Hence, given $\varepsilon > 0$, there is an N so that $n > n$ implies $| \varepsilon_n | < \varepsilon$ and hence for $n > N$

$$|(\mu_n/\mu)^n - 1 | \leq (1 + | \varepsilon_n |/n)^n - 1 \leq e^{\varepsilon} - 1.$$

In order to identify the limit in Theorem 2 it will be necessary for us to make more stringent assumptions about the outer factors g and h which appear in the factorization of f. It will also be necessary for us to use an identity due to Baxter [1] and Szegö [13]. In order to write down the Baxter-Szegö identity we first note that if $D_n \neq 0$ then there exists a unique trigonometric polynomial

$$v_n(\theta) = 1 + \sum_{k=1}^{n} \hat{v}_n(k) e^{2\pi i k\theta}$$

so that

$$\int_0^1 e^{2\pi i k\theta} v_n^*(\theta) f(\theta) \, d\theta = 0, \qquad 1 \leq k \leq n.$$

Clearly it is possible to obtain for the sequence $\{v_n\}$ results analogous to those we have obtained for the sequence $\{u_n\}$. The functions u_n and v_n can clearly be extended analytically into the unit disk (indeed over the entire complex plane) and the Baxter-Szegö identity says that if these polynomials do not vanish in the closed unit disk (which means in particular that they are outer factors) then

$$\frac{D_n}{\mu_n^{n+1}} = \exp\left\{ \sum_{k=1}^{\infty} k(\lambda u_n)\,\hat{}\,(k)\,(\lambda v_n)\,\hat{}\,(k)^* \right\} = \exp \frac{1}{\pi} \iint_D \left(\frac{u_n'(z)}{u_n(z)} \right) \left(\frac{v_n'(z)}{v_n(z)} \right)^* dx \, dy.$$

THEOREM 3. *Suppose f is a complex-valued function on the circle group which satisfies the following hypotheses:*

1. *f is continuous and $f(\theta) \neq 0$,*
2. *$f \in H_{1/2}$,*

3. $\Delta \arg f = 0$
4. $C \log f$ *is continuous.*

Then

$$\lim_{n\to\infty} D_n/\mu^{n+1} = \exp\{\textstyle\sum_{k=1}^{\infty} k(\lambda f)^{\wedge}(k)(\lambda f)^{\wedge}(-k)\}.$$

Proof. We here take $\log f = \log|f| + i \arg f$, where $\arg f$ is continuous on the circle group. The conditions 1 and 3 imply that

$$f = gh^*$$

where g, h, $1/g$ and $1/h$ are in H^p for some $p > 1$. This follows from the discussion at the beginning of Section 2 and the proof of Corollary 2. Indeed, up to non-essential multiplicative constants we may take

$$g = \exp \tfrac{1}{2}\{\log f + iC \log f\}$$
$$h = \exp \tfrac{1}{2}\{\log f^* + iC \log f^*\}.$$

This is always true for factorizations of the form (13) for functions satisfying (9) and (10). In this case the proof is particularly simple. Designate the right hand sides of the above expressions by g_1 and h_1 respectively. As is well known [4] any functions which can be written in this form are outer factors and indeed because of the assumptions 1 and 4 we have that g_1, h_1, $1/g_1$ and $1/h_1$ are continuous outer factors. Further it is clear that

$$f = gh^* = g_1 h_1^*$$

and hence

$$g/g_1 = (h_1/h)^* = \text{constant}.$$

Consequently, all of the conditions of (b) of theorem A are satisfied and in particular proposition 2 is valid for both u_n and v_n. In the case of v_n, of course, we must consider $s_n(\hat{h}(0)/h)$. It follows from our hypotheses and Corollary 2 that $\log f \in H_{1/2}$ and hence from Proposition 4 that g, h, $1/g$, $1/h$ are in $H_{1/2}$. Hence Proposition 1 is valid for these functions.

Since $1/g$ is continuous it follows from Propositions 1 and 2 that

$$\| u_n - \hat{g}(0)/g \|_\infty \to 0.$$

Since $1/g$ is outer and does not vanish on the circle group it follows that its analytic extension does not vanish in the closed unit disk. Hence using the maximum modulus principle and the uniform convergence of u_n to $\hat{g}(0)/g$ it follows that there is an n_1 so that $n > n_1$ implies that the analytic extension of u_n does not vanish in the closed unit disk. In the same way $v_n \to \hat{h}(0)/h$ uniformly and we may as well suppose that we have chosen n_1 large enough so that $n > n_1$ implies that the analytic extension of v_n does not vanish on the closed unit disk.

