PERIOD OF AN IRREDUCIBLE POSITIVE OPERATOR[1]

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I. Introduction

Let X be a non-empty set, $\mathcal{B}$, a σ-algebra of subsets of X and λ, a σ-finite measure on $\mathcal{B}$. Let $L_\infty(\lambda)$ be the collection of all real-valued, λ-essentially bounded $\mathcal{B}$-measurable functions defined on X, and let $\mathcal{A}(\lambda)$ be the collection of all finite, signed measures on $\mathcal{B}$ which are absolutely continuous to λ. Let M be an operator satisfying the following conditions:

M1. if $f \in L_\infty(\lambda)$ then $Mf \in L_\infty(\lambda)$,
M2. $f \in L_\infty(\lambda)$ and $f \geq 0$ a.e. (λ) imply $Mf \geq 0$ a.e. (λ),
M3. $f_n \in L_\infty(\lambda)$ and $f_n \downarrow 0$ a.e. (λ) imply $Mf_n \downarrow 0$ a.e. (λ).

Based on M1, M2 and M3 we can then define νM for any $\nu \in \mathcal{A}(\lambda)$ to be a signed measure satisfying

$$\int \nu M(dx)f(x) = \int \nu(dx)Mf(x)$$

for every $f \in L_\infty(\lambda)$. Then νM is again an element of $\mathcal{A}(\lambda)$. Such an operator is a λ-measurable Markov operator of E. Hopf if an additional condition $M1 \leq 1$ a.e. (λ) is satisfied (cf. [4]). An M satisfying M1, M2 and M3 shall be called a *λ-measurable positive operator* or simply, a *positive operator*. In this paper, the main concern is the "periodic" or "cyclic moving" behavior of sets. If X is discrete and λ is the measure which assigns measure 1 to every singleton then a positive operator M is just a non-negative matrix $M(i, j)$. If $M(i,j)$ is irreducible, a period for $M(i,j)$ may be defined in the same manner as that for a probability matrix. In [8] the present author has treated the period behavior of an ergodic conservative Markov operator. In this paper the "periodic" behavior of a positive operator is investigated. It is discovered that the *irreducibility* of M alone is enough to enable us to study the "cyclic moving" behavior. Notions of "λ-continuity" and the more general "quasi λ-continuity" for a positive operator are introduced. If an irreducible M is quasi λ-continuous then M has a positive integer δ as its period. This number δ is characterized by the following fact: the space X is partitioned into δ cyclic moving sets $C_1, C_2, \cdots, C_\delta$ each of which is irreducibly $M^{n\delta}$-closed for $n = 1, 2, \cdots$. This fact has been proved for a λ-continuous, egodic, conservative operator in [8]. This work, again, is inspired by Doeblin [2] and Chung [1] although the method used here is quite different. In Section III, positive operators with transition functions are studied. This kind of positive

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operators arises from Markov processes and branching processes. The relation between quasi λ-continuity of the operator and properties of transition functions is studied. It is shown for instance, that if the probability transition function satisfies a condition of Harris (see [6]) then the associated Markov operator is irreducible and quasi λ-continuous. This fact enables us to apply a result of Section II to establish a period for the operator.

II. Theory of periods for an irreducible positive operative

In this section all subsets of X are elements of $\mathcal{B}$ and all functions on X are $\mathcal{B}$-measurable. Unless otherwise indicated, for two sets $A, B, A \subset B, A = B$ are to mean $\lambda(A - B) = 0, \lambda(A \triangle B) = 0$ respectively. For two functions f, g on $X, f = g, f \leq g$ are to mean that the equality and the inequality, respectively, are satisfied except on a λ-null set. Occasionally we still indicate $=$ a.e. (λ) or $\leq$ a.e. (λ) for emphasis. A set A is null or non-null according as $\lambda(A) = 0$ or $\lambda(A) > 0$. *We shall always assume that* $\mathcal{B}$ *is non-trivial*, i.e., $\mathcal{B}$ contains at least one set A such that $\lambda(A) > 0$ and $\lambda(X - A) > 0$. For any set A, 1_A is to represent the function which is equal to 1 on A and 0 on the complement $\bar{A}$ of A. $\mathcal{a}^+(\lambda)$ is to denote the collection of all finite measures which are absolutely continuous to λ. For any $\nu \in \mathcal{a}^+(\lambda)$, the *support* of ν, supp ν is the set of all points $x \in X$ such that $(d\nu/d\lambda)(x) > 0$.

Definition 1. *A set C is M^k-closed,* where k is a positive integer, if $M^k 1_{\bar{C}} = 0$ a.e. (λ) on C where $\bar{C}$ is the complement of C. *A set is closed* if it is M-closed.

Lemma 1. *If $\{C_n\}$ is a sequence of M^k-closed sets then $\bigcap_n C_n$* and $\bigcup_n C_n$ *are M^k-closed. An M^k-closed set is also M^{km}-closed for* $m = 1, 2, \cdots$.

Proof. We shall prove the lemma for $k = 1$. For (λ) almost all $x \in \bigcap_n C_n$, we have $M1_{\bar{C}_n}(x) = 0$ for $n = 1, 2, \cdots$. Since $M1_{\bigcup_n \bar{C}_n} \leq \sum_n M1_{\bar{C}_n}$ and $\sum_n M1_{\bar{C}_n} = 0$ on $\bigcap_n C_n$, we have

$$0 = M1_{\bigcup_n \bar{C}_n} = M1_{\overline{\bigcap_n C_n}}$$

and $\bigcap_n C_n$ is M-closed. The fact that $\bigcup_n C_n$ is M-closed follows from the observation $M1_{\overline{\bigcup_n C_n}} \leq M1_{\bar{C}_n}$ for $n = 1, 2, \cdots$, therefore, $M1_{\overline{\bigcup_n C_n}} = 0$ on C_n for $n = 1, 2, \cdots$.

If C is M-closed, then $M1_{\bar{C}} = 1_{\bar{C}} \cdot M1_{\bar{C}}$, therefore, $M^2 1_{\bar{C}} = M(1_{\bar{C}} \cdot M1_{\bar{C}}) \leq (M1_{\bar{C}}) \cdot a$ where a is a number for which $M1 \leq a$. Hence $M^2 1_{\bar{C}} = 0$ on C and C is M^2-closed. Proceeding in the same manner, we arrive at the conclusion that C is M^m-closed for $m = 3, 4, \cdots$.

Definition 2. An M^k-closed set C is decomposable if there are two non-null M^k-closed sets A, B such that $A \cup B \subset C$ and $A \cap B = \emptyset$ (empty set). An M^k-closed set is indecomposable if it is not decomposable. An M^k-closed set C is irreducible if it is non-null and if $A \subset C, \lambda(A) > 0, \lambda(C - A) > 0$

imply A is not M^k-closed. M is irreducible if X, as an M-closed set, is irreducible.

