SINGULARLY FIBERED MANIFOLDS[1]

BY

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1. Introduction

Singular fiberings were first introduced by Seifert [8] in the fibering of 3-manifolds by 1-spheres. Singularities arose by the twisting of neighboring fibers around a center fiber. Montgomery and Samelson [7], Conner and Dyer [2], and Mahowald [6] have treated singularities caused by pinching a fiber to a point. Hu [4] defined a singularity by the breakdown of the covering homotopy property. This paper concerns singularities which generalize those of Seifert. Our singular fibers are homeomorphic to the non-singular ones and have a neighborhood which is a topological, but not a fiber, product of the fiber and a cell. We define the notion of singularity and classify singularities for a torus fiber. We are concerned mainly with the fibering of an $(n + 1)$-manifold by $(n - 1)$-tori over a closed surface. In this case we extend some results of Brody [1].

2. n-tori

An *n-torus*, T^n, is the topological product of n circles. Let

$$C^n = I_1 \times \cdots \times I_n$$

where $I_i = [0, 2\pi]$ is on the i-th axis of E^n. Let $\eta : C^n \to T^n$ be the identification of opposite faces of C^n. $\eta(C^n)$ will be the standard representation of T^n. Homology classes of η-images of k-dimensional faces of C^n form a set of generators of $H_k(T^n)$. We denote the 1-dimensional generators obtained from the oriented I_i as $S_1, S_2, \cdots, S_n$. Generators of $H_k(T^n)$ are carried by the Cartesian product of k carriers of the S_i. Since $H_*(T^n)$ is torsion-free, $H^*(T^n)$ is the exterior algebra on n 1-dimensional generators which we take as the duals S_i^* of S_i. Generators of $H^k(T^n)$ will be cup products of k distinct S_i^*. Defining

$$(S_i^* S_j^* \cdots S_k^*)^* = S_i \times S_j \times \cdots \times S_k$$

gives a specified orientation for generators of $H_k(T^n)$. Poincaré duality

$$\lambda : H_p(T^n) \to H^{n-p}(T^n)$$

is given by

$$\lambda(S_i \times S_j \times \cdots \times S_k) = S_l^* S_m^* \cdots S_q^*$$

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where $(S_i^* S_j^* \cdots S_k^*)(S_l^* S_m^* \cdots S_q^*) = S_1^* S_2^* \cdots S_n^*$. The intersection product

$$H_p(T^n) \times H_q(T^n) \to H_{p+q-n}(T^n)$$

is thus easily computed.

C^n induces a coordinate system of n-tuples of reals modulo 2π on T^n. A homeomorphism $h : T^n \to T^n$ is called *linear* if it is expressible as an $n \times n$ unimodular integral matrix using these coordinates. We write $h = | a_{ij} |$ and consider all matrices acting on the left. A *solid n-torus*, sT^{n+1}, is obtained from T^n by replacing the first circle by a disk. For polar coordinates (ρ, α_1) on a disk, a homeomorphism $h_s : sT \to sT$ is linear if h_s restricted to non-zero constant ρ is linear.

Linear $h = | a_{ij} |$ induces $h_1 : H_1(T^n) \to H_1(T^n)$ which, in terms of the S_i basis, is also given by $| a_{ij} |$. Thus

$$h^1 : H^1(T^n) \to H^1(T^n)$$

is given by $| a_{ij} |^t$. By computing the cup product and dualizing we see that

$$h_p : H_p(T^n) \to H_p(T^n)$$

is given by

$$h_p(S_i \times S_j \times \cdots \times S_k) = \sum D(ij \cdots k; rs \cdots t) S_r \times S_s \times \cdots \times S_t$$

where $D(ij \cdots k; rs \cdots t)$ is the determinant of the $p \times p$ submatrix of $| a_{ij} |$ formed by the rows $r, s, \cdots, t$ and the columns $i, j, \cdots, k$. The sum is to be taken over all p-dimensional generators. Then choosing a basis

$$A_i = (-1)^{i+1} S_1 \times \cdots \times \hat{S}_i \times \cdots \times S_n$$

for $H_{n-1}(T^n)$, we have $h_{n-1} = (\det | a_{ij} |) | a_{ij} |^{t,-1}$ where $t, -1$ indicates transpose inverse.

LEMMA 2.1. *A homeomorphism $f : T \to T$ can be extended to a map*

$$f_s : sT \to sT$$

iff $f_{n-1}(\ker i_{n-1}) = \ker i_{n-1}$ where $i : T \to sT$ is inclusion.

Proof. Since $f_{s*} i_* = i_* f_*$ and f_* is an isomorphism, the condition is necessary. Conversely, suppose $f_1 = | a_{ij} |$ and the condition is satisfied. Then since $\ker i_{n-1}$ is generated by A_i, $i \geq 2$, we have $a_{i1} = 0$, $i \geq 2$ and $a_{11} = \pm 1$. Define the homeomorphism g as $g = | a_{ij} |$. Then $g_1 = f_1$ and hence $f \simeq g$, f homotopic to g, (see for example, Hu [5, p. 198]). Now define f_s by deforming f to g on $1 \geq \rho \geq \frac{1}{2}$ and by

$$f_s(\rho, \alpha_1, \alpha_2, \cdots, \alpha_n) = (\rho, g(\alpha_1, \alpha_2, \cdots, \alpha_n)) \quad \text{for } \tfrac{1}{2} \geq \rho > 0.$$

For $\rho \leq \frac{1}{2}$, $f_s(\rho, \alpha_1, \cdots, \alpha_n)$ has i-th coordinate independent of α_1, so f_s is uniquely defined and continuous at $\rho = 0$.

