MODIFYING INTERSECTIONS

BY

Robert Wells[1]

1. Introduction

Suppose that M, $N \subset X$ are compact differential manifolds such that

(1) $\partial M, \partial N \subset \partial X$

(2) $\partial M \cap \partial N = \emptyset$

with $\dim M = r$, $\dim N = s$, $\dim X = m$ and $n = r + s - m$. The object of this paper is to investigate the possibility that there is an isotopy $\lambda_t : N \to X$ modulo boundaries, such that λ_0 is the inclusion $N \subset X$ and $\lambda_1(N) \cap M = \emptyset$. In the case that M, N, X are c-connected with $c \geq n + 1$ and $r \geq 2n + 3$ the obstruction to finding such an isotopy will be seen to be an element $\alpha(N, M:X)$ of the n-stem. Then the main theorem is that when $s \geq 2n + 3$ also, $\alpha(N, M:X)$ is the only obstruction. That is,

Theorem. *If M, N, X as above are c-connected with*

(1) $c \geq n + 1$

(2) $r, s \geq 2n + 3$

then there is an isotopy $\lambda_t : N \to X$ modulo boundaries, such that λ_0 is the inclusion $N \subset X$ and $\lambda_1(N) \cap M = \emptyset$, if and only if $\alpha(N, M:X) = 0$.

That this obstruction will sometimes be non-zero may be seen from the fact that if $X = S^n \times S^k$, $M = S^n \times$ pt. and N is the graph of an essential map $S^n \to S^k$ then no such isotopy exists.

I would like to thank Professors R. Lashof, A. Liulevicius, M. Rothenberg, and M. Mahowald for many helpful remarks on this paper.

2. Definition of $\alpha(N, M:X)$

In order to define $\alpha(N, M:X)$ and to find some of its properties, simple propositions are necessary. We will make only the assumptions in the first sentence of this paper and introduce further connectivity and dimension assumptions where needed. If A is a submanifold of the Riemann manifold B, $\nu(A:B)$ will be the normal bundle of A in B and $\tau(A)$ will be the tangent bundle of A.

Proposition 1. *Let M be c-connected, $c \geq n$ and $r \geq 2n + 3$. Let U be open in X with $U \cap M \neq \emptyset$. Then there is an isotopy $\lambda_t : N \to X$ modulo boundaries such that λ_0 is the inclusion $N \subset X$ and*

(1) *λ_1 is transverse regular along M*

(2) *$(\lambda_1(N) \cap M) \subset U$.*

Received April 29, 1966.

[1] The author was supported by the U. S. Army Research Office, Durham.

Proof. It is well known that there is an isotopy $\psi_t : N \to X$ modulo boundaries of $N \subset X$ such that ψ_t is transverse regular along M. Since M is c-connected, we may contract $K = M \cap \psi_1(N)$ to a point in $U \cap M$. Since $r \geq 2n + 3$, it follows that there is an isotopy $g_t : K \to M$ of $K \subset M$ such that $g_1(K) \subset U \cap M$. Then an application of the covering homotopy theorem for isotopies completes the proof.

Definition of $\alpha(N, M{:}X)$. Suppose now that $N \subset X$ is transverse regular along M with $N \cap M \subset U$ where $U \cap M$ is diffeomorphic to R^r. If N is c-connected with $c \geq n + 1$, we may extend $N \cap M \subset N$ to a map $C : C(N \cap M) \to N$ of the cone $C(N \cap M)$ of $N \cap M$. Then $C^*\nu(N{:}X)$ has a unique framing (up to orientation). This framing defines a canonical framing of

$$\nu(N{:}X) \mid N \cap M = \nu(N \cap M{:}U \cap M)$$

by restriction since any two maps C are homotopic in N modulo $N \cap M$. Since $U \cap M$ is diffeomorphic to R^r, the canonical framing defines in the usual way an element $\alpha(N, M{:}X)$ of $\pi_r(S^{r-n})$. In view of Proposition 1, the element $\alpha(N, M{:}X)$ is defined whenever M and N are c-connected, $c \geq n + 1$ and $r \geq 2n + 3$.

The next question is whether $\alpha(N, M{:}X)$ is well defined; it is answered by the next proposition.