It is clear that an irreducible M^k-closed set is indecomposably M^k-closed.

LEMMA 2. *If M is irreducible then $Mf > 0$, provided $f > 0$ and $f \in L_\infty(\lambda)$. It follows that $M^n 1 > 0$ for $n = 1, 2, \cdots$.*

Proof. Let $A = [x : M1(x) = 0]$. Then $M1_B = 0$ on A for every set B, hence every subset of A is closed. Since M is irreducible, either $\lambda(A) = 0$ or $\lambda(X - A) = 0$. If $\lambda(X - A) = 0$, then there is a set $D \subset A$ such that $\lambda(D) > 0$ and $\lambda(A - D) > 0$ since we assumed that $\mathcal{B}$ is non-trivial. D being closed clearly contradicts the hypothesis that M is irreducible. Hence $\lambda(A) = 0$ and $M1 > 0$ a.e. (λ). Now, let $f > 0$ a.e. (λ) and

$$E_n = [x : f(x) > 1/n], \quad G = [x : Mf(x) = 0] \quad \text{and} \quad D_n = [x : M1_{E_n}(x) = 0];$$

then $G \subset D_n$ for $n = 1, 2, \cdots$. Now $M1_{E_n} \uparrow M1$, hence $M1 = 0$ on G and $\lambda(G) = 0$ follows immediately.

LEMMA 3. *If a set C is decomposably M^k-closed then C is also decomposably M^{kn}-closed for an arbitrary positive integer n. If C is M^k-closed and indecomposably M^{kn}-closed where n is a positive integer, then C is also indecomposably M^k-closed.*

The above lemma follows immediately from Lemma 1.

LEMMA 4. *If μ, ν are elements of $\mathcal{A}^+(\lambda)$ such that μ is absolutely continuous to ν then* supp $\mu M^k \subset$ supp νM^k *for an arbitrary positive integer k.*

Proof. We shall prove for $k = 1$. Let $g = d\mu/d\nu$. Let

$$\begin{aligned} g_n(x) &= g(x), && \text{if } g(x) \le n; \\ &= n, && \text{otherwise.} \end{aligned}$$

Let μ_n be defined by $\mu_n(E) = \int_E g_n \, d\nu$. Then $\mu_n \le n\nu$, hence $\mu_n M \le n\nu M$ so that supp $\mu_n M \subset$ supp νM. Now for every set E, $\mu_n M(E) \uparrow \mu M(E)$, hence $d\mu_n M/d\lambda \uparrow d\mu M/d\lambda$. Hence

$$\operatorname{supp} \mu M = \textstyle\bigcup_n \operatorname{supp} \mu_n M \subset \operatorname{supp} \nu M.$$

We remark that, for two measures ν, μ in $\mathcal{A}^+(\lambda)$, ν is absolutely continuous to μ if and only if supp $\nu \subset$ supp μ. Thus, Lemma 4 may be stated as follows: νM^k is absolutely continuous to μM^k if ν is absolutely continuous to μ.

It follows from Lemma 4 that if supp $\nu =$ supp μ, then supp $\nu M^k =$ supp μM^k

DEFINITION 3. For any set A, define

$$F_0(A) = A, \qquad F_n(A) = \operatorname{supp} \nu M^n \qquad \text{for} \quad n = 1, 2, \cdots,$$
$$F(A) = \textstyle\bigcup_{n=0}^{\infty} F_n(A)$$

where ν is an element of $\mathcal{A}^+(\lambda)$ which has A as its support.

By Lemma 4, particular ν chosen in Definition 3 does not matter and $F_{n+1}(A) = F_1(F_n(A))$ for $n = 0, 1, 2, \cdots$.

The following lemma follows immediately from Lemma 4 and the fact that the support of the sum of several measures is equal to the union of the supports of measures.

LEMMA 5. *If A_1, A_2 are two sets such that $A_1 \subset A_2$ then $F_n(A_1) \subset F_n(A_2)$ for $n = 0, 1, 2, \cdots$, therefore $F(A_1) \subset F(A_2)$. If $\{A_i\}$ is a sequence of sets then*

$$F_n(\bigcap_i A_i) \subset \bigcap_i F_n(A_i) \quad \textit{and} \quad \bigcup_i F_n(A_i) = F_n(\bigcup_i A_i) \quad \textit{for} \quad n = 0, 1, 2, \cdots.$$

LEMMA 6. 1. *A set C is M^k-closed if and only if $C \supset F_k(C)$. If C is M^k-closed then $F_k(C) \supset F_{2k}(C) \supset \cdots$ and $F_n(C)$ is M^k-closed for $n = 0, 1, 2, \cdots$.*

2. *If a set C is M^k-closed then*

$$C \cup F_1(C) \cup \cdots \cup F_{k-1}(C) \quad \textit{and} \quad C \cap F_1(C) \cap \cdots \cap F_{k-1}(C)$$

are M-closed.

3. *For any set A, $F(A)$ is the smallest closed set containing A.*

Proof. If C is M^k-closed, then $M^k 1_E = 0$ on C for every subset E of $\bar{C}$. Hence, if $\nu \in \mathfrak{A}^+(\lambda)$ has C as its support then $\nu M^k(E) = 0$ for every subset E of $\bar{C}$. Hence $F_k(C) = \operatorname{supp} \nu M^k \subset C$. Conversely, if $F_k(C) \subset C$ and if $\nu \in \mathfrak{A}^+(\lambda)$, $\operatorname{supp} \nu \subset C$ then $\operatorname{supp} \nu M^k \subset F_k(C) \subset C$. Hence $\nu M^k(\bar{C}) = 0$ for every $\nu \in \mathfrak{A}^+(\lambda)$ with $\operatorname{supp} \nu \subset C$. This implies that $M^k 1_{\bar{C}} = 0$ a.e. (λ) on C. If C is M^k-closed, $C \supset F_k(C)$, then, by Lemma 5,

$$F_n(C) \supset F_{n+k}(C) = F_k(F_n(C)).$$

Hence $F_n(C)$ is also M^k-closed.

Let C be an M^k-closed set, then, by Lemma 5,

$$\begin{aligned} &F_1(C \cup F_1(C) \cup \cdots \cup F_{k-1}(C)) \\ &\qquad = F_1(C) \cup F_2(C) \cup \cdots \cup F_k(C) \subset C \cup F_1(C) \cup \cdots \cup F_{k-1}(C), \\ &F_1(C \cap F_1(C) \cap \cdots \cap F_{k-1}(C)) \\ &\qquad \subset F_1(C) \cap F_2(C) \cap \cdots \cap F_k(C) \subset C \cap F_1(C) \cap \cdots \cap F_{k-1}(C). \end{aligned}$$

Hence both sets $C \cup F_1(C) \cup \cdots \cup F_{k-1}(C)$ and $C \cap F_1(C) \cap \cdots \cap F_{k-1}(C)$ are M-closed.