3. Singular fibers

Let F and Y be topological manifolds and I, D, and S be the closed unit interval, a closed cell, and a sphere. Given a homeomorphism

$$f : Y \to S^q \times F^r,$$

define $[Y]_f$ to be the product fiber space with projection pf where

$$p : S \times F \to F$$

is the natural projection. $[Y]_f$ has fiber S and base F. Consider $I \times S^q \times F^r$ with $0 \times S \times F$ viewed as $[S \times F]_f$ for a given $f : S \times F \to S \times F$. Let $\eta : I \times S \times F \to C[f]$ be the identification map which pinches the S-fibers of $[S \times F]_f$ to points. Since $C[f]$ is homeomorphic to $\eta(1 \times f^{-1})(I \times S \times F)$, one can easily verify the following where we consider 0 as the center of D and $\eta(0 \times S \times F) = 0 \times F$.

LEMMA 3.1. *$C[f]$ is homeomorphic to $D^{q+1} \times F^r$ and has an induced partitioning into sets homeomorphic to F such that the decomposition space is D^{q+1} and $(D - 0) \times F$ is a product fiber space.*

We call $C[f]$ a *singularly fibered core of type f.* The *singular* or *center* fiber is $0 \times F$. All other fibers are *non-singular.* If f is the identity, then $C[f]$ is the product fiber space and the center fiber will also be non-singular.

If we choose $S^q = S^1$, $F^r = T^{n-1}$ and a linear $f : S \times T \to S \times T$ given by

$$f(\alpha_1, \alpha_2, \cdots, \alpha_n) = (y\alpha_1 + d\alpha_2, -x\alpha_1 + c\alpha_2, \alpha_3, \cdots, \alpha_n)$$

where c, d, x, and y are integers with $cy + dx = 1$, $c \neq 0$, then the resulting $C[f]$ is a *singularly fibered solid torus*, $sT^{n+1}(d/c)$. By 3.1 it is topologically a solid n-torus and is singularly fibered by $n - 1$-tori. For $n = 2$, the sT^3 are the fibered solid tori of Seifert [8].

Geometrically $sT^3(d/c)$ is the cylinder $D^2 \times I$ with the upper face rotated $2\pi\, d/c$ radians before being identified with the lower face. A similar geometric interpretation is possible for sT^{n+1}. Let $v_2/u_2, \cdots, v_n/u_n$ be pairs of relatively prime integers with $u_i \neq 0$. Consider $D^2 \times I^{n-1}$ embedded in E^{n+1} with D^2 the unit disk in the x_0, x_1-plane and $I_i = [0, 2\pi]$ in the x_i-axis, $i \geq 2$. Now rotate D^2 by $2\pi v_i/u_i$ about the x_i-axis before identifying the faces defined by $x_i = 0$ and $x_i = 2\pi$. Let $c = \text{lcm}\,(u_2, \cdots, u_n)$. Then this process of rotation and identification forms a topological $n - 1$ torus from $0 \times I^{n-1}$ and a topological $n - 1$ torus from exactly c of the $x \times I^{n-1} \subset D^2 \times I^{n-1}$. We denote this fibered solid torus by $sT(v_i/u_i)$. Note if $c = 1$, we have the product fiber space.

Choose coordinates $0 \leq \rho \leq 1$, $\alpha_i (\text{mod } 2\pi)$, on $sT(v_i/u_i)$ so that $(\rho, \alpha_1, \cdots, \alpha_n)$ is on the singular fiber iff $\rho = 0$, and $(\rho, \alpha_1, \cdots, \alpha_n)$ and $(\rho', \alpha_1', \cdots, \alpha_n')$ are on the same non-singular fiber iff $\rho = \rho'$ and $c\alpha_1 - \sum v_i c\alpha_i/u_i$ is congruent modulo 2π to $c\alpha_1' - \sum v_i c\alpha_i'/u_i$. We call such a system of coordinates *planar*. We assume all coordinates henceforth are planar. The subset $T(\rho)$ of sT defined by ρ equals a non-zero constant is a topological n-torus fibered as the product $S^1 \times T^{n-1}$. Then there exists an identification map η from $C^n \subset E^n$ to $T(\rho)$ given by

$$\eta(x_1, x_2, \cdots, x_n) = (\rho, \alpha_1, \cdots, \alpha_n)$$

where $\alpha_i = x_i(\text{mod } 2\pi)$. The η-preimage of a T^{n-1} fiber of $T(\rho)$ is a finite number of parallel equally spaced hyperplane sections in C^n.

In the following, F denotes both a fiber and its corresponding homology class and $\sim$ denotes "homologous to".

PROPOSITION 3.2. *Let F be a fiber on the boundary T^n of $sT^{n+1}(v_i/u_i)$. Then*

$$\pm F \sim cA_1 - v_2\, c/u_2\, A_2 - \cdots - v_n\, c/u_n\, A_n$$

on T^n.

Proof. We can assume $(1, 0, \cdots, 0) \in F$. Triangulate T^n via a triangulation of C^n which induces a triangulation on each planar section of F in C^n. Each of these sections separates C^n into two components. The chain sum of the closures of all simplices in the components which contain the point $(1, 0, 2\pi, \cdots, 2\pi)$ of C^n gives a chain on T^n with the desired boundary. Details of this construction and computation may be found in [9, pp. 45–50].