Proposition 2. *If M and N are c-connected, $c \geq n + 1, r \geq 2n + 3$ and U is such that $U \cap M$ is diffeomorphic to R^r and $\lambda_t, \lambda_t' : N \to X$ are isotopies modulo boundaries such that $\lambda_0 = \lambda_0' =$ inclusion $N \subset X$ and*
(1) *λ_1, λ_1' are transverse regular along M*
(2) *$\lambda_1(N) \cap M \subset U$ and $\lambda_1'(N) \cap M \subset U$*
then $\alpha(\lambda_1(N), M{:}X) = \alpha(\lambda_t'(N), M{:}X)$.

Proof. We may as well assume that $t \to \lambda_t$ and $t \to \lambda_t'$ are constant near 0 and 1, so we may glue them together at $t = 0$ and reparametrize to obtain an embedding

$$\Lambda : N \times I \to X \times I$$

transverse regular along $M \times I$ on $N \times [0, \varepsilon] \cup N \times [1 - \varepsilon, 1]$ for some $\varepsilon > 0$. We may replace Λ by another embedding Λ' agreeing with it on $N \times [0, \varepsilon] \cup N \times [1 - \varepsilon, 1]$ but transverse regular everywhere. Let J be the intersection $\Lambda'(N \times I) \cap M$ and ∂J its boundary. The ε-collar ∂J_ε is then

$$\lambda_1(N) \times [0, \varepsilon] \cup \lambda_1'(N) \times [1 - \varepsilon, 1],$$

which we may assume to be a subcomplex of J. Since M is c-connected, we obtain a homotopy g_t modulo ∂J_ε of $J \subset M \times I$ such that $g_1(J) \subset U \times I$. Since $r \geq 2n + 3$, we may as well replace g_t with an isotopy h_t modulo ∂J_ε such that $h_1(J) \subset U \times I$. An application of the covering homotopy theorem for isotopies extends this isotopy to an isotopy

$$H_t : N \times I \to X \times I \quad \text{modulo } N \times [0, \varepsilon] \cup N \times [1 - \varepsilon, 1]$$

such that H_t remains transverse regular along $M \times I$ and

$$L = H_1(N \times I) \cap (M \times I) \subset U \times I.$$

Let $\pi_1 : X \times I \to I$ be the projection; let $\pi = \pi_1 \mid L$ and let $\mathfrak{m}$ be the mapping cylinder of π. Using the maps

$$C : C(\lambda_1(N) \cap M) \to N \quad \text{and} \quad C' : C(\lambda_1'(N) \cap M) \to N$$

and the inclusion $L \subset H_1(N \times I)$, we obtain a map which extends to a map

$$\mathfrak{m} \xrightarrow{\Psi} H_1(N \times I)$$

since $H_1(N \times I)$ is c-connected with $c \geq n + 1$. Then

$$\psi^* \nu(H_1(N \times I):X \times I)$$

has a unique framing (up to orientation) which restricts to a framing of $\nu(L:U \cap M \times I)$. This framing in turn restricts to the canonical framings of

$$\nu(\lambda_1(N) \cap M:U \cap M) \quad \text{and} \quad \nu(\lambda_1'(N) \cap M:U \cap M).$$

In the usual way, the framing of $\nu(L:U \cap M \times I)$ then defines a homotopy from a representative of $\alpha(\lambda_1'(N), M:X)$. This fact proves Proposition 2.

COROLLARY 1. *If U and U' are open in X such that $U \cap M$ and $U' \cap M$ are diffeomorphic to R^r and λ_t, λ_t' are isotopies modulo boundaries of $N \subset X$ such that λ_1, λ_1' are transverse regular along M and*

$$\lambda_1(N) \cap M \subset U \quad \textit{and} \quad \lambda_1'(N) \cap M \subset U'$$

then

$$\alpha(\lambda_1(N), M:X) = \alpha(\lambda_1'(N), M:X).$$

Proof. Let $U = U_0, U_1, \cdots, U_k = U'$ be a chain of open sets such that $U_i \cap M$ is diffeomorphic to R^r and $U_i \cap U_{i+1} \cap M \neq \emptyset$. Choose successively isotopies $\lambda_t = \lambda_t^{(0)}, \lambda_t^{(1)}, \cdots, \lambda_t^{(k+1)} = \lambda_t'$ such that

$$\lambda_1^{(i)}(N) \subset U_{i-1} \cap U_i \cap M$$

for $i = 1, \cdots, k$. Then the corollary follows immediately since

$$\alpha(\lambda_1^{(i)}(N), M:X) = \alpha(\lambda_1^{(i+1)}(N), M:X)$$

for $i = 0, \cdots, k$ by the proposition.