For any set A. $M 1_{\overline{F_{n+1}(A)}} = 0$ on $F_n(A)$ for $n = 0, 1, 2, \cdots$. Hence $M 1_{\overline{F(A)}} = 0$ on $F_n(A)$ for $n = 0, 1, 2, \cdots$. Therefore $M 1_{\overline{F(A)}} = 0$ on $F(A)$ and $F(A)$ is closed. If C is an arbitrary closed set containing A, then $F(C) \supset F(A)$ by Lemma 5. However, $C \supset F(C)$. Hence $C \supset F(A)$. Thus $F(A)$ is the smallest closed set containing A.

The following lemma follows from Lemma 2 and Lemma 6.

LEMMA 7. *If M is irreducible and if A is non-null, then $F_n(A)$ is non-null for $n = 1, 2, \cdots$.*

COROLLARY 1. *M is irreducible if and only if*

$$X = [x : \textstyle\sum_{n=1}^{\infty} M^n 1_E(x) > 0]$$

for every non-null set E.

Proof. If M is not irreducible, then, there is a non-null closed set C such that $B = X - C$ is non-null. We have $M^n 1_B = 0$ on C for $n = 1, 2, \cdots$ so that

$$C \subset X - [x : \textstyle\sum_{n=1}^{\infty} M^n 1_B(x) > 0] \quad \text{and} \quad X \neq [x : \textstyle\sum_{n=1}^{\infty} M^n 1_B(x) > 0].$$

Suppose that M is irreducible. If there were a non-null set E such that

$$X - [x : \textstyle\sum_{n=1}^{\infty} M^n 1_E(x) > 0] = D$$

is non-null, then $M^n 1_E = 0$ on D so that $F_n(D) \cap E = \emptyset$ for $n = 1, 2, \cdots$. Hence

$$F(F_1(D)) \cap E = \textstyle\bigcup_{n=1}^{\infty} F_n(D) \cap E = \emptyset.$$

By Lemma 6 and Lemma 7 $F(F_1(D))$ is a non-null closed set which contradicts the supposition that M is irreducible.

LEMMA 8. *If M is irreducible and if C_1, C_2 are two non-null, disjoint, M^k-closed sets, then $F_n(C_1)$, $F_n(C_2)$ are also two non-null, disjoint, M^k-closed sets where n is an arbitrary positive integer.*

Proof. If C_1, C_2 are two non-null M^k-closed sets then $F_1(C_1)$, $F_1(C_2)$ are also two non-null, M^k-closed sets by Lemma 6 and Lemma 7. Now suppose that $F_1(C_1) \cap F_1(C_2)$ is non-null. Then $F_{k-1}(F_1(C_1) \cap F_1(C_2))$ is non-null by Lemma 7. However, by Lemma 5 and Lemma 6.

$$F_{k-1}(F_1(C_1) \cap F_1(C_2)) \subset F_k(C_1) \cap F_k(C_2) \subset C_1 \cap C_2 .$$

Hence $C_1 \cap C_2$ would be non-null. Hence the fact that $C_1 \cap C_2$ is null implies that $F_1(C_1) \cap F_1(C_2)$ is null. The conclusion for an arbitrary positive integer n follows easily from mathematical induction.

LEMMA 9. *Let M be irreducible. Then, if E is decomposably M^k-closed, so is $F_n(E)$; if E is indecomposably M^k-closed, so is $F_n(E)$. k, n are two arbitrary positive integers.*

Proof. If E is decomposably M^k-closed, then, there are two non-null M^k-closed sets B and C such that $B \cap C = \emptyset$ and $B \cup C \subset E$. By Lemma 8, $F_n(B)$ and $F_n(C)$ are also non-null, disjoint, M^k-closed sets. By Lemma 5, $F_n(B) \cup F_n(C) \subset F_n(E)$. Thus $F_n(E)$ is decomposably M^k-closed. If E is M^k-closed and $F_n(E)$ is decomposably M^k-closed then there are two non-null M^k-closed sets D and G such that $D \cup G \subset F_n(E)$, $D \cap G = \emptyset$. Let m be a positive integer such that $mk > n$. Then

$$F_{mk-n}(D) \subset F_{mk}(E), \qquad F_{mk-n}(G) \subset F_{mk}(E).$$

Both $F_{mk-n}(D)$ and $F_{mk-n}(G)$ are M^k-closed, non-null and mutually disjoint by Lemma 8. E is also M^{mk}-closed, hence $F_{mk}(E) \subset E$ by Lemma 6. Hence

$$F_{mk-n}(D) \cup F_{mk-n}(G) \subset E$$

and E is decomposably M^k-closed.

LEMMA 10. *If M is irreducible and $C_1, C_2, \cdots, C_n$ are M^k-closed, non-null and pairwise disjoint, then $n \leq k$.*

Proof. Let $G_m = C_m \cup F_1(C_m) \cup \cdots \cup F_{k-1}(C_m)$, $m = 1, 2, \cdots, n$. By Lemma 6, G_m are closed. $\bigcap_{m=1}^{n} G_m \neq \emptyset$ since M is indecomposable. Now

$$\bigcap_{m=1}^{n} G_m = \bigcup_{(i_1, \cdots, i_n)} \{F_{i_1}(C_1) \cap \cdots \cap F_{i_n}(C_n)\}$$

where $(i_1, \cdots, i_n)$ is an arbitrary n-tuple of integers lying between 0 and $k - 1$. There exists one n-tuple $(i_1, \cdots, i_n)$ such that

$$F_{i_1}(C_1) \cap \cdots \cap F_{i_n}(C_n)$$

is non-null. Hence $i_1, \cdots, i_n$ must be distinct integers, for to be other wise would imply that the set $F_{i_1}(C_1) \cap \cdots \cap F_{i_n}(C_n)$ is null by Lemma 8. Hence $n \leq k$.

LEMMA 11. *If M is irreducible and k is a positive integer then there is an indecomposably M^k-closed, non-null set.*

Proof. If X is not indecomposably M^k-closed, then there are two disjoint, non-null, M^k-closed sets $C_1^{(1)}$, $C_2^{(1)}$. If neither $C_1^{(1)}$ nor $C_2^{(1)}$ is indecomposably M^k-closed, then there are four pairwise disjoint, non-null, M^k-closed sets $C_1^{(2)}$, $C_2^{(2)}$, $C_3^{(2)}$, $C_4^{(2)}$, $\cdots$ etc. By Lemma 10, this process must stop after finitely many times and we obtain an indecomposably M^k-closed, non-null set.