COROLLARY 3.3. *Let F be a non-singular and F_0 the singular fiber of $sT^{n+1}(v_i/u_i)$. Then $\pm F \sim cF_0$ in sT*

Proof. F is homologous in sT to a fiber on the boundary. But $A_i \sim 0$ in sT for $i \geq 2$ and $F_0 \sim A_1$ in sT, so the result follows from 3.2.

COROLLARY 3.4. *A fiber preserving homeomorphism of non-trivially fibered tori takes the singular fiber onto the singular fiber.*

Proof. Suppose $h : sT \to sT'$. Let F_0, F_0' be the singular fibers and suppose $h(F_0) = F' \neq F_0'$ and $h(F_1) = F_0'$. $F_1 \sim cF_0$ implies $h(F_1) \sim ch(F_0)$ so $h(F_1) \sim cc'F_0'$. Thus $F_0' \sim cc'F_0'$ or $cc' = 1$. Therefore, $c = c' = 1$ and both tori are trivially fibered.

Seifert shows, [8, pp. 150–154], that $sT^3(v/u)$ and $sT^3(v'/u')$ are fiber preserving homeomorphic iff $|u| = |u'|$ and v and v' reduced mod $|u|$ to the interval $[-|u|/2, |u|/2]$ have the same absolute value. For $n \geq 3$, the corresponding result is as follows.

THEOREM 3.5. For $n \geq 3$, *$sT^{n+1}(v_i/u_i)$ is fiber preserving homeomorphic to $sT^{n+1}(1/c)$ iff $|c| = \text{lcm}(u_2, \cdots, u_n)$.*

Proof. If $|c| = \text{lcm}(u_2, \cdots, u_n)$ then a linear fiber preserving homeomorphism may be constructed taking $sT^{n+1}(v_i/u_i)$ onto $sT^{n+1}(1/c)$. See [9, pp.

59–63]. Conversely suppose

$$h : sT(v_i/u_i) \to sT'(1/c')$$

is given. Let $c = \operatorname{lcm}(u_2, \cdots, u_n)$. Then $F \sim cA_1 - \sum v_i\, c/u_i\, A_i$, so

$$h(F) \sim ch(A_1) - \sum v_i\, c/u_i\, h(A_i).$$

Now $h(A_1) \sim a_{11} A_1' + \sum a_{ij} A_j'$ with $a_{11} \neq 0$ since $h(A_1)$ does not bound on sT'. For $i \geq 2$, $A_i \sim 0$ on sT hence $h(A_i) \sim 0$ on sT'. Therefore, $h(A_i) \sim \sum a_{ij} A_j'$ on T' for $i, j \geq 2$. The $h(A_i)$ are a basis for $H_{n-1}(T')$ so $\det | a_{ij} | = \pm 1$ and $a_{11} = \pm 1$. By 3.4, $h(F) \sim F' \sim \pm c'A_1'$ so

$$\pm c'A_1' \sim c(a_{11} A_1' + \sum a_{ij} A_j') + \sum v_i\, c/u_i (\sum a_{ij} A_j').$$

The A_j' are homologously independent, so we may equate coefficients to obtain $\pm c' = ca_{11}$. Thus $c = | c' |$.

We define $sT(v_i/u_i)$ to be of *singularity type* $c > 0$ iff it is fiber preserving homeomorphic to $sT(1/c)$. The fibering is singular in the sense that if $c \neq 1$, $sT(1/c)$ does not have the covering homotopy property. It is easy to show that a homotopy of a homeomorphism onto the singular fiber followed by the projection cannot be lifted.

4. Singularly fibered manifolds

A *singularly fibered* $(q + r + 1)$*-manifold* consists of a topological $(q + r + 1)$-manifold M, a connected $q + 1$-manifold B, and a map $p : M \to B$ such that $p^{-1}(b)$ is homeomorphic to a space F^r for all $b \in B$ and such that each $b \in B$ has a closed $(q + 1)$-cell neighborhood $\bar{V}$ with $p^{-1}(\bar{V})$ fiber preserving homeomorphic to some $C[f]$. Suppose $S^q = S^1$, $F^r = T^{n-1}$ and f is linear. Then a singularly fibered M^{n+1} is partitioned into fibers $F = T^{n-1}$ so that each F has a closed neighborhood fiber preserving homeomorphic to some $sT^{n+1}(1/c)$. We call such a neighborhood, $N(F)$, a *solid fiber neighborhood.* The projection of $N(F)$ in B is a *base fiber neighborhood.* Here B is a closed surface.

Seifert's lemmas 1 through 4 and Theorem 1 can be easily shown to hold for fibered M^{n+1}. In particular, if $N(F)$ and $N'(F)$ are two solid fiber neighborhoods of F, then there is a fiber preserving isotopy of M taking $N(F)$ onto $N'(F)$, leaving F pointwise fixed and which is the identity outside a fixed open set containing $N(F) \cup N'(F)$.

Let $N(F)$ be a solid fiber neighborhood of F in M. $M'' = M - \operatorname{int} N(F)$ is a manifold with an n-torus boundary T''. We say F has been *excised* from M to obtain M''. Given any singularly fibered sT with boundary T and a fiber preserving homeomorphism $f : T \to T''$, let M_1 be M'' with sT adjoined by f. We say M'' has been *completed* to M_1. Hereafter a double prime shall always refer to a manifold with boundary.