We shall presently show that

$$\| \lambda u_n - \lambda(1/g) \|_{1/2}' \quad \text{and} \quad \| \lambda v_n - \lambda(1/h) \|_{1/2}'$$

go to zero as $n \to \infty$. From this it follows that

$$\sum_{k=1}^{\infty} k(\lambda u_n)^\wedge(k)(\lambda v_n)^\wedge(k)^* \to \sum_{k=1}^{\infty} k(\lambda g)^\wedge(k)(\lambda h)^\wedge(k)^*$$
$$= \sum_{k=1}^{\infty} k(\lambda f)^\wedge(k)(\lambda f)^\wedge(-k).$$

Taken in conjunction with the Baxter-Szegö identity this tells us that

$$\lim_{n\to\infty} D_n/\mu_n^{n+1} = \exp\{\sum_{k=1}^{\infty} k(\lambda f)^\wedge(k)(\lambda f)^\wedge(-k)\}.$$

However, from Corollary 3 we know that $(\mu_n/\mu)^{n+1} \to 1$, and hence the proof of our theorem will be complete.

We shall only establish the fact that $\| \lambda u_n - \lambda(1/g)\|'_{1/2} \to 0$ since the proof for $\lambda v_n \to \lambda(1/h)$ is the same. First of all we notice, using Proposition 2, that

$$\| u_n - \hat{g}(0)/g \|'_{1/2} \leq \| u_n - s_n(\hat{g}(0)/g)\|'_{1/2} + \| s_n(\hat{g}(0)/g) - \hat{g}(0)/g \|'_{1/2}$$
$$= O\{\| s_n(1/g) - 1/g \|'_{1/2}\} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, we may write, for $n > n_1$,

$$\| \lambda u_n - \lambda(1/g)\|'_{1/2} = \left[\frac{1}{\pi}\iint_D \left|\frac{u_n'(z)}{u_n(z)} + \frac{g'(z)}{g(z)}\right|^2 dx\, dy\right]^{1/2}$$
$$\leqq \left[\frac{1}{\pi}\iint_D \left|\frac{u_n'(z)}{u_n(z)} - \left(\frac{\hat{g}(0)}{g(z)}\right)'\frac{1}{u_n(z)}\right|^2 dx\, dy\right]^{1/2}$$
$$+ \left[\frac{1}{\pi}\iint_D \left|\frac{g'(z)}{g(z)}\right|^2 \left|\frac{\hat{g}(0)}{u_n(z)g(z)} - 1\right|^2 dx\, dy\right]^{1/2}.$$

For $n > n_1$ the first integral of this last sum is

$$O\{\| u_n - \hat{g}(0)/g \|'_{1/2}\},$$

which we have noted above goes to zero, and since $\hat{g}(0)/u_n(z)g(z) \to 1$ uniformly in the closed unit disk and $\log g \in H_{1/2}$ we have shown that $\lambda u_n \to \lambda(1/g)$.

Remarks. If in Theorem 3 we replace condition 4 by the condition

4′. $C \log |f|$ *is continuous*

then it is an immediate consequence of Theorem 2 that the limit of D_n/μ^{n+1} exists. This follows from the fact that g/g^* and h/h^* are continuous and hence the conditions of Theorem 2 are fulfilled.

In the simple situation of Theorem 3, that is to say where the functions g and h are continuous, it is of course possible to use Reich's theorem [11] instead of our more general Theorem A to obtain the needed estimates corresponding to (17) and (19).

We should perhaps also note that the limit of D_n/μ^{n+1} can be alternatively written in the form

$$\exp \frac{1}{\pi}\iint_D \left(\frac{g'(z)}{g(z)}\right)\left(\frac{h'(z)}{h(z)}\right)^* dx\, dy.$$

5. The fact that Theorem 3 contains Hirschman's theorem is quite clear.

It will still take a small amount of work to show that it contains the last result of §1. To do this it will be enough to show that the condition α' implies our condition 4, namely that $C \log f$ is continuous. We begin with a slightly more general result.

PROPOSITION 5. *If $f \in L^1$, is continuous at $\varphi \in [0, 1]$, and*

$$\int_0^1 \left| \frac{f(\theta) - f(\varphi)}{\theta - \varphi} \right| d\theta < \infty, \tag{23}$$

then the conjugate function Cf is continuous at φ if and only if

$$\int_{-2|h|}^{2|h|} \frac{f(\varphi - t) - f(\varphi + h)}{t + h} \, dt = o(1) \quad as \quad h \to 0. \tag{24}$$

Proof. Our proof is a modification of a proof given in Zygmund [14, p. 122] for a similar situation. It will be more convenient to work on the interval $[-\frac{1}{2}, \frac{1}{2}]$. The condition (23) is clearly equivalent with

$$\int_{-1/2}^{1/2} \left| \frac{f(\theta) - f(\varphi)}{\tan \pi(\theta - \varphi)} \right| d\theta < \infty.$$

