ON REPRESENTABILITY OF CONTRAVARIANT FUNCTORS OVER NON-CONNECTED CW COMPLEXES

BY

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1. Introduction

We assume throughout that each space under consideration has a prescribed base point and that all maps and homotopies preserve this base point. Let $\mathcal{W}$ denote the category of spaces admitting the structure of a finite CW complex, the base point being a vertex, and all (continuous) maps. Let $\mathcal{W}'$ denote the full subcategory of connected spaces.

Recall that a space B_F is a *classifying space* for a based-set-valued contravariant function F defined on a category $\mathcal{C}$ of spaces if F is naturally equivalent to the functor $[\ \ , B_F]$, the homotopy classes $[f]$ of maps f into B_F. If B_F exists we say that F is *representable*. In [2], E. H. Brown, Jr., has given a set of conditions on F which will imply that F is representable when $\mathcal{C} = \mathcal{W}'$. In [6] we showed that if F mapped $\mathcal{W}'$ into the category $\mathcal{A}$ of abelian groups, then we could take B_F to be a weakly homotopy abelian and weakly homotopy associative H-space such that F and $[\ \ , B_F]$ are naturally equivalent as functors into $\mathcal{A}$. In this note we show how to extend this to representability for functors F defined on the larger category $\mathcal{W}$.

Since this paper was written the paper [7] of Brown has appeared in which he formalizes the methods of [2] to obtain a very general representability theorem. In particular, his result covers the case in which the domain category of the functor F in question is $\mathcal{W}$. However, our main result (1.1) is quite different in that it relates the classifying spaces of F and its restriction to $\mathcal{W}'$.

Before giving the precise statement of the main theorem we must recall some definitions. Maps f and g from a space X to a space Y are said to be *weakly homotopic* if the induced maps f_* and g_* from $[K, X]$ to $[K, Y]$ are equal for every finite CW complex K. Here, $f_*[\varphi] = [f\varphi]$ for $[\varphi] \in [K, X]$. An H-structure map $\mu : B \times B \to B$ on an H-space B is *weakly homotopy associative* if $\mu(\mu \times 1)$ and $\mu(1 \times \mu)$ are weakly homotopic maps from $B \times B \times B$ to B. Let

$$T : B \times B \to B \times B$$

be defined by $T(x, y) = (y, x), x, y \in B$. Then μ is said to be *weakly homotopy abelian* if μT and μ are weakly homotopic. Suppose that ν is an H-structure for a space A and $f : A \to B$ is a map; we say that f is a *weak homomorphism* if $\mu(f \times f)$ and $f\nu$ are weakly homotopic maps.

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Let (X, A) be a pair in $\mathcal{W}$ with A non-empty and having the same base point as X. Let $i : A \subset X$ be the inclusion map and $q : X \to X/A$ the identification map which collapses A to the base point of X. Now it is easy to see that if $F : \mathcal{W} \to \mathcal{A}$ is a representable contravariant functor, then the sequence

$$F(X/A) \xrightarrow{F(q)} F(X) \xrightarrow{F(i)} F(A)$$

is exact. Dold [3] has called any contravariant functor $F : \mathcal{W} \to \mathcal{A}$ satisfying this condition *half exact.*

For the hypothesis of the main theorem (1.1) below we let $F : \mathcal{W} \to \mathcal{A}$ be a contravariant functor and let F' be its restriction to $\mathcal{W}'$. Suppose B'_F is a connected space with a weakly homotopy associative and weakly homotopy abelian H-structure μ' such that the functors F' and $[\ \ , B'_F]$ from $\mathcal{W}'$ to $\mathcal{A}$ are naturally equivalent. Regard $F(S^0)$ as a discrete space with base point 0, set $B_F = B'_F \times F(S^0)$, and let $\mu : B_F \times B_F \to B_F$ be the map defined by

$$\mu(a, x, b, y) = (\mu'(a, b), x + y), \qquad a, b \in B'_F,\ x, y \in F(S^0)$$

Then μ is an H-structure on B_F which is weakly homotopy associative and weakly homotopy abelian. Hence the functor $[\ \ , B_F]$ maps $\mathcal{W}$ into $\mathcal{A}$ and its restriction to $\mathcal{W}'$ is naturally equivalent of F'.

(1.1) THEOREM. *If $F : \mathcal{W} \to \mathcal{A}$ is as above and is half exact in the sense of Dold, then there is a natural equivalence $\Phi : [\ \ , B_F] \cong F$ of functors from $\mathcal{W}$ to $\mathcal{A}$.*

Suppose that $G : \mathcal{W} \to \mathcal{A}$ is another such functor and B'_F is a countable CW complex. If $T : F \to G$ is a natural transformation, then there exists a weak homomorphism $f : B'_F \to B'_G$ such that the weak homomorphism

$$f \times T_{S^0} : B_F \to B_G$$

represents T, i.e. $\Phi T = (f \times T_{S^0})_* \Phi$.