PROPOSITION 4.1. *The result of excising F from M is independent of the $N(F)$ used.*

Proof. Let $M_i'' = M - \operatorname{int} N_i(F)$, $i = 1, 2$. $N_i(F) \cap N_2(F)$ contains a $N_3(F)$. Then there is a fiber preserving isotopy taking $N_i(F)$ onto $N_3(F)$.

PROPOSITION 4.2. *M'' obtained from M by excising a non-singular fiber is independent of the fiber excised.*

Proof. Given non-singular F_1 and F_2 over b_1 and b_2, let τ be an arc in B from b_1 to b_2 which does not contain singular points; i.e. points b where $p^{-1}(b)$ is a singular fiber. Let σ be a closed disk in B containing τ in its interior and without singular points. Then $p^{-1}(\sigma)$ is a solid fiber neighborhood of both F_1 and F_2.

Let F'' be a fiber on the boundary T'' of M''. A subgroup $G \subset H_{n-1}(T'')$ is *allowable* if G is of rank $n - 1$, the homology class of F'', $[F'']$, is not in G and there exists an $(n - 1)$-torus $P'' \subset T''$ such that $[P'']$ and G generate $H_{n-1}(T'')$.

LEMMA 4.3. *If G is allowable, then there is exactly one type of singularly fibered sT^{n+1} such that there exists a fiber preserving homeomorphism $h : T \to T''$ with $h_*(\ker i_*) = G$ where $i : T \to sT$ is inclusion of the boundary.*

Proof. Let F'', P'' also denote the homology classes of F'' and P'', etc. Suppose P'', Q_2'', $\cdots$, Q_n'' is a basis for $H_{n-1}(T'')$ where Q_2'', $\cdots$, Q_n'' is a basis for G and suppose $F'' \sim b_{11} P'' + \sum b_{i1} Q_i''$. Since $F'' \notin G$, $b_{11} \neq 0$. Note that $| b_{11} |$ is independent of the Q_i'' basis and the choice of P'', so $| b_{11} |$ is uniquely determined by G and the homology class of a fiber on T''. Let D_2'', $\cdots$, D_n'' be a collection of $(n - 1)$-tori on T'' such that F'', D_2'', $\cdots$, D_n'' are a basis for $H_{n-1}(T'')$. Suppose $D_j'' \sim b_{1j} P'' + \sum b_{ij} Q_i''$ for $j = 2, \cdots, n$. Then $| b_{ij} |$ is an $n \times n$ unimodular matrix.

Consider sT^{n+1} fibered by $v_i/u_i = -b_{i1}/b_{11}$ for $i = 2, \cdots, n$. Then on the boundary T,

$$F \sim b_{11} A_1 + b_{21} A_2 + \cdots + b_{n1} A_n .$$

Choose $(n - 1)$-tori D_j on T for $j = 2, \cdots, n$ such that

$$D_j \sim b_{1j} A_1 + b_{2j} A_2 + \cdots + b_{nj} A_n .$$

A choice of planar coordinates on T'' defines A_i''. Suppose

$$P'' \sim a_{11} A_1'' + \cdots + a_{n1} A_n'' \quad \text{and} \quad Q_j'' \sim a_{1j} A_1'' + \cdots + a_{nj} A_n'' .$$

Then $| a_{ij} |$ is unimodular and we define $h = | a_{ij} | : T \to T''$. Computation shows $h(F) = F''$, $h(D_i) = D_i''$, $h(A_1) = P''$ and $h(A_j) = Q_j''$ for $i, j \geq 2$. Thus $h_*(\ker i_*) = G$.

Suppose sT' is fibered by $v_i/u_i = -c_{i1}/c_{11}$ has a fiber preserving homeomorphism $h' : T' \to T''$ with $h'_*(\ker i'_*) = G$. Then

$$F' \sim c_{11} A'_1 + \cdots + c_{n1} A'_n$$

and

$$h'(F') \sim F'' \sim c_{11} h'(A'_1) + c_{21} h'(A'_2) + \cdots + c_{n1} h'(A'_n).$$

Choosing $P'' = h'(A'_1)$ and $Q''_i = h'(A'_i)$, $i \geq 2$, we have

$$F'' \sim c_{11} P'' + \sum c_{i1} Q''_i.$$

Hence $| b_{11} | = | c_{11} |$ so sT' is fiber preserving homeomorphic to the constructed T.

The converse of 4.3 is not true. There exists an infinite number of distinct allowable groups G_r, $r \geq 3$, which determine $sT(1/c)$. For example, if F'', $D''_2, \cdots, D''_n$ generate $H_{n-1}(T'')$, then G_r generated by $-F'' + cD''_2$, $rD''_2 + D''_3$, $D''_4, \cdots, D''_n$ is allowable and determines $sT(1/c)$. Note that $G_r = G_s$ iff $r = s$.

Let G be allowable and $h : T \to T''$ be a linear fiber preserving homeomorphism with $h_*(\ker i_*) = G$. The *completion of M'' along G*, $M(G)$ is $M'' \cup sT$ with $h(x)$ identified with x for $x \in T$, the boundary of sT.

PROPOSITION 4.4. *Given planar coordinates on T'', $M(G)$ is uniquely determined by G and M''.*

Proof. Let $g : T' \to T''$ be another linear homeomorphism with $g_*(\ker i_*) = G$ which gives $M'(G)$. Then $(h^{-1}g)_* : \ker i_* \to \ker i_*$ so $h^{-1}g$ can be extended to a linear homeomorphism $f_s : sT' \to sT$. Then $f : M'(G) \to M(G)$ defined by $f \mid M'' =$ identity and $f \mid sT' = f_s$ is a fiber preserving homeomorphism.