We can write

$$Cf(\varphi) = \int_{-1/2}^{1/2} \frac{f(\varphi - t) - f(\varphi)}{\tan \pi t} \, dt,$$

$$Cf(\varphi + h) = \int_{-1/2}^{1/2} \frac{f(\varphi - t) - f(\varphi + h)}{\tan \pi(t + h)} \, dt.$$

Let us set

$$I_1(h) = \int_{-2|h|}^{2|h|} \frac{f(\varphi - t) - f(\varphi)}{\tan \pi t} \, dt, \qquad I_2(h) = \int_{-2|h|}^{2|h|} \frac{f(\varphi - t) - f(\varphi + h)}{\tan \pi(t + h)} \, dt,$$

$$I(h) = I_1(h) + I_2(h).$$

Then we write

$$\begin{aligned} &Cf(\varphi + h) - Cf(\varphi) \\ &= \left(\int_{-1/2}^{-2|h|} + \int_{2|h|}^{1/2} \right) [f(\varphi - t) - f(\varphi)] [\cot \pi(t + h) - \cot \pi t] \, dt \\ &\qquad - [f(\varphi + h) - f(\varphi)] \left(\int_{-1/2}^{-2|h|} + \int_{2|h|}^{1/2} \right) \cot \pi(t + h) \, dt + I(h). \end{aligned}$$

By a direct computation the second set of integrals on the right is $O(1)$ and by the continuity of f at φ, the corresponding term is $o(1)$ as $h \to 0$.

Let us now concentrate our attention on the first set of integrals. We may write

$$\begin{aligned} &\left| \int_{2|h|}^{1/2} [f(\varphi - t) - f(\varphi)] [\cot \pi(t + h) - \cot \pi t] \, dt \right| \\ &\qquad \leq \int_{2|h|}^{1/2} \left| \frac{f(\varphi - t) - f(\varphi)}{\sin \pi t} \right| \left| \frac{\sin \pi h}{\sin \pi(t + h)} \right| dt. \end{aligned} \tag{26}$$

Let $m(t)$ be a positive monotone decreasing function on $(0, \frac{1}{2}]$ so that $m(0+) = \infty$ and

$$\int_0^{1/2} \left| \frac{f(\varphi - t) - f(\varphi)}{\sin \pi t} \right| m(t)\, dt < \infty.$$

This is always possible since $[f(\varphi - t) - f(\varphi)]/\sin \pi t$ is a summable function of t. Given $\varepsilon > 0$, choose δ so that $1/m(\delta) < \varepsilon$ and then h sufficiently small so that $0 < 2 \mid h \mid < \delta$ and

$$\int_\delta^{1/2} \left| \frac{f(\varphi - t) - f(\varphi)}{\sin \pi t} \right| \left| \frac{\sin \pi h}{\sin \pi(t + h)} \right| dt < \varepsilon. \tag{27}$$

Further, we have

$$\begin{aligned} &\int_{2|h|}^{\delta} \left| \frac{f(\varphi - t) - f(\varphi)}{\sin \pi t} \right| \left| \frac{\sin \pi h}{\sin \pi(t + h)} \right| dt \\ &\qquad \leq \frac{1}{m(\delta)} \int_{2|h|}^{\delta} \left| \frac{f(\varphi - t) - f(\varphi)}{\sin \pi t} \right| m(t)\, dt \\ &\qquad < \varepsilon \int_0^{1/2} \left| \frac{f(\varphi - t) - f(\varphi)}{\sin \pi t} \right| m(t)\, dt. \end{aligned} \tag{28}$$

If we combine (27) and (28) we see the left side of (26) is $o(1)$ as $h \to 0$. Analogously we can work with the integral over $[-\frac{1}{2}, -2 \mid h \mid]$. Hence we see that the first set of integrals in (25) is $o(1)$ as $h \to 0$.

It remains to examine $I(h)$. Since $[f(\varphi + t) - f(\varphi)]/\tan \pi t$ is a summable function of t, $I_1(h) = o(1)$ as $h \to 0$. Hence Cf is continuous at φ if and only if $I_2(h) = o(1)$ which is clearly equivalent with (24).

Corollary 4. *If f is continuous and there is a non-negative, continuous, doubly periodic function $m(\theta, \varphi)$ with $m(\theta, \theta) = 0$ and an $M > 0$ so that for every $\varphi \in [0, 1]$*

$$\int_0^1 \left| \frac{f(\theta) - f(\varphi)}{\theta - \varphi} \right| \frac{d\theta}{m(\theta, \varphi)} \leq M, \tag{29}$$

then (24) is satisfied for f and hence Cf is continuous.