Example 1. An Eilenberg-MacLane space of type (G, n), G abelian and $n \geq 1$, is a classifying space for the reduced singular cohomology functor $\tilde{H}^n(\ \ ; G)$ on both $\mathcal{W}$ and $\mathcal{W}'$ since $\tilde{H}^n(S^0; G) = 0$.

Example 2. Consider $\tilde{K} : \mathcal{W} \to \mathcal{A}$ as defined in [1], for example. Now $\tilde{K} \mid \mathcal{W}'$ has a countable connected CW complex B_U as classifying space and there exists a weakly homotopy associative and weakly homotopy abelian H-structure on B_U representing the addition on $\tilde{K} \mid \mathcal{W}'$; see [6] for example. Since $\tilde{K}$ satisfies the half exactness property and $\tilde{K}(S^0) = Z$, the integers, it follows from theorem (1.1) that $Z \times B_U$ is a classifying space for $\tilde{K}$ on $\mathcal{W}$, a well-known fact. Similarly for $\widetilde{KO}$.

The organization of this note is as follows. In Section 2 we prove two elementary lemmas concerning half exact functors. The next section introduces a technical device, due essentially to Dold [3], which will be the crux of the proof of (1.1) given in Section 5. Section 4 establishes the needed results about representability of natural transformations,

2. Elementary properties of half exact functors

Let $X \vee Y$ denote the disjoint union of the spaces X and Y with their base points identified. Let $i_1 : X \to X \vee Y$ and $p_1 : X \vee Y \to X$ be the canonical injection and projection maps, and similarly for i_2 and p_2. The following two results are due to Dold [3].

(2.1) LEMMA. *If $F : \mathcal{W} \to \mathcal{A}$ is half exact, then*

$$F(X \vee Y) = F(p_1)F(X) \oplus F(p_2)F(Y).$$

Proof. Apply F to the commutative diagram

$$\begin{tikzcd}
X \arrow[dr, "i_1"] \arrow[dd, "1"'] & & Y \arrow[dl, "i_2"'] \arrow[dd, "1"] \\
& X \vee Y \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X = X \vee Y/Y & & X \vee Y/X = Y
\end{tikzcd}$$

and use [4, Lemma 13.1, p. 32].

(2.2) LEMMA. *Let $r : X \to A$ be a retraction of X onto A in $\mathcal{W}$, and let $q : X \to X/A$ be the identification map. Then*

$$F(X) = F(r)F(A) \oplus F(q)F(X/A).$$

Proof. The Puppe sequence [5] of the inclusion map $i : A \subset X$ gives rise to the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[dr, "1"'] & X \arrow[r, "q"] \arrow[d, "r"] & X/A \arrow[r, "p"] & \Sigma A \arrow[r, "\Sigma i"] \arrow[dr, "1"'] & \Sigma X \arrow[r] \arrow[d, "\Sigma r"] & \cdots \\
& A & & & \Sigma A &
\end{tikzcd}$$

Here Σ denotes the reduced suspension functor. Using [5, Th. 5], one easily shows that the half exact functor F takes the row into an exact sequence of abelian groups. But $F(i)$ and $F(\Sigma i)$ are epimorphisms since $F(i)F(r) = F(1)$. Hence $F(p) = 0$. We thus obtain the following commutative diagram in which the row is exact:

$$\begin{tikzcd}
0 & F(A) \arrow[l] & F(X) \arrow[l, "F(i)"'] & F(X/A) \arrow[l, "F(q)"'] & 0 \arrow[l] \\
& & F(A) \arrow[ul, "1"] \arrow[u, "F(r)"'] & &
\end{tikzcd}$$

This splitting establishes the lemma.

3. Special wedges of 0-spheres

Let $X \in \mathcal{W}$ have path components $X_0, X_1, \cdots, X_n$ where the base point $*$ belongs to X_0. We write $X = X_0 + \cdots + X_n$. Let S_j^0, $j = 1, \cdots, n$, be a 0-sphere consisting of the point $*$ and a vertex $x_j \in X_j$. The subcomplex $A = S_1^0 \vee \cdots \vee S_n^0$ of X will be called a *special wedge* of 0-spheres for X. Notice that if we let x_j be the base point of X_j, then

$$X/A = X_0 \vee X_1 \vee \cdots \vee X_n$$

belongs to $\mathcal{W}'$. Moreover, the map $r : X \to A$ defined by $r(X_0) = \{*\}$, $r(X_j) = \{x_j\}$, $1 \leqq j \leqq n$, is a retraction.

Concerning naturality properties of special wedges we have the following result.