LEMMA 12. *Let M be irreducible and let C be a non-null, indecomposably M^k-closed set. Consider the following sequence of sets:*

$$C, F_1(C), F_2(C), F_3(C), \cdots. \tag{1}$$

Let δ be the smallest positive integer such that $C \cap F_\delta(C)$ is non-null; then

1. *for all non-negative integers m, n*

$$F_m(C) \cap F_{m+\delta}(C) \cap \cdots \cap F_{m+n\delta}(C) \tag{2}$$

are non-null, indecomposably M^k-closed,

2. *if $F_l(C) \cap F_m(C)$ is non-null then δ divides $m - l$. It follows that δ divides k and C, $F_1(C) \cdots, F_{\delta-1}(C)$ are pairwise disjoint.*

Proof. It is clear that (2) is indecomposably M^k-closed. To show that (2) is non-null we shall show that (2) is non-null for $m = 0$ and then apply Lemmas 5 and 7.

We know that $C \cap F_\delta(C)$ is non-null. Assume that

$$C \cap F_\delta(C) \cap \cdots \cap F_{l\delta}(C)$$

is non-null; then

$$F_\delta(C \cap F_\delta(C) \cap \cdots \cap F_{l\delta}(C)) \subset F_\delta(C) \cap F_{2\delta}(C) \cap \cdots \cap F_{(l+1)\delta}(C)$$

so that

$$F_\delta(C) \cap F_{2\delta}(C) \cap \cdots \cap F_{(l+1)}(\delta)$$

is non-null. If $C \cap F_\delta(C) \cap \cdots \cap F_{(l+1)\delta}(C)$ were null then

$$C \cap F_\delta(C) \cap \cdots \cap F_{l\delta}(C) \quad \text{and} \quad F_\delta(C) \cap F_{2\delta}(C) \cap \cdots \cap F_{(l+1)\delta}(C)$$

would be two disjoint, non-null M^k-closed sets both contained in $F_\delta(C)$ which contradicts the fact that all sets in (1) are indecomposably M^k-closed (Lemma 9).

Suppose $F_l(C) \cap F_m(C)$ is non-null and $m - l > 0$. Let $m - l = n\delta + d$ where n, d are non-negative integers such that $0 \leq d < \delta$. By the preceeding result, $F_l(C) \cap F_{l+n\delta}(C)$ is non-null. Then $F_l(C) \cap F_{l+n\delta}(C) \cap F_m(C)$ is non-null for if it were otherwise then $F_l(C) \cap F_{l+n\delta}(C)$ and $F_l(C) \cap F_m(C)$ would be two disjoint, non-null, M^k-closed subsets of $F_l(C)$ which is impossible. If $d > 0$, then $C \cap F_d(C)$ is null, which in turn implies that $F_{l+n\delta}(C) \cap F_m(C)$ is null (Lemma 8). Hence $d = 0$ and δ divides $m - l$.

THEOREM 1. *If M is irreducible and k is a positive integer, then there is a unique positive integer $\delta = \delta(k)$, which divides k, such that*

1. *X is partitioned into δ non-null, indecomposably M^k-closed sets $C_1, C_2, \cdots, C_\delta$ with $F_1(C_1) = C_2$, $F_1(C_2) = C_3, \cdots, F_1(C_\delta) = C_1$,*
2. *each C_i, $i = 1, \cdots, \delta$, is also indecomposably M^δ-closed but not M^d-closed for $d = 1, \cdots, \delta - 1$,*
3. *$\{C_1, C_2, \cdots, C_\delta\}$ consists of all non-null indecomposably M^k-closed sets.*

Proof. By Lemma 11, there exists a non-null, indecomposably M^k-closed set C. Consider the sequence of sets, C, $F_1(C)$, $F_2(C)$, $\cdots$. Let δ be the smallest positive integer such that $C \cap F_\delta(C)$ is non-null. By Lemma 12, δ divides k. Let $k = \delta l$. Let

$$C_0 = C \cap F_\delta(C) \cap \cdots \cap F_{(l-1)\delta}(C).$$

Then C_0 is M^δ-closed by Lemma 6. C_0 is non-null by Lemma 12. Since X is irreducibly closed,

$$X = C_0 \cup F_1(C_0) \cup \cdots \cup F_{\delta-1}(C_0).$$

Since $C_0 \subset C$, $F_1(C_0) \subset F_1(C)$, $\cdots$, $F_{\delta-1}(C_0) \subset F_{\delta-1}(C)$,

$$X = C_0 \cup F_1(C_0) \cup \cdots \cup F_{\delta-1}(C_0) = C \cup F_1(C) \cup \cdots \cup F_{\delta-1}(C).$$

Sets $C, F_1(C), \cdots, F_{\delta-1}(C)$ are pairwise disjoint by Lemma 12; hence $C = C_0$ and C is M^δ-closed; therefore $C \supset F_\delta(C)$. Now $C_0 \subset F_\delta(C)$, hence $C = F_\delta(C)$. Let $C_1 = C$, $C_2 = F_1(C_1)$, $F_1(C_2) = C_3, \cdots, C_\delta = F_{\delta-1}(C)$; then $C_1, \cdots, C_\delta$ satisfy conclusion 1 of Theorem 1. Since $C_1 = C$ is M^δ-closed, $C_2, \cdots, C_\delta$ are also M^δ-closed. They are indecomposably M^δ-closed since they are indecomposably M^k-closed (Lemma 3 and Lemma 9). None of C_i is M^d-closed if

$d < \delta$ for C_i and $F_d(C_i)$ are disjoint. Now suppose that C' is an arbitrary non-null indecomposably M^k-closed set. Then $C' \subset C_d$ for some integer d, $1 \leq d \leq \delta$. Let us proceed for C' as we did for C and let δ' be the δ for C'. Since $F_n(C') \subset F_{n+d}(C)$ for $n = 1, 2, \cdots$, δ' must be an integral multiple of δ by Lemma 12. Interchanging the rules of C and C', we arrive at the conclusion that δ must be an integral multiple of δ'. Hence $\delta = \delta'$ and

$$X = C' \cup F_1(C') \cup \cdots \cup F_{\delta-1}(C').$$

Since $C' \subset C_d$, $F_1(C') \subset C_{d+1}, \cdots, F_{\delta-1}(C') \subset C_{d-1}$, we have $C' = C_d$ and $\{C_1, \cdots, C_\delta\}$ consists of all non-null-indecomposably M^k-closed set.

Corollary 2. *If M is irreducible, then every non-null indecomposably M^k-closed set is also an irreducibly M^k-closed set for every positive integer k.*

Proof. If C is a non-null indecomposably M^k-closed set, then C must be one of C_i, say C_d, of Theorem 1. If C' is a non-null, M^k-closed set contained in C, then C' is also indecomposably M^k-closed; therefore it is also one of C_i, say $C_{d'}$, of Theorem 1. d' must equal d for, if not, $C_d \cap C_{d'} = \emptyset$. Hence C contains no smaller non-null M^k-closed subset. Hence C is irreducibly M^k-closed.