Proof. Let us again work on $[-\frac{1}{2}, \frac{1}{2}]$. To show that condition (24) is satisfied it will be enough to show that

$$\int_{-2|h|}^{2|h|} \left| \frac{f(\varphi - t) - f(\varphi + h)}{t + h} \right| dt = o(1) \quad \text{as} \quad h \to 0. \tag{30}$$

However, from (29) the condition (30) is almost trivial. Indeed, set $n(t, \varphi) = m(\varphi - t, \varphi)$; clearly $n(t, \varphi)$ is a continuous periodic function and $n(0, \varphi) = 0$. Now given φ and ε take h sufficiently small so that $n(t + h, \varphi + h) < \varepsilon/M$ for $\mid t \mid \leq 2 \mid h \mid$. Then

$$\begin{aligned}\int_{-2|h|}^{2|h|} &\left| \frac{f(\varphi - t) - f(\varphi + h)}{t + h} \right| dt \\ &\leq \int_{-2|h|}^{2|h|} \left| \frac{f(\varphi - t) - f(\varphi + h)}{t + h} \right| \frac{n(t + h, \varphi + h)}{n(t + h, \varphi + h)} dt \\ &< \varepsilon.\end{aligned}$$

Of course Corollary 4 can be proved directly since the condition (29) will make the integral defining Cf uniformly convergent.

Some special cases of condition (29) are as follows: If $0 < \alpha < 1$ and there is an $M > 0$ so that for all $\varphi \in [0, 1]$

$$\int_0^1 \frac{|f(\theta) - f(\varphi)|^\alpha}{|\theta - \varphi|} d\theta \leq M,$$

then (29) is satisfied. Indeed, choose $m(\theta, \varphi) = |f(\theta) - f(\varphi)|^{1-\alpha}$. Also, if $p > 1$ and for all φ

$$\int_0^1 \left| \frac{f(\theta) - f(\varphi)}{\theta - \varphi} \right|^p d\theta \leq M$$

then it is easy to show there is an $0 < \alpha < 1$ for which the previous inequality is satisfied. Of course in this last case the continuity of Cf is immediate.

COROLLARY 5. *If f satisfies* 1 *and* 3 *of Theorem* 3 *and in addition* (29) *of Corollary* 4 *is satisfied then* $C \log f$ *is continuous.*

Proof. From the proof of Corollary 2 there is an analytic trigonometric polynomial p so that if $\gamma = \exp(-p)$ and

$$f_1 = \gamma f,$$

then $\| Af_1 \|_\infty < \pi/2$. Further, it is clear that

$$|f_1(\theta) - f_1(\varphi)| = O\{|f(\theta) - f(\varphi)| + |\gamma(\theta) - \gamma(\varphi)|\}.$$

Since γ is continuously differentiable it is clear that (30) is true for γ. Since (30) is true for f, it is true for f_1.

Now, $\log f = \operatorname{Log} f_1 + p$, where as before $\operatorname{Log} z$ is the principal branch of the logarithm function. Since

$$|\operatorname{Log} f_1(\theta) - \operatorname{Log} f_1(\varphi)| = O\{|f_1(\theta) - f_1(\varphi)|\}$$

it follows that (30) is true for $\operatorname{Log} f_1$. Since p is continuously differentiable (30) is true for p. Hence (30) is true for $\log f$. Since $\log f$ is continuous the proof is completed by applying Proposition 5.

6. As Szegö has pointed out, for f real D_n/μ^{n+1} is non-decreasing and hence the limit will always exist, although it may not necessarily be finite. Using results obtained in [6] we can establish the following:

If $0 < m \leq f \leq M < \infty$ then D_n/μ^{n+1} goes to a finite limit if and only if $f \in H_{1/2}$.

Because of the boundedness conditions on f, it has a factorization $f = |g|^2$, where g is outer. It follows from Propositions 3 and 4 that if $f \in H_{1/2}$ then $1/g \in H_{1/2}$. The sufficiency is then a consequence of Theorem 2.

To prove the necessity we notice that since f is real, $\mu_n > \mu$ for all n and hence $1 - (\mu_n/\mu)$ is always non-positive. From the proof of Theorem 2 it follows that a necessary and sufficient condition for the convergence of D_n/μ^{n+1} to a finite limit is that

$$\sum_{n=0}^{\infty} | 1 - \mu_n/\mu | < \infty .$$

We have shown in [6] that $f \geq m > 0$ implies there is a constant α so that

$$\sum_{k>n} |(1/g)\hat{}(k)|^2 < \alpha | 1 - \mu_n/\mu |.$$

Summing over n we arrive at the fact that

$$\sum_{k=1}^{\infty} k\, |(1/g)\hat{}(k)|^2 \leq \alpha \sum_{n=0}^{\infty} | 1 - \mu_n/\mu | < \infty ;$$

i.e., $1/g \in H_{1/2}$. Propositions 3 and 4 imply that $f \in H_{1/2}$.

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