(3.1) LEMMA. *Let A be a special wedge of 0-spheres for the finite CW complex X, let Y be a finite CW complex, and let $f : X \to Y$ be a map. Then there exist a special wedge of 0-spheres C for Y and a map $g : (X, A) \to (Y, C)$ such that g is homotopic to f.*

Proof. We write

$$X = X_0 + \cdots + X_n, \quad Y = Y_0 + \cdots + Y_m, \quad \text{and} \quad A = \{x_0, x_1, \cdots, x_n\}$$

where $x_0 = *$ and x_j is a vertex of X_j. By the cellular approximation theorem we may assume that the map f is cellular; in particular, each $f(x_j)$ is a vertex of Y.

The set C is constructed as follows. Let $f(x_0) \in C$. Let $f(x_k) \in C$ if and only if $f(x_k)$ does not belong to the path component of $f(x_i)$ for all i less than k. Finally, if $Y_j \cap f(X)$ is empty, choose any vertex $y_j \in Y_j$ and let $y_j \in C$. Clearly C is a special wedge of 0-spheres for Y.

It remains to construct the map $g : X \to Y$. Let k be an integer such that $f(x_i) \in C$ for $i < k$ and $f(x_k) \notin C$. Then there exists $j < k$ and a path $\gamma : I \to Y$ with $\gamma(0) = f(x_k)$ and $\gamma(1) = f(x_j)$. Define the map

$$F : X_k \times \{0\} \cup \{x_k\} \times I \to Y$$

by $F(x, 0) = f(x)$, $x \in X_k$, and $F(x_k, t) = \gamma(t)$, $t \in I$. By the homotopy extension theorem there exists an extension map $F : X_k \times I \to Y$. Define the map $g_1 : X \to Y$ by $g_1(x) = f(x)$ if $x \notin X_k$, $g_1(x) = F(x, 1)$ if $x \in X_k$. Then $g_1 \simeq f$ and $g_1(x_i) \in C$ for $i < k + 1$. Similarly, there exists a map $g_2 : X \to Y$ such that $g_2 \simeq g_1$ and $g_2(x_i) \in C$ for $i < k + 2$. Continuing in this way we obtain a map $g : X \to Y$ such that $g \simeq f$ and $g(A) \subset C$.

4. Natural transformations

(4.1) PROPOSITION. *Let $T : [\ \ , B_1] \to [\ \ , B_2]$ be a natural transformation of functors from $\mathcal{W}'$ to $\mathfrak{A}$. Let μ_i be a multiplication on B_i representing the multiplication on the functor $[\ \ , B_i]$, $i = 1, 2$. If B_1 is a countable connected CW complex, then there is a weak homomorphism $f : B_1 \to B_2$ such that $f_* = T$.*

Proof. The existence of f is established in [2, Lemma 2.1]. (The hypothesis that $Y' \in \mathcal{C}_\omega$ in [2] is not necessary.) It remains to show that f is a weak homomorphism. For this it suffices to show that if A is a connected finite subcomplex of B_1 containing the base point, then $\mu_2(f' \times f') \simeq f\mu_1'$ as maps from $A \times A$ to B_2 where $f' = f \mid A$ and $\mu_1' = \mu_1 \mid A \times A$. Let p_1 and p_2 denote the canonical projections of $A \times A$ onto A followed by inclusion into B_1 and let $\Delta : A \to A \times A$ be the diagonal map defined by $\Delta(a) = (a, a)$, $a \in A$. Then

$$\begin{aligned}[f\mu_1'] &= f_*[\mu_1'] = f_*([p_1] + [p_2]) = f_*[p_1] + f_*[p_2] \\ &= [\mu_2(fp_1 \times fp_2)\Delta] = [\mu_2(f' \times f')]\end{aligned}$$

as we were to show.

5. Proof of (1.1)

We assume the notation and hypothesis of (1.1) except that the subscripts on B_F, etc., will be suppressed when confusion won't arise.

(5.1) LEMMA. *If A is a wedge product of 0-spheres, then there is an isomorphism $\Phi : [A, B] \cong F(A)$ of abelian groups. Moreover, if C is also a wedge product of 0-spheres and $f : A \to C$ is a map, then the diagram*

$$\begin{array}{ccc} [C, B] & \xrightarrow{\Phi} & F(C) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle F(f)} \\ [A, B] & \xrightarrow{\Phi} & F(A) \end{array}$$

is commutative.