Definition 4. Let M be irreducible and $\delta(k)$ be the number of distinct, non-null, indecomposably M^k-closed sets. Let

$$\delta = \sup\,[\delta(k) : k = 1, 2, 3, \cdots]. \tag{3}$$

δ may be a positive integer or $+\infty$. If δ is finite, we say that M *has a period* $= \delta$. If $\delta = 1$, we say that M is *aperiodic.*

Lemma 13. *Let M be irreducible and m, n be two positive integers such that m divides n. Let $\delta(m)$, $\delta(n)$ be the numbers of non-null, distinct, indecomposably M^m-closed sets and M^n-closed sets, respectively. Then $\delta(m)$ divides $\delta(n)$. Let $l = \delta(n)/\delta(m)$. Then each non-null, indecomposably M^m-closed set is partitioned into l non-null, indecomposably M^n-closed sets.*

Proof. Let C be a non-null, indecomposably M^m-closed set. Consider the sequence of sets: $C, F_1(C), F_2(C), \cdots$. By Lemma 12 and Theorem 1, X is partitioned into $\delta(m)$ sets $C, F_1(C), \cdots, F_{\delta(m)-1}(C)$ and $F_k(C) \cap F_j(C) \neq \emptyset$ implies that $\delta(m)$ divides $k - j$. Let D be a non-null indecomposably M^n-closed set. Then $D \subset F_j(C)$ for some j, say $j = 0$. Consider the sequence of sets: $D, F_1(D), F_2(D), \cdots$. Then X is partitioned into $\delta(n)$ sets $D, F_1(D), \cdots, F_{\delta(n)-1}(D)$ and $D = F_{\delta(n)}D$. Since $D \subset C$, $F_{\delta(n)}(D) \subset F_{\delta(n)}(C)$, $C \cap F_{\delta(n)}(C) \neq \emptyset$. Hence $\delta(m)$ divides $\delta(n)$. Let $l = \delta(n)/\delta(m)$. Now, sets $D, F_1(D), \cdots, F_{\delta(n)-1}(D)$ are $M^{\delta(n)}$-closed. Let

$$\begin{aligned}
C_1 &= D \cup F_{\delta(m)}(D) \cup \cdots \cup F_{(l-1)\delta(m)}(D),\\
C_2 &= F_1(D) \cup F_{\delta(m)+1}(D) \cup \cdots \cup F_{(l-1)\delta(m)+1}(D),\\
&\vdots\\
C_{\delta(m)} &= F_{\delta(m)-1}(D) \cup F_{\delta(m)}(D) \cup \cdots \cup F_{\delta(n)-1}(D).
\end{aligned}$$

Then sets $C_1, C_2, \cdots, C_{\delta(m)}$ are $M^{\delta(m)}$-closed by Lemma 6. It is clear that $C_1, C_2, \cdots, C_{\delta(m)}$ are pairwise disjoint, therefore, all distinct. Each C_i must be indecomposably M^m-closed for to be otherwise would imply that the number of distinct M^m-closed sets is greater than $\delta(m)$. Hence $C_1, C_2, \cdots, C_{\delta(m)}$ constitute the totality of all non-null, indecomposably M^m-closed sets. Each C_i is partitioned into l non-null, indecomposably M^n-closed sets by definition.

THEOREM 2. *Let M be irreducible. M has a period $= d$ if and only if* (I) *is true.*

(I) *X is partitioned into d sets $C_1, C_2, \cdots, C_d$, such that $F_1(C_1) = C_2$, $F_1(C_2) = C_3, \cdots, F_1(C_d) = C_1$ and each C_i is irreducibly M^{dn}-closed for $n = 1, 2, \cdots$.*

M does not have a period if and only if (II) *is true.*

(II) *There is an increasing sequence of positive integers $m_1, m_2, \cdots$, such that each m_i divides its successor $m_{i+1}(m_{i+1} = m_i \cdot l_{i+1}$ where l_{i+1} is a positive integer) and for every i, X is partitioned into m_i non-null, indecomposably M^{m_i}-closed sets $C_1^{(i)}, \cdots, C_{m_i}^{(i)}$ and each $C_j^{(i)}$ is partitioned into l_{i+1} $C_k^{(i+1)}$ sets.*

Proof. If (I) is true, then $\delta(k)$ of Theorem 1 is equal to d provided $k = n\,d$ where n is a positive integer. By Lemma 13, $\delta(n) \leq \delta(n\,d) = d$. Hence the δ given by (3) is equal to d. Hence M has a period $= d$. Conversely, if M has period $= d$, then there is a positive integer k such that $\delta(k) = d$ and X is partitioned into d non-null sets $C_1, C_2, \cdots, C_d$, each of which is both indecomposably M^k-closed and indecomposably M^d-closed, such that $F_1(C_1) = C_2$, $F_1(C_2) = C_3, \cdots, F_1(C_d) = C_1$. The fact that each C_i is indecomposably M^d-closed implies that $\delta(d) = d$. By Lemma 13, $\delta(n\,d) \geq d$, hence $\delta(n\,d) = d$ Hence each C_i is indecomposably M^{nd}-closed; therefore, irreduciblyM^{nd}-closed by Corollary 2.

It is clear that (II) implies that the δ given by (3) is $+\infty$, hence, M does not have a period. Conversely, if M does not have a period then there is an increasing sequence of positive integers $n_1, n_2, \cdots$ such that $\lim_{i\to\infty} \delta(n_i) = +\infty$. Let $k_i = n_1 \cdot \cdots \cdot n_i$; then $\lim_{i\to\infty} \delta(k_i) = +\infty$. Let $m_i = \delta(k_i)$. Applying Lemma 13, we conclude that the sequence $m_1, m_2, \cdots$ satisfies the requirement of (II).

LEMMA 14. *Let M be irreducible and possess no period. Let the sequence of positive integers $\{m_i\}$ and the sequence of partitions $\{C_1^{(i)}, \cdots, C_{m_i}^{(i)}\}$ of X be as in* (II) *of Theorem 2. Let $\lambda(X) = 1$ and*

$$a_i = \max\,[\lambda(C_k^{(i)}) : k = 1, 2, \cdots, m_i].$$

Then $\{a_i\}$ is a decreasing sequence which converges to 0.

