AN ANALYTIC CONTINUATION FOR CERTAIN FUNCTIONS DEFINED BY DIRICHLET SERIES

BY

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Introduction

The problem which motivated this research is that of obtaining Ω results of the type obtained by Hardy [4] and Landau [6]. Erdös and Fuchs [2] and later Bateman, Kolbecker, and Tull [1] have generalized the classical Ω result for the circle problem.

Richert [8] showed that the classical Ω results for the divisor problems are also indicative of results in a class of problems involving the multiplication of Dirichlet series. In particular, his results concerned series whose generating functions have only finitely many poles in a strip to the left of the region of convergence.

In this paper we present some preliminary work necessary to extend his results to a class of Dirichlet series and Laplace Transforms whose generating functions are analytic in a strip to the left of the region of convergence with the possible exception of a bounded region. This class of functions is interesting because of the unity of exposition it allows and because there are available in this class examples which show why certain restrictions imposed by Richert are necessary if a proof is to be given along his lines.

1. An analytic continuation

Let $A(\omega)$ be a complex function of the real variable ω which is of bounded variation on every finite interval. Further, we assume $A(\omega) = 0$ for $\omega \leq 0$. Define

$$Z(s) = \int_0^\infty e^{-\omega s}\, dA(\omega)$$

for every value of $s = \sigma + it$ for which the integral converges.

We say $Z(s)$ belongs to the class $\Sigma(\sigma_0, P)$, where $0 \leq P < \sigma_0$, in case the following three conditions are satisfied by Z:

(i) $\int_0^\infty e^{-\omega s}\, dA(\omega)$ has abscissa of convergence σ_0;

(ii) for $\lambda > P$ there exists a $T_\lambda \geq 0$ such that $Z(s)$ has an analytic continuation into the region $\sigma \geq \lambda$, $|t| \geq T_\lambda$;

(iii) for each $\lambda > P$ there exists a $\beta_\lambda \geq 0$ such that $Z(s) = O(|t|)^{\beta_\lambda})$ as $t \to \infty$ uniformly for $\lambda \leq \sigma < R$, for all $R > \lambda$.

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For any complex function $C(\omega)$ which is of bounded variation on $[0, K]$ for every $K \geq 0$, we define $C^{(0)}(\omega) = C(\omega)$ and if κ is a positive integer we let

$$C^{(-\kappa)}(\omega) = \int_0^\omega C^{(-\kappa+1)}(u)\, du.$$

$C^{(-\kappa)}(\omega)$ is a multiple of the κ^{th} Riesz Mean of C.

We now list some hypotheses which we shall use in the statement of Theorem 1. Theorem 1 gives the relationship between the availability of an analytic continuation for $Z(s)$ and the asymptotic behavior of $A(\omega)$ as $\omega \to \infty$.

H1. $\sigma_0 = \text{glb}\, \{\gamma \mid A(\omega) = O(e^{\gamma\omega}) \text{ as } \omega \to \infty\}$.

H2(λ, G). There exists a positive integer $\kappa = \kappa_\lambda$ such that

$$A^{(-\kappa)}(\omega) - G^{(-\kappa)}(\omega) = O(e^{\lambda\omega}) \text{ as } \omega \to \infty.$$

H3(M, G). $G(\omega) = 0$ for $\omega \leq M$ and $G(\omega)$ is continuous for $\omega \geq 0$.

H4(M, G). Define $\text{Arg}\, z$ so that $-\pi < \text{Arg}\, z \leq \pi$. For $\phi \geq 0$ let

$$R_\phi = \{z \mid |\text{Arg}\, z| \leq \phi, |z| \geq M\}.$$

There exists a $\phi > 0$ such that there is a function which is analytic on some domain containing R_ϕ and agrees with $G(\omega)$ on R_0.

H5(M, G). Given $\varepsilon > 0$, if $\phi > 0$ is sufficiently small, $z \in R_\phi$ implies $G(z) = O(\exp{(\sigma_0 + \varepsilon)}\, \text{Rl}\, z)$.