Proof. First, suppose $A = S^0$ and $x \in S^0$ is not the base point. Let

$$p_2 : B = B' \times F(S^0) \to F(S^0)$$

be the projection map and define the function Φ by

$$\Phi[f] = p_2 f(x) \in F(S^0), \qquad [f] \in [S^0, B].$$

Since B' is connected and $F(S^0)$ is discrete it follows that Φ is a well-defined bijection. To show that Φ is an isomorphism we simply note that if $[f]$ and $[g]$ are in $[S^0, B]$ then

$$\begin{aligned}\Phi([f] + [g]) &= \Phi[\mu(f \times g)\Delta] = p_2\,\mu(f(x), g(x)) \\ &= (p_2 f(x)) + (p_2\, g(x)) = \Phi[f] + \Phi[g].\end{aligned}$$

More generally, let $A = S_1^0 \vee \cdots \vee S_n^0$ where S_j^0 is a 0-sphere. If $p_j : A \to S_j^0$ denotes the projection map, $1 \leqq j \leqq n$, then $p_j^*[S_j^0, B]$ and $F(p_j)F(S_j^0)$ are canonically isomorphic groups since p_j^* and $F(p_j)$ are monomorphisms. From (2.1) we conclude that

$$\begin{aligned}[A, B] &= p_1^*[S_1^0, B] \oplus \cdots \oplus p_n^*[S_n^0, B] \\ &\cong F(p_1)F(S_1^0) \oplus \cdots \oplus F(p_n)F(S_n^0) \\ &= F(A).\end{aligned}$$

Let Φ denote this isomorphism.

It remains to prove naturality. Let $C = S_1^0 \vee \cdots \vee S_m^0$ and let $f : A \to C$ be a map. If $A = S^0$ then either $f = 0$ or f is a homeomorphism onto some $S_j^0 \subset C$. Easy computations of f^* and $F(f)$ show that $F(f)\Phi = \Phi f^*$. More generally, let $A = S_1^0 \vee \cdots \vee S_n^0$. Now the homomorphism

$$F(f) : F(C) \to F(A) = F(p_1)F(S_1^0) \oplus \cdots \oplus F(p_n)F(S_n^0)$$

is given by

$$F(f)(\alpha) = (F(f \mid S_1^0)\alpha, \cdots, F(f \mid S_n^0)\alpha), \qquad \alpha \in F(C)$$

Similarly for $f^* : [C, B] \to [A, B]$. Therefore if $\alpha \in [C, B]$ then

$$\begin{aligned} \Phi f^*(\alpha) &= (\Phi(f \mid S_1^0)^*\alpha, \cdots, \Phi(f \mid S_n^0)^*\alpha) \\ &= (F(f \mid S_1^0)\Phi\alpha, \cdots, F(f \mid S_n^0)\Phi\alpha) \\ &= F(f)\Phi(\alpha), \end{aligned}$$

which completes the proof of (5.1).

Proof of (1.1). Let $X \in \mathcal{W}$ and let A be a special wedge of 0-spheres for X. Let $Y \in \mathcal{W}$ and let $f : X \to Y$ be a map. By Lemma (3.1), there exist a special wedge of 0-spheres C for Y for a map $g : (X, A) \to (Y, C)$ homotopic to f. Let $g' : X/A \to Y/C$ be the induced map. By hypothesis there is a natural equivalence between $[\ \ , B] \mid \mathcal{W}'$ and F'. From (2.2) and (5.1) we have the following commutative diagram:

$$\begin{array}{ccccccc} [Y, B] \cong & [C, B] \oplus [Y/C, B] & \cong & F(C) \oplus F(Y/C) & \cong & F(Y) \\ \downarrow{\scriptstyle g^*} & \downarrow{\scriptstyle (g|A)^* \oplus g'^*} & & \downarrow{\scriptstyle F(g|A) \oplus F(g')} & & \downarrow{\scriptstyle F(g)} \\ [X, B] \cong & [A, B] \oplus [X/A, B] & \cong & F(A) \oplus F(X/A) & \cong & F(X). \end{array}$$

Since $f^* = g^*$ and $F(f) = F(g)$, the first part is proved. Let

$$\Phi : [\ \ , B] \cong F(\ \ )$$

denote the above composition.

Next, let $G : \mathcal{W} \to \mathfrak{A}$ and $T : F \to G$. By (4.1) there exists a weak homomorphism $f : B'_F \to B'_G$ representing the restriction $T' : F' \to G'$ of T. Let $h = f \times T_{S^0} : B_F \to B_G$. Now if $X \in \mathcal{W}$ and A is a special wedge of 0-spheres for X, then the diagram

$$\begin{array}{ccccccc} [X, B_F] \cong & [A, F(S^0)] \oplus [X/A, B'_F] & \cong & F(A) \oplus F'(X/A) & \cong & F(X) \\ \downarrow{\scriptstyle h_*} & \downarrow{\scriptstyle (T_{S^0})_* \oplus f_*} & & \downarrow{\scriptstyle T_A \oplus T'_{X/A}} & & \downarrow{\scriptstyle T_X} \\ [X, B_G] \cong & [A, G(S^0)] \oplus [X/A, B'_G] & \cong & G(A) \oplus G'(X/A) & \cong & G(X) \end{array}$$

is commutative and the horizontal composite maps are both Φ. This completes the proof.

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