A simple closed curve $\omega \subset T^n$ is *linear* with respect to coordinates $\alpha_1, \alpha_2, \cdots, \alpha_n$ if ω is the image of a line in E^n under the map $x_i \to x_i (\bmod 2\pi)$. Two linear curves are *parallel* if they are images of parallel lines in E^n. Let the boundary T'' of M'' be fibered by linear simple closed curves parallel to a given ω and let η be the natural map defined on M'' which identifies these S^1-fibers to points. Then $\eta(M'') = M(\omega)$ is the *completion of M'' along ω*.

THEOREM 4.5. *For each fixed planar coordinate system on T'' there exists a bijective correspondence between linear ω with intersection product of the homology class of ω and the homology class of a fiber $F'' \subset T''$ not zero, i.e., $[\omega]\cdot[F''] \neq 0$, and allowable subgroups G. If ω and G correspond, then $M(\omega)$ is fiber preserving homeomorphic with $M(G)$ where G determines the completion by adding $sT(1/c)$ with $c = [\omega]\cdot[F'']$.*

Proof. Suppose $\omega \simeq a_{11} S''_1 + \cdots + a_{n1} S''_n$ and choose $| a_{ij} |$ to be integral unimodular. Let $| b_{ij} | = | a_{ij} |^{t,-1}$ and choose $n - 1$-tori Q''_j on T'' so that

$$Q''_j = b_{1j} A''_1 + \cdots + b_{nj} A''_n, \qquad j \geq 2.$$

Then $Q''_2, \cdots, Q''_n$ generate a subgroup G. G is independent of the choice of $| a_{ij} |$ as long as the first column is $a_{11}, a_{21}, \cdots, a_{n1}$. The assumption $[\omega]\cdot[F''] \neq 0$ insures $[F''] \notin G$. Hence G is a uniquely determined allowable subgroup.

Conversely, given G generated by $Q''_j \sim b_{1j} A''_1 + \cdots + b_{nj} A''_n$ for $j \geq 2$, choose

$$P'' \sim b_{11} A''_1 + \cdots + b_{n1} A''_n$$

so that P'' and G generate $H_{n-1}(T'')$. Then define $| a_{ij} | = |b_{ij}|^{t,-1}$ and $\omega \simeq a_{11} S''_1 + \cdots + a_{n1} S''_n$. F'' not in G insures $[\omega]\cdot[F''] \neq 0$. Computation shows ω is independent of the basis chosen for G.

To define fiber preserving homeomorphism of $M(\omega)$ onto $M(G)$ we first extend the planar coordinates on T'' to be a neighborhood $[1, 3] \times T'' \subset M''$. Suppose $| b_{ij} |$ expresses P'', Q''_j in terms of the A''_i. Define $f'' : M'' \to M'' + sT$ by

$$f'' \mid M'' - [1, 3] \times T'' = \text{identity},$$

$f'' : [2, 3] \times T'' \to [1, 3] \times T''$ given by

$$(\rho, \alpha_1, \alpha_2, \cdots, \alpha_n) \to (2\rho - 3, \alpha_1, \cdots, \alpha_n)$$

and $f'' : [1, 2] \times T'' \to sT$ given by

$$(\rho, \alpha_1, \cdots, \alpha_n) \to (\rho - 1, | b |^t (\alpha_1, \cdots, \alpha_n)^t).$$

Let $h : T \to T''$ given by $h = | b_{ij} |^{t,-1} = | a_{ij} |$ be used to adjoin sT along G to M''. Then f'' induces a fiber preserving homeomorphism

$$f : M(\omega) \to M(G).$$

Suppose $F \sim c_1 A_1 + \cdots + c_n A_n$ on T. Since $h^{-1}(\omega) \simeq \pm S_1 \subset T$, we have

$$c = [\omega]\cdot[F''] = [h^{-1}(\omega)]\cdot[F] = [\pm S_1]\cdot[c_1 A_1 + \cdots + c_n A_n] = \pm c_1 .$$

Thus sT is of type c.

5. Manifold classes and fundamental groups

Let τ be a closed path in B based at b_0 which contains no singular points. Since M minus all singular fibers has the covering homotopy property, $F_0 = p^{-1}(b_0)$ can be isotopically deformed onto itself with projection image τ in B. This we call a *translation* of F_0 over τ. Any two translations of F_0 over τ give isotopic homeomorphisms of F_0. Hence the induced isomorphism $\chi(\tau) : H_1(F_0) \to H_1(F_0)$ is well defined. $\chi(\tau)$ depends only on the homotopy class of τ and has the property $\chi(\tau'\tau) = \chi(\tau')\chi(\tau)$ for the path τ followed by τ'. For fixed coordinates on F_0, $\chi : \Pi_1(B, b_0) \to \mathrm{Gl}\,(n - 1, Z)$ is a homomorphism. Change of coordinates on F_0 or basepoint in B alters χ by an inner automorphism. Hence M determines a unique inner automorphism class of $\chi : \Pi_1(B) \to \mathrm{Gl}\,(n - 1, Z)$. The *characteristic* of M is defined to be this class of equivalent integral representations of $\Pi_1(B)$ of degree $n - 1$.