Thus when M and N are c-connected with $c \geq n + 1$ and $r \geq 2n + 3$, the element $\alpha(N, M:X) = \alpha(\lambda_1(N), M:X)$, for λ_t as above, is well defined and we have

COROLLARY 2. *If $\alpha(N, M:X) \neq 0$ then there is no isotopy λ_t modulo boundaries of $N \subset X$ such that $\lambda_1(N) \cap M = \emptyset$.*

In other words, $\alpha(N, M:X)$ is some kind of obstruction to the existence of λ_t. Under suitable connectivity assumptions it is actually the only obstruction to the existence of λ_t.

THEOREM. *If X, M, N are c-connected and*
(1) $c \geq n + 1$
(2) $r, s \geq 2n + 3$
then there is an isotopy $\lambda_t : N \to X$ modulo boundaries, such that $\lambda_1(N) \cap M = \emptyset$ and λ_0 is the inclusion $N \subset X$, if and only if $\alpha(N, M:X) = 0$.

3. Spherical modifications

The main step in the proof of the theorem will be Proposition 3 below, which deals with isotopies and spherical modifications. We will regard a spherical modification as being a manifold L with two boundaries, which admits a Morse function with a single critical point, of some index $p + 1$, equal to 0 on one boundary and a positive constant on the other. Let K be the boundary on which the function is 0 and K' the other. We say L is a *spherical modification from K to K'.* Then there will be an embedding $f : S^p \to K$ of the p-sphere, which extends to an embedding $F : D^{p+1} \to L$ of the $(p + 1)$-disc, such that $K \cup F(D^{p+1})$ is a deformation retract of L. The bundle $\nu(f(S^p):K)$ has a canonical framing which is the restriction of the unique (up to orientation) framing of $\nu(F(D^{p+1}):L)$—it defines a unique frame-preserving bundle homotopy class of one-to-one bundle homomorphisms $f' : S^p \times \varepsilon^{q+1} \to \tau(K)$ covering f, where $p + q + 1 = n = \dim K$, ε is the trivial line bundle over a point and for any space A, the bundle $A \times \varepsilon^t$ has the framing induced by that of ε^t. We will call f a *sphere attaching map* and f' *the bundle attaching map over f.* It is known that f and f' determine the spherical modification L up to diffeomorphism. A spherical modification L will *preserve a framing*

$$\nu(K) \xrightarrow{\sigma} \varepsilon^t$$

of the stable normal bundle of K if and only if there is a framing

$$\nu(L) \xrightarrow{\Sigma} \varepsilon^t$$

such that $\Sigma \mid \nu(K) = \sigma$. A spherical modification L will *preserve a framing*

$$\tau(K) \times \varepsilon^t \xrightarrow{\sigma'} \varepsilon^{n+t}$$

of the stable tangent bundle of K if and only if there is a framing

$$\tau(L) \times \varepsilon^{t-1} \xrightarrow{\Sigma'} \varepsilon^{n+t}$$

such that $\Sigma' \circ \gamma = \sigma'$; here

$$\gamma : \tau(K) \times \varepsilon^t \to \tau(L) \times \varepsilon^{t-1}$$

is the bundle homotopy class of bundle maps defined by

$$\gamma(v, r_1, \cdots, r_t) = (v + r_1 \tilde{u}, r_2, \cdots, r_t)$$

where $v \in \tau(K) \subset \tau(L)$, $r_i \in R$, and $\tilde{u}$ is the unit inward normal field of L along K for some Riemann metric. It is known that if σ and σ' are dual framings of the stable and normal tangent bundles respectively (i.e., $\sigma \oplus \sigma'$ is the standard framing of $\tau(R^l)$ restricted to K for any embedding $K \subset R^l$) then L preserves σ if and only if L preserves σ'.