H6(G). Given $\varepsilon > 0$, the variation of $G(\omega)$ from ω_1 to ω_2

$$VG[\omega_1, \omega_2] = O(|e^{\omega_2(\sigma_0+\varepsilon)} - e^{\omega_1(\sigma_0+\varepsilon)}|)$$

as $\omega_1, \omega_2 \to \infty$.

THEOREM 1. *$Z(s)$ belongs to the class $\Sigma(\sigma_0, P)$ if and only if $A(\omega)$ satisfies* H1 *and there exists for each λ such that $\sigma_0 > \lambda > P$, a function $G_\lambda(\omega)$ and an $M_\lambda \geq 0$ satisfying* H2, H3, H4, H5, *and* H6.

Remark. The M that the theorem produces may be chosen arbitrarily.

Proof. We first suppose that $Z(s)$ is in $\Sigma(\sigma_0, P)$. Choose $\lambda > P$ and let κ be an integer greater than β_λ. Then from [10, Theorem 8.1] we have

$$A^{(-\kappa)}(\omega) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\sigma_0+1-iT}^{\sigma_0+1+iT} e^{\omega s} \frac{Z(s)}{s^{\kappa+1}}\, ds.$$

Let $T > T_\lambda$, $1 > \delta > 0$. We apply Cauchy's Theorem to the above integral over the contour obtained by connecting each of the following list of points to its successor with a line segment: $\sigma_0 + 1 - iT, \lambda - iT, \lambda - iT_\lambda, \sigma_0 + \delta - iT_\lambda$, $\sigma_0 + \delta + iT_\lambda, \lambda + iT_\lambda, \lambda + iT, \sigma_0 + 1 + iT, \sigma_0 + 1 - iT$. The part of the contour between $\lambda - iT_\lambda$ and $\lambda + iT_\lambda$ will be denoted by C. Notice that the integral over C is independent of δ. We then obtain that

$$A^{(-\kappa)}(\omega) = \frac{1}{2\pi i}\int_C e^{\omega s}\frac{Z(s)}{s^{\kappa+1}}\,ds + O(e^{\lambda\omega}).$$

Pick $M \geq 0$. Let

$$\begin{aligned} G(\omega) &= 0 && \text{for } \omega \leq M \\ &= \frac{1}{2\pi i}\int_C (e^{\omega s} - e^{Ms})Z(s)\,\frac{ds}{s} && \text{for } \omega \geq M. \end{aligned}$$

Then

$$G(\omega) = \frac{1}{2\pi i}\int_C e^{\omega s}\frac{Z(s)}{s}\,ds + O(1)$$

and after integrating κ times we have

$$G^{(-\kappa)}(\omega) = \frac{1}{2\pi i}\int_C e^{\omega s}\frac{Z(s)}{s^{\kappa+1}}\,ds + O(\omega^{\kappa}).$$

Hence $A^{(-\kappa)}(\omega) - G^{(-\kappa)}(\omega) = O(e^{\lambda\omega})$. It is clear that A and G satisfy H1, H2 and H3, and since $\int_C e^{sz}Z(s)\,ds/s$ is an entire function of z, H4 is satisfied. Also M satisfies the remark.

To verify H5 we note that for $z \in R_\phi$, $\phi < \pi/2$, $\varepsilon > 0$

$$G(z) = O(\exp(\max_{s\in C} \mathrm{Rl}\,(zs))).$$

Now

$$\mathrm{Rl}\, zs = x\sigma - yt \leq x(\sigma_0 + \varepsilon/2) + |\, y \,|\, T_\lambda$$

since the value of G is independent of δ and we have here chosen $\delta = \varepsilon/2$. For ϕ sufficiently small $z \in R_\phi$ implies $|\, y \,| < (\varepsilon/2T_\lambda)x$ and thus

$$\mathrm{Rl}\,(zs) \leq (\sigma_0 + \varepsilon)\,\mathrm{Rl}\, z.$$

To verify H6 we consider for $\omega \geq M$

$$G(\omega) = \frac{1}{2\pi i}\int_C e^{\omega s}Z(s)(ds/s) + k_1$$

where k_1 is a constant. Then

$$G'(\omega) = \frac{1}{2\pi i}\int_C e^{\omega s}Z(s)\,ds = O(e^{\omega(\sigma_0+\varepsilon)}).$$

and

$$\begin{aligned} VG[\omega_1, \omega_2] &= \int_{\omega_1}^{\omega_2} |\, G'(\omega) \,|\, d\omega \\ &= O(e^{\omega_2(\sigma_0+\varepsilon)} - e^{\omega_1(\sigma_0+\varepsilon)}). \end{aligned}$$

This completes the proof of one-half of the equivalence.