We consider two manifolds to be in the same *class* if they have the same base and characteristic. The class, $\{M\}$, of M is independent of any singular fibers. Manifolds may be constructed with a pre-given base and characteristic. The construction follows the method of Seifert [8, p. 173–174] and is detailed in [9, p. 113–124]. Note that excision and completion does not alter the class of a manifold.

To classify singularly fibered $(n+1)$-manifolds we first classify the manifold classes and then seek the manifolds within each class. First we make the *conjecture* (n): Two homeomorphisms $T^n \to T^n$ are isotopic iff they are homotopic. Conjecture (1) is clearly true. Conjecture (2) is a special case of a theorem of R. Baer that homotopic homeomorphisms of any closed surface are isotopic. Since two homeomorphisms on T^n are homotopic iff the induced maps on $H_1(T^n)$ are the same, conjecture (n) would imply the space of homeomorphisms of T^n with a fixed coordinate system has exactly one linear homeomorphism in each path-component. Conjecture (n) would also imply the map f_s of Lemma 2.1 can be a homeomorphism.

THEOREM 5.1. *Suppose conjecture $(n-1)$ is true. Then there is a bijective correspondence between classes $\{M\}$ of closed singularly fibered $(n+1)$-manifolds and fibered $(n+1)$-manifolds M_0'' with an n-torus boundary T'' and without singular fibers. The corresponding M_0'' is the class space of $\{M\}$. The base B_0'' of M_0'' is the base B of M with an open disk removed. The characteristic of M_0'' is the same as the characteristic of M. Any member of $\{M\}$ may be obtained from M_0'' by excision and completion.*

Proof. Given M_0'' we may complete it to M_0 and obtain a unique class $\{M_0\}$. Conversely, let M be any member of $\{M\}$. By excising all singular fibers and completing with non-singular fibers we obtain a manifold M_0 of $\{M\}$. Excise one non-singular fiber to obtain M_0'' which has the proper base and characteristic. To show M_0'' is well defined, it suffices to show that if M_1 in $\{M\}$ contains no singular fibers and we excise a non-singular fiber to obtain M_1'', then M_1'' is fiber preserving homeomorphic to M_0''. Construction of this homeomorphism follows the method of Seifert [8, p. 171–173]. Both M_1'' and M_0'' are first viewed as $P'' \times T^{n-1}$, where P'' is a polygon which becomes $B_0'' = B_1''$ by identifications on its edges, and corresponding identifications are made on fibers over the edges of P''. Conjecture $(n-1)$ is then used to isotopically deform the identifications giving M_1'' to those giving M_0''. Details may be found in [9, p. 131–135].

COROLLARY 5.2. *Compact $(n+1)$-manifolds, fibered in the usual sense by $(n-1)$-tori, with n-torus boundaries, are in a bijective correspondence with closed surfaces B and equivalence classes of integral representations of $\Pi_1(B, b_0)$ of degree $n-1$.*

COROLLARY 5.3. $(S^2 - \operatorname{int} D^2) \times T^{n-1}$ *is the class space of the unique class of singularly fibered manifolds with base S^2.*

COROLLARY 5.4. *Manifolds of a single class are either all orientable or all non-orientable.*

There are only two classes of orientable 4-manifolds with base a sphere with one crosscap. If γ is a generator of $\Pi_1(B, b_0)$, then $\gamma^2 \simeq 0$ and $\chi(\gamma)^2 = C^2 = I$. Since the manifolds are orientable, $\det C = -1$. Then the two conjugate classes of χ are represented by

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

In cases other than this one and the case $B = S^2$, there are an infinite number of non-conjugate χ and hence an infinite number of distinct manifold classes.

Suppose $M \in \{M_0\}$ has $r \geq 0$ singular fibers $\{F_j\}$ of types $\{c_j\}$. To obtain M from the class space M_0'', we first excise r non-singular fibers to obtain M'' with $r + 1$ n-tori boundary components T_j''. For a proper choice of linear ω_j on each T_j'', the identification of the S^1 parallel to the ω_j on all T_j'' will produce M. A necessary condition on the ω_j is $[\omega_j] \cdot [F_j''] = c_j$ where F_j'' is a fiber on T_j''. To obtain other necessary conditions on the ω_j, we compute the fundamental group of M.

Cut the base B'' on M'' into a polygon P'' in such a way that the boundaries $J_j = p(T_j'')$ lie in the interior of P''. Let b be the vertex of the polygon and let the edges be A_i, B_i, $i = 1, \cdots, h$ if B is a sphere with h handles or C_i, $i = 1, \cdots, q$ if B is a sphere with q crosscaps. Let τ_j, $j = 0, 1, \cdots, r$ be paths from b to a point of J_j. Fix a coordinate system on $T^{n-1} = S_2 \times \cdots \times S_n$ and consider these coordinates on each $x \times T^{n-1}$ of $P'' \times T^{n-1}$. A choice of coordinates on each J_j gives planar coordinates on each $T_j'' = J_j \times T^{n-1}$. Then M'' is obtained from $P'' \times T^{n-1}$ by identifications

$$A_i \times T^{n-1} \equiv A_i^{-1} \times T^{n-1} \quad \text{and} \quad B_i \times T^{n-1} \equiv B_i^{-1} \times T^{n-1}, \quad i = 1, \cdots, h$$