Finally, let $N \subset X$ be transverse regular along M, let L be a spherical modification of $K = N \cap M$ and let $\lambda_t : N \to X$ be an isotopy of $N \subset X$ modulo boundaries. Then λ_t defines a map $\Lambda : N \times I \to X \times I$. The quadruple (λ_t, N, M, X) will be said to *realize* L if and only if

(1) Λ is transverse regular along $M \times I$

(2) $L_1 = \Lambda(N \times I) \cap (M \times I)$ is diffeomorphic to L.

Notice that under the connectivity and dimension assumptions of the theorem we may assume $L_1 \subset U \times I$ and the normal bundle of L in $X \times I$ may be given a framing that extends the canonical framing of $\nu(K)$. Thus L preserves the canonical framing of $\nu(K)$. Suppose that

$$\Sigma : \nu(L)^{r-n} \to \varepsilon^{r-n}$$

is a framing extending the canonical framing $\sigma : \nu(K)^{r-n} \to \varepsilon^{r-n}$. Then the isotopy λ_t will be said to *realize* the *framed spherical modification* (L, Σ) if and only if the canonical framing of the normal bundle of

$$L_1 = \Lambda(N \times I) \cap (M \times I)$$

in $U \times I$ corresponds to the framing Σ under some diffeomorphism

$$U \times I \to R^r \times I.$$

Given these definitions, the theorem will follow immediately from the following proposition.

PROPOSITION 3. *If X, M, N are c-connected and $N \subset X$ is transverse regular along M and*

(1) $c \geq n + 1$

(2) $r, s \geq 2n + 3$

then any framed spherical modification preserving the canonical framing of $\nu(K)$ may be realized by (λ_t, N, M, X) where $\lambda_t : N \to X$ is an isotopy of $N \subset X$ modulo boundaries.

4. Capping the sphere attaching map

It will be convenient to introduce a rather special Riemann metric in the proof of Proposition 3, as permitted by the following lemma.

LEMMA 2. *Suppose that A is a submanifold of B and B is a submanifold of*

C with $a = \dim A < b = \dim B < c = \dim C$ and A, B, C compact. Suppose that there is a Riemann metric g on C and an open subset $D \subset C$ such that

(1) *A has no boundary outside of D*

(2) *$A - D$ is compact*

(3) *There is an $\varepsilon > 0$ such that $v \in \nu(D \cap A:B)$ and $|v| = 1$ implies that $\exp tv \in B$ for $0 \leq t \leq \varepsilon$. Then, if $F \subset U$ is closed and contains ∂A, there is a Riemann metric g' on C which agrees with g on a neighborhood of F and which has the property that $v \in \nu'(A:B)$ and $|v|' = 1$ implies that $\exp'(tv) \in B$ for $0 \leq t \leq \varepsilon$.*

Here ν', $\exp'$ and $|\ \ |'$ refer to normal bundles, exponential maps and norms with respect to g'. The proof is a straightforward exercise with Riemann metrics and the covering isotopy theorem, and will be omitted.

With this lemma available, we begin the proof of Proposition 3. Let

$$(U, x_1, \cdots, x_m)$$

be a coordinate system such that $(U, U \cap M) \xrightarrow{x} (R^m, R^r)$ is a diffeomorphism. We may assume that

$$N \cap U = \{y \in U \mid (x_1(y), \cdots, x_r(y), 0, \cdots, 0) \in x(K)\},$$

where $K = N \cap M \subset U$ after applying the argument of Proposition 1.

Let L be a spherical modification of K preserving the canonical framing of $\nu(K)$. Let $f : S^p \to K$ be a sphere attaching map for L. By the lemma above, we may give $M \cap U$ a Riemann metric $\bar{g}$ such that

$$\overline{\exp} : \bar{\nu}(f(S^p):K)_\delta \to K$$

for some $\delta > 0$, where $\overline{\exp}$, $\bar{\nu}$ and $\overline{|\ \ |}$ refer to the exponential, normal bundle and norm with respect to $\bar{g}$ and $\bar{\nu}(\ \ )_\delta$ is the subset of $\bar{\nu}(\ \ )$ consisting of all w such that $\overline{|w|} \leq \delta$.

