DECOMPOSITIONS OF E^3 WITH A NULL SEQUENCE OF STARLIKE EQUIVALENT NON-DEGENERATE ELEMENTS ARE E^3

BY

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Donald V. Meyer has recently proven [4] that if an upper semi-continuous decomposition of E^3 has only a null sequence of non-degenerate elements and if each of these is a tame 3-cell then the decomposition space is E^3. We generalize this result to include any null sequence of continua provided only that each is equivalent, under a space homeomorphism, to a starlike continuum. Thus our result includes not only tame 3-cells but tame disks, triods, whiskbrooms and any combination of these.

There are still several very interesting unsolved questions in this area. For example, is the decomposition space E^3 if the non-degenerate elements

(1) form a sequence of sets, each equivalent, under a space homeomorphism, to a starlike set? This question is not answered even when each element is a tame cell.

(2) form a null sequence of strongly cellular sets [1]? i.e., for each $g \in G$ there is a cell C in E^3 with $g \in \text{Int } C$, and a homotopy $f : C \times [0, 1] \to C$ such that

(a) $h(x, 0) = x$, for all $x \in C$ and $h(x, t) = x$ for all $x \in g$, $t \in [0, 1]$

(b) $h|_{Bd C \times [0,1)}$ is a homeomorphism onto $C - g$

(c) $h(C \times 1) = g$.

It is known that cellular (in place of strongly cellular) in question 2 is not enough to insure that the decomposition space is E^3 [2]. The answer to question 1 is yes if each element is taken to be starlike [3].

We will use standard notation. A collection of disjoint continua G filling up E^3 is called upper semi-continuous if for each $g \in G$ and each neighborhood U of g there is a neighborhood V of g such that if $g' \in G$ and $g' \cap V \neq \emptyset$ then $g' \subset U$. The decomposition space G' is defined by letting a set $U' \subset G$ be open in G' if the set $U = \bigcup_{g \in U'} g$ is open in E^3. H denotes the collection of all non-degenerate elements of G and $H^* = \bigcup_{g \in H} g$. A continuum g is starlike with respect to $p \in g$ if every line through p intersects g in either an interval or the point p. A null sequence of sets is a sequence such that given $\varepsilon > 0$ there are only a finite number of sets in the sequence whose diameters are greater than ε.

THEOREM. *Let G be an upper semi-continuous decomposition of E^3 such that H is a null sequence of continua and each continuum $g \in H$ is equivalent (under a space homeomorphism) to a starlike continuum. Then G' is E^3.*

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Proof of theorem. As shown in the proof of Theorem 1 of [3], the theorem is an immediate consequence of the following lemma.

LEMMA. *Let G be as above. Let $\varepsilon > 0$ be given and let U be a neighborhood of H^* in E^3. Then there exists a homeomorphism h of E^3 onto itself such that $h|_{E^3-U}$ is the identity and* diam $h(g) < \varepsilon$ *for each g in H.*

Proof of lemma. Since H is a null sequence there are only a finite number of elements of H whose diameters are $\geq \varepsilon$. Let g be such an element. We will describe a homeomorphism of E onto itself which is the identity outside U and outside a very small neighborhood of g and which shrinks g to diameter $<\varepsilon$ while not expanding any other element of H to have diameter $\geq \varepsilon$. Clearly, a finite composition of such homeomorphisms will be the one we are seeking in the lemma.

Since g is equivalent to a starlike continuum there is a homeomorphism f of E^3 onto itself such that $f(g)$ is starlike with respect to some point $f(p)$, and for some neighborhood V of g we can find a $\delta > 0$ such that if $|f(x) - f(y)| < \delta$ and $x, y \in V$, then $|x - y| < \varepsilon$. Let k be an integer so that

$$k \cdot \delta/16 > \operatorname{diam} f(g) + \delta/8$$

and let $S_1, S_2, \cdots, S_k$ be neighborhoods of $f(g)$ such that

(1) $S_k \subset U \cap V \cap S_{\delta/16}(f(g))$;

(2) $f(g) \subset S_i \subset \bar{S}_i \subset S_{i+1}$;

(3) if $f(g') \cap S_i \neq \emptyset$ for some $g' \in H$ then diam $f(g') < \delta/16$ and if $i \neq k$, $f(g') \subset S_{i+1}$;

(4) each S_i is ideally starlike with respect to $f(p)$, i.e. if r is a ray from p, then $r \cap S_i$ is one point.

One may find such neighborhoods since G is upper semi-continuous, H is a null sequence and $f(g)$ is starlike.

Let $R_i = \{x \mid |x - f(p)| < i \cdot \delta/16\}$ $i = 1, 2, \cdots, k$.

As in the proof of Lemma 4 of [1] we will define a homeomorphism h' of E^3 onto itself by defining it on each ray r from the point $f(p)$. Let $S_k \cap r = s_k$ and $R_k \cap r = r_k$. Then let

$$h'(s_k) = r_k \quad \text{if } r_k \text{ is closer to } f(p) \text{ than } s_k$$

$$= s_k \quad \text{if not.}$$

Extend h' to all of r by taking $[s_i, s_{i+1}]$ linearly onto $[h'(s_i), h'(s_{i+1})]$ $(i = 1, 2, \cdots, k-1)$, $[f(p), s_1]$ linearly onto $[f(p), h'(s_1)]$ and let h' be the identity on $[s_k, \infty)$. Clearly h' is a homeomorphism of E^3 onto itself which is the identity on $E^3 - f(U)$. We need only show that h' "shrinks" without "expanding". Clearly $h'(f(g))$ has diameter $<\delta$. Let g' be such that $f(g') \cap S_k \neq \emptyset$. To see that diam $h'(f(g')) < \delta$ we observe that if $f(g')$ is moved at all by h' it is moved toward p. Since diam

$f(g') < \delta/16$, $f(g')$ is contained in the annulus between R'_{i-1} and R_{i+1} for some i. Since by the definition of $S_1, S_2, \cdots, S_k$ it is also contained between S'_{j-1} and S'_{j+1}, for some j, we know $h'f(g')$ is in the annulus between R'_{j-1} and R'_{j+1} and $j \leq i$. Since the angular size (from $f(p)$) of $f(g')$ is unchanged by h' we immediately verify that diam $h'f(g') < \delta$. $f^{-1}h'f$ is the homeomorphism we are seeking.

References

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