or

$$C_i \times T^{n-1} \equiv C_i' \times T^{n-1}, \qquad i = 1, \cdots, q.$$

On each T_j'' there is a linear simple closed curve ω_j with the property that if T_j'' is fibered by one-spheres parallel to ω_j, then shrinking these to points is equivalent to replacing the original excised F_j. Note that it is necessary to choose ω_j to correspond not only to the singularity type of F_j, but also to the way in which F_j is embedded in M. Suppose the ω_j are specified by

$$\omega_j \simeq w_{1j} J_j + w_{2j} S_2 + \cdots + w_{nj} S_n .$$

To compute $\Pi_1(M)$ we first find $\Pi_1(P'' \times T^{n-1})$ and then $\Pi_1(M'')$. Let J_j also denote the path $\tau_j^{-1} J_j \tau_j$ in P''. The product space $P'' \times T^{n-1}$ has $\Pi_1(P'' \times T^{n-1})$ generated by $J_0, J_1, \cdots, J_r, S_2, \cdots, S_n$ with commutativity relations $S_k J_j = J_j S_k$ and $S_k S_l = S_l S_k$. The identification forming

M'' from $P'' \times T^{n-1}$ introduces new generators and relations. The new generators are $A_i, B_i, i = 1, \cdots, h$ or $C_i, i = 1, \cdots, q$.

$$B_h^{-1}A_h^{-1}B_h A_h \cdots B_1^{-1}A_1^{-1}B_1 A_1 = J_r \cdots J_0 \quad \text{or} \quad C_q^2 \cdots C_1^2 = J_r \cdots J_0$$

holds on B'' hence also in M''. Translation in M'' of the fiber F over b along the path A_i induces an isomorphism $\chi(A_i) : H_1(F) \to H_1(F)$ which we express in terms of the basis $S_2, \cdots, S_n$ as $\chi(A_i) = |a_{ij}|$. Let $\chi(A_i)S_k$ denote $a_{2k} S_2 + \cdots + a_{nk} S_n$ written multiplicatively. Then in translating F over A_i, $S_k \subset F$ is taken into $\chi(A_i)S_k$. Thus in M'' there is the relation

$$S_k = A_i^{-1}(\chi(A_i)S_k)A_i \quad \text{or} \quad A_i S_k A_i^{-1} = \chi(A_i)S_k .$$

Likewise we have the corresponding relations for the B_i (or C_i). Closing M'' to M creates no new generators, but each ω_j on T_j'' gives the relation $J_j^w S_2^s \cdots S_n^t = I$ where $w = w_{1j}$ and the exponent of S_k is w_{kj}. Thus $\Pi_1(M)$ can be given as follows.

THEOREM 5.5. *Let M be a singular fibered manifold with base a sphere with h handles (or q crosscaps) obtained from the class space with r fibers excised by shrinking*

$$\omega_j \simeq w_{1j} J_j + w_{2j} S_2 + \cdots + w_{nj} S_n$$

on T_j'' for $j = 0, 1, \cdots, r$. Then $\Pi_1(M)$ is given by

Generators:

$$\begin{aligned} &A_i, B_i, && i = 1, \cdots, h \\ &(\text{or} \quad C_i, && i = 1, \cdots, q) \\ &J_j, && j = 0, \cdots, r \\ &S_k, && k = 2, \cdots, n \end{aligned}$$

Relations:

$$B_h^{-1}A_h^{-1}B_h A_h \cdots B_1^{-1}A_1^{-1}B_1 A_1 = J_r \cdots J_0$$

$$(\text{or} \quad C_q^2 \cdots C_1^2 = J_r \cdots J_0)$$

$$S_k J_j = J_j S_k , \qquad S_k S_l = S_l S_k$$

$$A_i S_k A_i^{-1} = \chi(A_i)S_k , \qquad B_i S_k B_i^{-1} = \chi(B_i)S_k$$

$$(\text{or} \quad C_i S_k C_i^{-1} = \chi(C_i)S_k)$$

$$J_j^w S_2^s \cdots S_n^t = I \qquad \text{for all} \quad i, j, k, l$$

where $w = w_{1j}$ and the exponent of S_k is w_{kj}.

COROLLARY 5.6. *$\Pi_1(B)$ is a quotient of $\Pi_1(M)$.*

COROLLARY 5.7. *If $\Pi_1(M)$ is finite, B is the sphere or the projective plane.*

6. Classification in special cases

By making additional assumptions on B and χ, we may determine the homeomorphism and fiber homeomorphism classes in certain cases.

THEOREM 6.1. *Let M'' with boundary T'' and without singular fibers have an orientable base and trivial characteristic. Suppose*

$$\omega \simeq J_0 + a_2 S_2 + \cdots + a_n S_n \quad \textit{and} \quad \omega' \simeq J_0 + a_2' S_2 + \cdots + a_n' S_n$$

on T''. Then $M(\omega)$ is fiber preserving homeomorphic to $M(\omega')$ with the identity induced on the base iff

$$\gcd(a_2, \cdots, a_n) = \gcd(a_2', \cdots, a_n').$$

Proof. $M(\omega)$ homeomorphic to $M(\omega')$ implies the first homology groups are isomorphic. Computation of these groups using Theorem 5.5 with the commutativity relations added shows that $\gcd(a_2, \cdots, a_n) = \gcd(a_2', \cdots, a_n')$ is a necessary condition. Conversely, suppose the gcd's are equal to d. Let $|a_{ij}|$ and $|a_{ij}'|$ be $n \times n$ unimodular integral matrices with

$$|a_{ij}|(a_1, \cdots, a_n)^t = (0, 0, \cdots, 0, d) = |a_{ij}'|(a_1', \cdots, a_n')^t.$$

Then $|b_{ij}| = |a_{ij}'|^{-1}|a_{ij}|$ is such that

$$|b_{ij}|(a_1, \cdots, a_n)^t = (a_1', \cdots, a_n')^t.$$

Thus $h : M'' \to M''$ defined by $h(b \times t) = b \times |b_{ij}| t$ for $b \in B$ and $t \in T^{n-1}$ is a fiber preserving homeomorphism with $h(\omega) = \omega'$. Hence h induces a fiber preserving homeomorphism $M(\omega) \to M(\omega')$.