Proof. It is clear that $\{a_i\}$ is a decreasing sequence. Let $\lim_{i\to\infty} a_i = a$. a_i is equal to $\lambda(C_k^{(i)})$ for some $k = k_i$. By rearranging the indices k we may assume $k_i = 1$ for $i = 1, 2, \cdots$. If $a > 0$, then there would be a subsequence $\{i_j\}$ of the sequence $\{i\}$ such that $C_1^{(i_j)} \cap C_1^{(i_j+1)} \neq \emptyset$ for $j = 1, 2, \cdots$, which, in turn, implies $C_1^{(i_j)} \supset C_1^{(i_{j+1})}$ for $j = 1, 2, \cdots$. Let $D = \bigcap_{j=1}^{\infty} C^{(i_j)}$; then $\lambda(D) = a$. Consider the following sequence of sets: $D, F_1(D), F_2(D), \cdots$. Since $\lambda(D) > 0$, every set in this sequence is non-null by Lemma 7. Now, $D \subset C_1^{(i_j)}$ and

$$C_1^{(i_j)}, F_1(C_1^{(i_j)}), \cdots, F_{m_{i_j}-1}(C_1^{(i_j)})$$

are pairwise disjoint, hence

$$D, F_1(D), \cdots, F_{m_{i_j}-1}(D)$$

are pairwise disjoint. Since this is true for $j = 1, 2, \cdots$ and since $\lim_{j\to\infty} m_{i_j} = \infty$, we conclude that the sets in the sequence $D, F_1(D), F_2(D), \cdots$ are pairwise disjoint. However, $\bigcup_{n=1}^{\infty} F_n(D) = F(F_1(D))$ is a non-null closed set by Lemma 6. $D \subset X - \bigcup_{n=1}^{\infty} F_n(D)$ and D being non-null contradict the fact that X is irreducibly closed. Hence $a = 0$.

THEOREM 3. *If M is irreducible and if M does not have a period, then the measure λ on $\mathcal{B}$ is non-atomic.*

Proof. If $\lambda(X)$ is not equal to 1, we replace it by an equivalent measure which assigns measure 1 to X. The new measure is non-atomic if and only if the original one is non-atomic. Hence it is sufficient to prove the theorem for the case that $\lambda(X) = 1$. Let the sequence of positive integers $\{m_i\}$ and the sequence of partitions $\{C_1^{(i)}, \cdots, C_{m_i}^{(i)}\}$ of X be as in (II) of Theorem 2. If $\mathcal{B}$ had a λ-atom A then

$$\lambda(A) \leq \max [\lambda(C_k^{(i)}): k = 1, \cdots, m_i]$$

which contradicts the conclusion of Lemma 14. Hence λ is non-atomic.

DEFINITION 5. A positive operator M is said to be λ-*continuous* if there is a real-valued, $\mathcal{B} \times \mathcal{B}$ measurable function $m(x, y)$ such that

$$Mf(x) = \int m(x, y) f(y) \lambda\, (dy)$$

for every $f \in L_\infty(\lambda)$. The function $m(x, y)$ is called the *density function* of M with respect to measure λ. The iterates M^n of a λ-continuous positive operator M are also λ-continuous with density functions $m^{(n)}(x, y)$ defined inductively by

$$m^{(1)}(x, y) = m(x, y),$$

$$m^{(n+1)}(x, y) = \int m^{(n)}(x, z) \lambda\, (dz) m(z, y).$$

DEFINITION 6. A positive operator M is said to be *quasi* λ-*continuous* if there is a positive integer r such that M^r is the sum of two positive operators M_1, M_2 one of which is non-zero and λ-continuous.

It is clear that a λ-continuous M is quasi λ-continuous.

THEOREM 4. *An irreducible, quasi λ-continuous positive operator M has a period.*

Proof. Suppose r is the positive integer such that $M^r = M_1 + M_2$ where M_1, M_2 are two positive operators for which M_1 is non-zero and λ-continuous. Let $m_1(x, y)$ be the density function of M_1 with respect to λ. Without loss of generality we may assume $\lambda(X) = 1$. Let E be the subset of $X \times X$,

$$E = [(x, y) : m_1(x, y) > 0].$$

Since M_1 is not zero, $\lambda \times \lambda(E) > 0$. Now, if M did not have a period, then (II) of Theorem 2 would be satisfied. Let the sequence $\{m_i\}$ and the sequence of partitions $\{C_1^{(i)}, \cdots, C_{m_i}^{(i)}\}$ of X be as in (II) of Theorem 2. We have $F_r(C_1^{(i)}) = C_{r+1}^{(i)}$, $F_r(C_2^{(i)}) = C_{r+2}^{(i)}, \cdots$, etc. (Here we let $C_n^{(i)} = C_k^{(i)}$ if $n > m_i$, $1 \le k \le m_i$, $n = k + lm_i$, k, l, n are positive integers.) If $\nu \in \mathfrak{A}^+(\lambda)$ has $C_j^{(i)}$ as its support, then $\nu M^r(X - C_{r+j}^{(i)}) = 0$. On the other hand

$$\int \nu\,(dx) \int_{X - C_{r+j}^{(i)}} m_1(x, y)\lambda\,(dy) = \nu M_1(X - C_{r+j}^{(i)}) \le \nu M^r(X - C_{r+j}^{(i)}).$$

Hence

$$\int \nu\,(dx) \int_{X - C_{r+j}^{(i)}} m_1(x, y)\lambda\,(dy) = 0. \tag{4}$$

Since $m_1(x, y)$ is non-negative a.e. $(\lambda \times \lambda)$, (4) implies that $m_1(x, y) = 0$ a.e. $(\lambda \times \lambda)$ on $C_j^{(i)} \times (X - C_{r+j}^{(i)})$. This is true for $j = 1, 2, \cdots, m_i$. Hence

$$\lambda \times \lambda[E - \textstyle\bigcup_{j=1}^{m_i} C_j^{(i)} \times C_{r+j}^{(i)}] = 0$$

so that

$$\lambda \times \lambda(E) \le \lambda \times \lambda(\textstyle\bigcup_{j=1}^{m_i} C_j^{(i)} \times C_{r+j}^{(i)}). \tag{5}$$

Let $a_i = \max\,[\lambda(C_j^{(i)}) : j = 1, 2, \cdots, m_i]$; then $\lambda \times \lambda(\bigcup_{j=1}^{m_i} C_j^{(i)} \times C_{r+j}^{(i)}) \le a_i$. By Lemma 14, $a_i \downarrow 0$. This fact, together with (5), implies that $\lambda \times \lambda(E) = 0$ which contradicts $\lambda \times \lambda(E) > 0$. Hence M must possess a period.

III. Positive operators with transition functions

We call a real-valued function $M(x, A)$ of two variables, $x \in X$, $A \in \mathcal{B}$, a *transition function* if the following two conditions are satisfied.

(T1) For every fixed $x \in X$, $M(x, \cdot)$ is a measure.
(T2) For every fixed set $A \in \mathcal{B}$, $M(\cdot, A)$ is a $\mathcal{B}$ measurable function.

This is a generalization of a *probability transition function* of a Markov process. If the measurable space $(X, \mathcal{B})$ is the space of all types of a branching process, then the first moment function of the process is a transition function. We shall always assume that (T3) is satisfied by a transition function.

(T3) There is a number a such that $M(x, X) \le a$ for all $x \in X$.