We now assume that $A(\omega)$ satisfies H1, H2, H3, H4, H5 and H6.

Pick λ with $\sigma_0 > \lambda > P$ and let $\kappa = \kappa_\lambda$,

$$B(\omega) = B_\lambda(\omega) = A(\omega) - G_\lambda(\omega) = A(\omega) - G(\omega).$$

We set

$$Y(s) = \int_0^\infty e^{-\omega s}\, dB(\omega) \quad \text{and} \quad W(s) = \int_0^\infty e^{-\omega s}\, dG(\omega).$$

By H1 and H5 both these integrals converge for $\sigma > \sigma_0$ and by [10, Theorem 2.3a]

$$Z(s) = s \int_0^\infty e^{-\omega s} A(\omega)\, d\omega \tag{1}$$

$$Y(s) = s \int_0^\infty e^{-\omega s} B(\omega)\, d\omega \tag{2}$$

$$W(s) = s \int_0^\infty e^{-\omega s} G(\omega)\, d\omega \tag{3}$$

for $\sigma > \sigma_0$ and these integrals all converge absolutely for $\sigma > \sigma_0$. For $\sigma > \sigma_0$

$$Z(s) = Y(s) + W(s).$$

We wish to find analytic continuations for $Y(s)$ and $W(s)$.

By H2, $B^{(-\kappa)}(\omega) = O(e^{\lambda\omega})$ and hence

$$s^{\kappa+1} \int_0^\infty e^{-s\omega} B^{(-\kappa)}(\omega)\, d\omega \tag{4}$$

converges absolutely for $\sigma > \lambda$. Now, integrating (2) by parts κ times we have

$$s^{\kappa+1} \int_0^\infty e^{-s\omega} B^{(-\kappa)}(\omega)\, d\omega = s \int_0^\infty e^{-s\omega} B(\omega)\, d\omega$$

for $\sigma > \sigma_0$. Thus (4) provides an analytic continuation for $Y(s)$ into the region $\sigma > \lambda$.

We consider now

$$W(s)s^{-1} = \int_M^\infty e^{-s\omega} G(\omega)\, d\omega. \tag{5}$$

Fix $\varepsilon > 0$. Let C_R be the boundary of the set $R \geq |z| \geq M$, $0 \leq \operatorname{Arg} z \leq \phi$ where ϕ is from H5. Applying the Cauchy Theorem to the right side of (5) around C_R, we have for $s = \sigma > \sigma_0 + \varepsilon$

$$\begin{aligned} W(\sigma)\sigma^{-1} = \lim_{R\to\infty} \Big\{ \int_M^R & G(\omega e^{i\phi}) \exp(-\omega\sigma e^{i\phi}) e^{i\phi}\, d\omega \\ & + iM \int_0^\phi G(e^{i\theta}M) \exp(-\sigma e^{i\theta}M) e^{i\theta}\, d\theta \\ & - iR \int_0^\phi G(e^{i\theta}M) \exp(-\sigma e^{i\theta}R) e^{i\theta}\, d\theta \Big\}. \end{aligned}$$

Thus, by H5

$$(6)\quad \begin{aligned} W(\sigma)\sigma^{-1} &= \int_M^\infty G(\omega e^{i\phi}) \exp(-\omega\sigma e^{i\phi})e^{i\phi}\, d\omega \\ &\qquad + iM \int_0^\phi G(e^{i\theta}M) \exp(-\sigma e^{i\theta}M)e^{i\theta}\, d\theta \\ &= g_1(\sigma, \phi) + g_2(\sigma, \phi). \end{aligned}$$