Consider all fibered M_0^4 without singular fibers, with trivial characteristic and orientable base. Then each M_0^4 arises from $M_0''^4 = B'' \times T^2$ by shrinking

$$\omega \simeq J_0 + a_2 S_2 + a_3 S_3$$

on the boundary. Let $d = \gcd(a_2, a_3)$ and suppose B is a sphere with $h \geq 0$ handles.

COROLLARY 6.2. *The integers d and h are a complete system of topological invariants for the spaces M_0^4. If conjecture $(n-1)$ is true, then d and h are a complete system of topological invariants for the spaces M_0^{n+1} with orientable base, trivial characteristic and no singular fibers.*

Note that for $n = 2$, this corollary gives Brody's result [1, p. 164] for Seifert's manifolds $(O, o; h \mid p)$. A similar result, allowing one singular fiber, is given by the following.

THEOREM 6.3. *Let M'' have base a sphere with two open disks removed and no singular fibers. Suppose*

$$\omega_0 \simeq J_0 + \sum a_i S_i, \qquad \omega_1 \simeq cJ_1 + \sum b_i S_i,$$

$$\omega_0' \simeq J_0 + \sum a_i' S_i, \quad \textit{and} \quad \omega_1' \cong cJ_1 + \sum b_i' S_i,$$

$|c| \geq 2$ give two fiberings of the two boundary components of M''. Then $M(\omega_0, \omega_1)$ is fiber preserving homeomorphic to $M(\omega_0', \omega_1')$ iff the matrices

$$A = \begin{pmatrix} 1 & a_2 \cdots a_n \\ -c & b_2 \cdots b_n \end{pmatrix} \quad \textit{and} \quad A' = \begin{pmatrix} 1 & a_2' \cdots a_n' \\ -c & b_2' \cdots b_n' \end{pmatrix}$$

are equivalent, i.e., there exist unimodular matrices X and Y so that $XAY = A'$.

Proof. Using Theorem 5.5 we see that $H_1(M(\omega_0, \omega_1))$ can be given by generators A_i, B_i, J_0, and S_j with $i = 1, \cdots, h, j = 2, \cdots, n$, and the relations $J_0 + \sum a_j S_j = 0$, $-cJ_0 + \sum b_j S_j = 0$. Fox [3, Theorem 3.6] shows the torsion coefficients of $H_1(M(\omega_0, \omega_1))$ are the invariant factors of the matrix A. Hence the two matrices must have the same invariant factors and thus are equivalent. To prove the converse, first note that A and A' are equivalent iff there exists an $n \times n$ unimodular matrix Z with integer entries and first column $(1, 0, \cdots, 0)^t$ such that $AZ = A'$. See [9, p. 154–157] for the construction of Z. The base B'' is an annulus which we coordinatize by (ρ, α_1), $1 \leq \rho \leq 2$, $0 \leq \alpha_1 < 2\pi$, with $(1, \alpha_1) \in J_0$, $(2, \alpha_1) \in J_1$. Then we define $h : M'' \to M''$ by

$$h(\rho, \alpha_1, \cdots, \alpha_n) = (\rho, Z^t(\alpha_1 \cdots \alpha_n)^t), \quad \text{for} \quad (\alpha_2, \cdots, \alpha_n) \in T^{n-1}.$$

Fibers are preserved since Z^t has first row $(1, 0, \cdots, 0)$. Computation shows $h(\omega_0) = \omega_0'$ and $h(\omega_1) = \omega_1'$. Thus h induces a fiber preserving homeomorphism $M(\omega_0, \omega_1) \to M(\omega_0', \omega_1')$.

This theorem is an extension to higher dimensions of Brody's theorem [1, Theorem 3.1] for the case $h = 0$.

References

1. E. J. Brody, *On the fibred spaces of Seifert*, Quart. J. Math. Oxford Ser. (2), vol. 13 (1962), pp. 161–171.
2. P. E. Conner and E. Dyer, *On the singular fiberings by spheres*, Michigan Math. J., vol. 6 (1959), pp. 303–311.
3. R. H. Fox, *Free differential calculus, II*, Ann. of Math. (2), vol. 59 (1954), pp. 196–210.
4. S. T. Hu, *On fiberings with singularities*, Michigan Math. J., vol. 6 (1959), pp. 131–149.
5. ———, *Homotopy theory*, New York, Academic Press, 1959.
6. M. Mahowald, *On singular fiberings*, Canad. J. Math., vol. 15 (1963), pp. 214–219.
7. D. Montgomery and H. Samelson, *Fiberings with singularities*, Duke Math. J., vol. 13 (1946), pp. 51–56.
8. H. Seifert, *Topologie Dreidimensionaler Gefaster Räume*, Acta Math., vol. 60 (1933), pp. 147–238.
9. M. C. Thornton, *Singularly Fibered Manifolds*, Ph.D. dissertation, University of Illinois, 1965.

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