Extend $\bar{g}$ to g on all of U by setting

$$g = \bar{g} + (dx_{r+1})^2 + \cdots + (dx_m)^2.$$

Then also $\exp : \nu(K:M)_\delta \to M$ for some $\delta > 0$.

With respect to g, let $\tilde{u}$ be a unit field along K in M normal to K. Let $\tilde{v}$ be the field $\partial/\partial x_{r+1}$; it is normal to K in N. Using $\tilde{u}$ and $\tilde{v}$ define embeddings

$$f_1 : D^{p+1} \to M \quad \text{and} \quad f_2 : D^{p+1} \to N$$

extending f as follows. For $\delta > 0$ small enough define

$$f_1 : [1 - \delta, 1]S^p \to M \cap U$$

by $f_1(ts) = \exp(1 - t)\tilde{u}(f(s))$, where $t \in [1 - \delta, 1]$ and $s \in S^p$. Using Poincaré duality we see that $M \cap U - K$ is n-connected so f_1 extends to an embedding

$$f_1 : [0, 1]S^p \to M \cap U$$

since $r \geq 2n + 3$ and the Whitney theorems apply. Similarly, define

$$f_2 : [1 - \delta, 1]S^p \to N$$

by $f_2(ts) = \exp [1 - t]\tilde{v}(f(s))$. Again, by Poincaré duality, $N - f(S^p)$ is n-connected so f_2 extends to an embedding $f_2 : D^{p+1} \to N$ since $s \geq 2n + 3$ and the Whitney theorems apply. Finally, we use the lemma above to extend a suitable restriction of the Riemann metric g to a metric on all of X such that

$$\exp : \nu(f_2(D^{p+1}):N)_\delta \to N \qquad \text{for small } \delta > 0$$

as well as

$$\exp : \nu(K:M)_\delta \to M$$

and

$$\exp : \nu(f(S^p):K) \to K.$$

By a dilatation of the metric, we may assume $\delta > \frac{2}{3}$.

The unit field $\tilde{v}$ is already extended all over $M \cap U$ and consequently over $f_1(D^{p+1})$. To extend $\tilde{u}$ to all of $\nu(N:X) \mid f_2(D^{p+1})$, observe that, in coordinates, for $(x_1, \cdots, x_r, 0, \cdots, 0) \in x(K)$ there are functions $u_1, \cdots, u_r$ on $x(K)$ such that

$$\tilde{u}(x_1, \cdots, x_r, 0, \cdots, 0)$$
$$= u_1(x_1, \cdots, x_r)\partial/\partial x_1 + \cdots + u_1(x_1, \cdots, x_r)\partial/\partial x_r .$$

extend $\tilde{u}$ over a neighborhood of $f(S^r)$ in $f_2(D^{p+1})$ by

$$\tilde{u}(x_1, \cdots, x_r, s, 0, \cdots, 0)$$
$$= u_1(x_1, \cdots, x_r)\partial/\partial x_1 + \cdots + u_r(x_1, \cdots, x_r)\partial/\partial x_r .$$

Then since $\pi_p(S^{n-s-1}) = 0$ the extension continues to all of $\nu(N:X) \mid f_2(D^{p+1})$. Call the extended field u also. It is easy to see that if $y \in f(S^p)$ and s, t are small enough then

$$(*) \qquad \exp (s\tilde{v}(\exp t\tilde{u}(y))) = \exp (t\tilde{u}(\exp s\tilde{v}(y))).$$

Observe that $f_1(D^{p+1}) \cup f_2(D^{p+1})$ is a topological $(p + 1)$-sphere. We wish to embed a homeomorph of D^{p+2} in X so its boundary sphere coincides with this topological sphere. We now construct this homeomorph and carry out some related constructions.