Formally substituting s for σ we obtain a function $g_1(s, \phi)$. The integrand of the associated integral is dominated by

$$\exp(((\sigma_0 + \varepsilon)\cos\phi - \sigma\cos\phi + t\sin\phi)\omega)$$

Now, the coefficient of ω in the exponent is $< -\delta < 0$ when s is in the region defined by

$$(7)\qquad t < -(\sigma_0 + \varepsilon)\cot\phi + \sigma\cot\phi - \delta/\sin\phi$$

The line $t = \cot\phi(\sigma - (\sigma_0 + \varepsilon))$ is the boundary for the union of all such regions. The integral for $g_1(s, \phi)$ converges uniformly in each region defined by (7). Thus $g_1(s, \phi)$ is analytic to the region L_ε :

$$-\phi_\varepsilon - \pi/2 < \operatorname{Arg}(s - \sigma_0 - \varepsilon) < -\phi_\varepsilon + \pi/2.$$

Since $g_2(s, \phi)$ is entire and

$$W(s)s^{-1} = g_1(s, \phi) + g_2(s, \phi)$$

for s real and greater than σ_0, $g_1(s, \phi) + g_2(s, \phi)$ is an analytic continuation for $W(s)s^{-1}$ into L_ε.

If we let C_R' be the boundary of the set

$$R \geq |z| \geq M, \qquad 0 \geq \operatorname{Arg} z \geq -\phi,$$

and proceed as before we get an analytic continuation for $W(s)s^{-1}$ into the region U_ε.

$$\phi_\varepsilon - \pi/2 < \operatorname{Arg}(s - \sigma_0 - \varepsilon) < \phi_\varepsilon + \pi/2.$$

Thus we have an analytic continuation for $W(s)s^{-1}$ into

$$\Omega = \{s \mid \sigma > \sigma_0\} \cup (\textstyle\bigcup_{\varepsilon>0} U_\varepsilon) \cup (\bigcup_{\varepsilon>0} L_\varepsilon)$$

which excludes only a bounded portion of the half plane $\sigma > \lambda$. The analytic continuations for $Y(s)$ and $W(s)$ now give us an analytic continuation for $Z(s)$ into the region

$$\{s \mid \sigma > \lambda\} \cap \Omega.$$

We now investigate the behavior of $W(s)s^{-1}$ as $t \to -\infty$ when $a \leq \sigma \leq b$. Here a and b are finite real numbers. We choose $T_0(a)$ large enough so that

$t < -T_0$ implies s is in the region of convergence of

$$\begin{aligned} g_1(s, \phi) &= \int_M^\infty G(\omega e^{i\phi}) \exp(-\omega s e^{i\phi}) e^{i\phi}\, d\omega \\ &= O\left(\int_M^\infty \exp(((\sigma_0 + \varepsilon)\cos\phi - \sigma\cos\phi + t\sin\phi)\omega)\, d\omega\right) \\ &= O\left(\frac{e^{Mt\sin\phi}}{t\sin\phi}\right) \\ &= O(|t|^{-1}) \qquad \text{as} \quad t \to -\infty. \end{aligned}$$

Also, we have

$$\begin{aligned} g_2(s, \phi) &= iM \int_0^\phi G(e^{i\theta}M) \exp(-se^{i\theta}M) e^{i\theta}\, d\theta \\ &= O\left(\int_0^\phi e^{Mt\sin\theta}\, d\theta\right) = O\left(\int_0^\phi e^{2Mt\theta/\pi}\, d\theta\right) \\ &= O(|t|^{-1}) \qquad \text{as} \quad t \to -\infty. \end{aligned}$$

Therefore, using the symmetry of the situation, we have

$$W(s)s^{-1} = O(|t|^{-1}) \tag{8}$$

uniformly for $|t| \geq T_0(a)$, $a \leq \sigma \leq b$.

From (8) and the absolute convergence of (4) we have

$$Y(s) = O(|t|^{\kappa+1}) \quad \text{and} \quad Z(s) = O(|t|^{\kappa+1})$$

uniformly for $|t| \geq T_0(a)$, $a \leq \sigma \leq b$ where $\lambda < a$.