If $M(x, X)$ is a probability transition function, then the stronger condition (T3′) is satisfied:

(T3′) $M(x, X) = 1$ for all $x \in X$.

$M^{(n)}(x, A)$, $n = 1, 2, \cdots$, are defined inductively as follows:

$$M^{(1)}(x, A) = M(x, A),$$

$$M^{(n+1)}(x, A) = \int M^{(n)}(x, dy)M(y, A).$$

$M^n(x, A)$ are also transition functions. For a bounded, $\mathcal{B}$ measurable function f, we define Mf by

$$Mf(x) = \int M(x, dy)f(y). \tag{6}$$

For a bounded, countably addition set function ν defined on $\mathcal{B}$, we define νM by

$$\nu M(A) = \int \nu\,(dx)M(x, A). \tag{7}$$

νM is also a bounded, countably additive set function and Mf is also a bounded $\mathcal{B}$-measurable function. $M^n f$ and νM^n are then given by

$$M^n f(x) = \int M^{(n)}(x, dy)f(y),$$

$$\nu M^n(A) = \int \nu\,(dx)M^{(n)}(x, A).$$

Furthermore, if ν is absolutely continuous to a finite measure π, then νM is absolutely continuous to πM. Let π be an arbitrary finite measure and let $\lambda = \sum_{n=0}^{\infty} (2a)^{-n}\pi M^n$. Then, if ν is absolutely continuous to λ, so is νM; and if $f \in L_\infty(\lambda)$, so is Mf. Thus a λ-measurable positive operator is generated.

We call a positive operator M given by (6) a *positive operator with a transition function.* A λ-continuous positive operator is a positive operator with a transition function. If $\mathcal{B}$ is generated by a countable collection, and if M is a positive operator with a transition function then the transition function $M(x, A)$ is uniquely determined up to a set of λ-measure 0 by M in the sense that, if M has another transition function $M'(x, A)$ then $M(x, \cdot) = M'(x, \cdot)$ for (λ) almost all x.

Let M be a positive operator with a transition function $M(x, A)$. Define a measure η on $\mathcal{B} \times \mathcal{B}$ as follows. If E is a $\mathcal{B} \times \mathcal{B}$-measurable subset of $X \times X$,

$$\eta(E) = \int \lambda\,(dx) \int M(x, dy)1_E(x, y).$$

η is uniquely determined by the operator M as

$$\eta(A \times B) = \int_A \lambda\,(dx)M1_B(x)$$

or all rectangles $A \times B$ in $\mathcal{B} \times \mathcal{B}$. η is called the *measure* associated with M. It is clear that η is absolutely continuous to $\lambda \times \lambda$ if and only if M is λ-continuous. For the general case, η may be decomposed into two parts η_C and η_S where η_C is absolutely continuous to $\lambda \times \lambda$ and η_S is singular to $\lambda \times \lambda$. Let $m_1(x, y)$ be a derivative of η_C with respect to $\lambda \times \lambda$ and let us define a λ-continuous operator M_1 by

$$M_1 f(x) = \int m_1(x, y) f(y) \lambda\,(dy). \tag{8}$$

This λ-continuous positive operator M_1 is characterized by two facts: (1) $M_1 \le M$; (2) if N is a λ-continuous positive operator such that $N \le M$, then $N \le M_1$. M_1 is called the λ-*continuous* part of M.

THEOREM 5. *If $\mathcal{B}$ is generated by a countable collection, M is a positive operator with a transition function $M(x, A)$, M_1 is the λ-continuous part of M and $m_1(x, y)$ is a density function of M_1 with respect to λ, then there is a set $Z \in \mathcal{B}$ with $\lambda(X - Z) = 0$ such that $x \in Z$ implies that $m_1(x, \cdot)$ is a derivative of the λ-continuous part of $M(x, \cdot)$ with respect to λ. Furthermore, $M = M_1 + M_2$ where M_2 is a positive operator with a transition function $M_2(x, A)$ such that $M_2(x, \cdot)$ is singular to λ for every $x \in Z$.*

Proof. If $\mathcal{B}$ is generated by a countable collection, then there is a sequence of finite subalgebras $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots$ such that $\mathcal{B}$ is generated by $\bigcup_{n=1}^{\infty} \mathcal{B}_n$. Each $\mathcal{B}_n$ is generated by a partition $B_1^{(n)}, \cdots, B_{i_n}^{(n)}$ of X. We shall define a sequence of functions $\{f_n(x, y)\}$ as follows.

$$\begin{aligned} f_n(xy) &= M(x, B_i^{(n)})/\lambda(B_i^{(n)}), &&\text{if } \ y \in B_i^{(n)},\ \lambda(B_i^{(n)}) > 0,\\ &= 0, &&\text{if } \ y \in B_i^{(n)},\ \lambda(B_i^{(n)}) = 0. \end{aligned}$$

For any $A \in \mathcal{B}$,

$$\int_{A \times B_i{}^{(n)}} f_n(x, y) \lambda \times \lambda\,(d(x, y)) = \int_A M(x, B_i^{(n)}) \lambda\,(dx) = \eta(A \times B_i^{(n)}).$$

If we restrict the domain of definition of η and $\lambda \times \lambda$ to $\mathcal{B} \times \mathcal{B}_n$, then f_n is the derivative of η with respect to $\lambda \times \lambda$. Since $\bigcup_{n=1}^{\infty} \mathcal{B} \times \mathcal{B}_n$ generates $\mathcal{B} \times \mathcal{B}$, $\{f_n\}$ converges a.e. $(\lambda \times \lambda)$ to the derivative of η_C with respect to $\lambda \times \lambda$, which is $m_1(x, y)$ of (8). On the other hand, for each fixed x, the a.e. (λ) limit of the sequence $\{f_n(x, \cdot)\}$ is the derivative of the λ-continuous part of $M(x, \cdot)$ with respect to λ (See Example 2.7, pp. 616 of [3]). Hence, there is a set $Z \in \mathcal{B}$ with $\lambda(X - Z) = 0$ such that if $x \in Z$, $m_1(x, \cdot)$ is the derivative of the λ-continuous part of $M(x, \cdot)$ with respect to λ. Now for $x \in Z$, $A \in \mathcal{B}$, define

$$M_2(x, A) = M(x, A) - \int m_1(x, y) \lambda\,(dy)$$

and for $x \notin Z$, define $M_2(x, \cdot)$ arbitrarily. Thus $M_2(x, \cdot)$ is singular to λ if

$x \in Z$ and

$$Mf(x) - M_1 f(x) = \int M_2(x, dy) f(y)$$

for $x \in Z$ and $M = M_1 + M_2$ where

$$M_2 f(x) = \int M_2(x, dy) f(y).$$

LEMMA 15. *Let M be a positive operator with a transition function $M(x, A)$. If the λ-continuous part of M is 0, then there is a set $Z \in \mathcal{B}$ with $\lambda(X - Z) = 0$ such that $M(x, \cdot)$ is singular to λ for every $x \in Z$. The converse is also true if $\mathcal{B}$ is generated by a countable collection.*