This completes the proof of Theorem 1.

2. The Carlson ν function

In this section we assume the integral defining $Z(s)$ has an abscissa of absolute convergence α_1 and P is as in the previous section.

Definition. Let $\sigma > P$, $0 < \rho \leq 1$. Then the Carlson function, $\nu_\rho(\sigma)$ is defined by

$$\nu_\rho(\sigma, Z) = \nu_\rho(\sigma) = \inf\left\{\xi > 0 \mid \left(\frac{1}{T}\int_{-T}^{*T} |Z(\sigma + it)|^{1/\rho}\, dt\right)^\rho = O(T^\xi)\right\}$$

where $\int_{-T}^{*T}$ means $\int_{-T}^{T_0} + \int_{T_0}^{T}$ with T_0 sufficiently large. Also we let $\nu_0(\sigma) = \lim_{\rho\to 0+} \nu_\rho(\sigma)$.

This definition differs from that given in [7] only in that $\nu_\rho(\sigma)$ is by definition nonnegative here. The following theorem is proved in exactly the same way as that for ordinary Dirichlet series which is found in [7].

Theorem 2. *On the region of the (ρ, σ) plane defined by $\sigma > P$, $0 \leq \rho \leq 1$,*

$\nu_\rho(\sigma)$ is a continuous function, a convex function, a decreasing function of σ, a decreasing function of ρ, equal to 0 for $\sigma > \alpha_1$, and $\nu_\rho(\sigma) + \rho$ is an increasing function ρ.

Now, as in [7] it can be shown that $\nu_0(\sigma)$ is the maximum of the well-known Lindelöf μ function and 0.

From [8], $\nu_\rho(\sigma, W) = 0$. Thus, by Minkowski's Inequality, for $0 < \rho \leq 1$,

$$\nu_\rho(\sigma, Z) \leq \text{Max}\,(\nu_\rho(\sigma, Y), \nu_\rho(\sigma, W)) \leq \nu_\rho(\sigma, Y)$$

and

$$\nu_\rho(\sigma, Y) \leq \text{Max}\,(\nu_\rho(\sigma, Z), \nu_\rho(\sigma, W)) \leq \nu_\rho(\sigma, Z).$$

Therefore, $\nu_\rho(\sigma, Z) = \nu_\rho(\sigma, Y)$ for $\sigma > P$, $0 \leq \rho \leq 1$.

DEFINITION. For $0 \leq \rho \leq 1$, let

$$\gamma_\rho(Y) = \inf\{\sigma > \lambda \mid \nu_\rho(\sigma, Y) = 0\}$$

$$\gamma_\rho(Z) = \inf\{\sigma > P \mid \nu_\rho(\sigma, Z) = 0\}.$$

It is clear that $\lim_{\lambda \to P} \gamma_\rho(Y_\lambda) = \gamma_\rho(Z)$. The following two theorems are proved in [7].

THEOREM 3. *For $0 \leq \rho \leq 1$, γ_ρ is a continuous, convex and decreasing function, and for $0 \leq \rho_1 \leq \rho_2 \leq 1$, $\sigma > \gamma_{\rho_2}$,*

$$\nu_{\rho_1}(\sigma) \leq \rho_2 - \rho_1.$$

THEOREM 4. *For $0 \leq \rho \leq 1$, $P < \sigma_1 < \sigma < \gamma_\rho$,*

$$\nu_\rho(\sigma) \leq \frac{\gamma_\rho - \sigma}{\gamma_\rho - \sigma_1}\,\nu_\rho(\sigma_1).$$

These elementary results show, among other things, that the behavior of Y and Z on vertical lines is the same. Furthermore, if $Z(s)$ is a Dirichlet series it is possible to show that some very strong results of Richert concerning $\nu_\rho(\sigma, Z)$ are now valid in the region $\sigma > P$ whereas they were previously known only in the half plane where $Z(s)$ is analytic except for finitely many poles. The proofs of these theorems rest on the extension of the theory of strong Riesz summability to the associated $Y(s)$.

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