Proof. If the λ-continuous part of M is 0, then the measure of η associated with M is singular to $\lambda \times \lambda$. There is a set $S \in \mathcal{B} \times \mathcal{B}$ with $\lambda \times \lambda(S) = 0$ such that $\eta(S \cap E) = \eta(E)$ for every $E \in \mathcal{B} \times \mathcal{B}$. Let $S_x = [y : (x, y) \in S]$. Since $\lambda \times \lambda(S) = 0$, there is a set Z_1 with $\lambda(X - Z_1) = 0$ such that $x \in Z_1$ implies $\lambda(S_x) = 0$. Now

$$0 = \eta(X \times X - S) = \int \lambda\,(dx) M(x, X - S_x).$$

Hence there is set $Z_2 \in \mathcal{B}$ with $\lambda(X - Z_2) = 0$ such that $x \in Z_2$ implies $M(x, X - S_x) = 0$. Hence if $x \in Z = Z_1 \cap Z_2$, then $M(x, X - S_x) = 0$, $\lambda(S_x) = 0$ and the singularity of $M(x, \cdot)$ to λ follows.

If $\mathcal{B}$ is generated by a countable collection then the converse follows from Theorem 5.

THEOREM 6. *Let M be a positive operator with a transition function $M(x, A)$. If M is not quasi λ-continuous, then there is a set $Z \in \mathcal{B}$ with $\lambda(X - Z) = 0$ such that, for every $x \in Z$, $M^{(n)}(x, \cdot)$ is singular to λ for $n = 1, 2, 3, \cdots$. The converse is also true if $\mathcal{B}$ is generated by a countable collection.*

Proof. M is not quasi λ-continuous if and only if the λ-continuous part of M^n is zero for $n = 1, 2, \cdots$. This fact, together with Lemma 15, implies Theorem 6.

COROLLARY 3. *Let $M(x, A)$ be a transition function, π be a non-zero finite measure on $\mathcal{B}$ and $\lambda := \sum_{n=0}^{\infty} (2a)^{-n} \pi M^n$. Let M be the λ-measurable positive operator given by (6). If for (λ) almost all x, $\sum_{n=1}^{\infty} M^{(n)}(x, H) > 0$ for every set H with $\pi(H) > 0$, then M is irreducible and quasi λ-continuous, therefore, possesses a period by Theorem 4.*

Proof. Clearly $\pi \in \mathfrak{A}^+(\lambda)$. Let G be the support of π. Then

$$\lambda(\pi - F(G)) = 0.$$

If E is a non-null subset of G, then $\sum_{n=1}^{\infty} M^n 1_E > 0$ a.e. (λ) for $\sum_{n=1}^{\infty} M^n 1_E(x) = \sum_{n=1}^{\infty} M^{(n)}(x, E)$. Now, if E is a non-null subset of $F_k(G)$ then

$$H = G \cap [x : M^k 1_E(x) > 0]$$

is a non-null set. Hence $\sum_{n=1}^{\infty} \int_H M^{(n)}(x, dy)M^k 1_E(y) > 0$ for every $x \epsilon Z$. Hence $\sum_{n=k+1}^{\infty} M^n 1_E > 0$ a.e. (λ). Applying Corollary 1, we arrive at the irreductibility of M. To show the quasi λ-continuity of M, we set

$$\nu(x, A) = \sum_{n=1}^{\infty} (2a)^{-n} M^{(n)}(x, A).$$

For each fixed x, $\nu(x, \cdot)$ is a finite measure and $\nu(x, A) > 0$ if and only if $\sum_{n=1}^{\infty} M^{(n)}(x, A) > 0$. If M is not quasi λ-continuous, then, by Theorem 6, $\nu(x, \cdot)$ is singular to λ for (λ) almost all x. But $\nu(x, H) > 0$ for every non-null subset H of G. This fact implies the restriction of λ to subsets of G is absolutely continuous to the same restriction of $\nu(x, \cdot)$ for (λ) almost all x. This is incompatible with the statement that $\nu(x, \cdot)$ is singular to λ for (λ) almost all x. Hence M is quasi λ-continuous.

Now we turn to a probability transition function. We shall write $P(x, A)$ instead of $M(x, A)$ and operator P instead of M. A complete theory of Markov process with a discrete parameter under a condition (D) of Doeblin is given in Chapter V of [3]. In [3] the special case (c) is treated first. Combining (D) and (c) one obtained a period for the probability transition function. T. E. Harris gave a condition (H) on the probability transition function in 1956. An extensive amount of theory of Markov process was developed by T. E. Harris [6] and S. Orey [9] based on condition (H). In both cases the existence of a finite period is established after a considerable amount of knowledge of $P^{(n)}(x, A)$ *is obtained.*

Condition (D). There is a finite measure π on $\mathcal{B}$, a positive integer k and a positive number ε such that

$$P^{(k)}(x, A) \leq 1 - \varepsilon \qquad \text{for all } x \tag{9}$$

whenever $\pi(A) \leq \varepsilon$.

Special Case (c). $\mathrm{Sup}_{n \geqq 1} P^{(n)}(x, A) > 0$ for all $x \epsilon X$ whenever $\pi(A) > 0$.

Condition (H). There is a non-zero finite measure π on $\mathcal{B}$ such that
(10) $\pi(A) > 0$ implies that the probability that A is visited infinitely many times is 1 for all starting point $x \epsilon X$.

Under either condition obtain a λ-measurable Markov operator P by letting $\lambda = \sum_{n=0}^{\infty} 2^{-n} \pi P^n$. Clearly, (9) implies that $P^{(k)}(x, \cdot)$ is not singular to π, therefore, not singular to λ, for all x. Hence, by Theorem 6, P is quasi λ-continuous under Condition (D). (c) is equivalent to (11).

(11) $\pi(A) > 0$ implies that $\sum_{n=1}^{\infty} P^{(n)}(x, A) > 0$ for all $x \epsilon X$.

Hence, by Corollary 3, (c) alone implies that P is irreducible and quasi λ-continuous, therefore, possesses a period. (c) is a much weaker condition than (10). Hence, under condition (H), we also have a irreducible, quasi λ-continuous P. We summarize these facts in the following.

COROLLARY 4. *If the probability transition function satisfies Condition*

(D) *then the* λ*-measurable Markov operator* P is quasi λ*-continuous. If Condition* (H) *is satisfied by the probability transition function then* (c) *is also satisfied.* (c) *implies that* P *is irreducible and quasi* λ*-continuous and, therefore, possesses a period.*

We remark that, in the above corollary, we do not assume that $\mathcal{B}$ is generated by a countable collection as was the case in [6] and [9].

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