5. A $(p + 2)$-cell and a model of the isotopy

Let $t \to (\alpha(t), \beta(t))$ be a C^∞ path $[0, 1] \to [0, 1]^2$ such that

(1) $\alpha(t) = t$ and $\beta(t) = 1 - t^2$ for $0 \leq t \leq \frac{1}{3}$
(2) $\alpha(t) = 1$ and $\beta(t) = 1 - t$ for $\frac{2}{3} \leq t \leq 1$
(3) $\alpha'(t) > 0$ for $0 \leq t \leq \frac{2}{3}$
(4) $\beta'(t) < 0$ for $0 < t \leq 1.$

Let $\mathfrak{D}^{p+2} \subset R^{p+1} \times R$ be

$$\{(x, t) \mid |x| = \alpha(s), 0 \leq t \leq \beta(s), s \in [0, 1]\}$$

and

$$\mathfrak{S}^{p+1} = \{(x, t) \mid |x| = \alpha(s), t = \beta(s), s \in [0, 1]\} \subset \mathfrak{D}^{p+2}.$$

Then $\mathfrak{D}^{p+2}$ is the desired homeomorph of D^{p+2}. Its boundary consists of $D^{p+1} \times 0 \cup \mathfrak{S}^{p+1}$. An explicit diffeomorphism $A : D^{p+1} \to \mathfrak{S}^{p+1}$ is given by

$$A(x) = \left(\frac{\alpha(|x|)}{|x|} x, \beta(|x|)\right) \quad \text{for } |x| > 0$$

$$A(x) = (x, 1 - |x|^2) \qquad \text{for } |x| < \tfrac{1}{3}$$

Let $\tilde{U}$ be the field along $\mathfrak{S}^{p+1}$ defined by

$$\tilde{U}(x, t) = \left(\frac{\beta'(s)x}{|x| \sqrt{\alpha'(s)^2 + \beta'(s)^2}}, \frac{-\alpha'(s)}{\sqrt{\alpha'(s)^2 + \beta'(s)^2}}\right)$$

for $x \neq 0$ and $(|x|, t) = (\alpha(s), \beta(s))$; and

$$\tilde{U}(x, t) = \left(\frac{-2x}{\sqrt{1 + 4|x|^2}}, \frac{-1}{\sqrt{1 + 4|x|^2}}\right)$$

for $0 \leq |x| < \frac{1}{3}$.

We will also need later the following: h is a C^∞ function $[0, 1] \to [\frac{1}{9}, 1]$ such that

(1) $h(s) = \frac{1}{9} + s^2$ for $0 \leq s \leq \frac{1}{3}$
(2) $h(s) = 1$ for $\frac{2}{3} \leq s \leq 1$
(2) $h'(s) > 0$ for $0 < s < \frac{2}{3}$.

Define an embedding $\varphi : D^{p+1} \times D^{q+1+k} \to \mathfrak{D}^{p+2} \times R^{q+1+k}$ by

$$\varphi(x, y) = \left(\frac{\alpha(|x|)}{|x|} x, h(|y|)\beta(|x|), y\right) \qquad \text{for } 0 < |x|$$

and

$$\varphi(x, y) = (x, h(|y|)(1 - |x|^2), y) \qquad \text{for } |x| \leq \tfrac{1}{3}.$$

Then $\varphi(D^{p+1} \times D^{q+1+k})$ is a submanifold of $\mathfrak{D}^{p+2} \times R^{q+1+k}$ which agrees with $\mathfrak{S}^{p+1} \times R^{q+1+k}$ on $|y| > \frac{2}{3}$ and is isotopic to $\mathfrak{S}^{p+1} \times R^{q+1+k}$ modulo that subset via the isotopy defined by

$$\eta_t(x, y) = \left(\frac{\alpha(|x|)}{|x|} x, (1 - t)h(|y|)\beta(|x|) + t\beta(x|), y\right) \qquad \text{for } 0 < |x|$$

and similarly defined for $|x| \leq \frac{1}{3}$. The set $\varphi(D^{p+1} \times D^{q+1})$ is also a submanifold. On both these manifolds the function $(x, y) \to h(|y|)\beta(|x|)$ defines a C^∞ function with a single non-degenerate critical point of index $p + 1$ at the level $\frac{1}{9}$.

Finally, the isotopy $\zeta_t(x, 0, y) = (x, t, y)$ of $R^{p+1} \times 0 \times R^{q+1+k}$ in $R^{p+1} \times R \times R^{q+1+k}$ for $0 \leq t \leq \frac{2}{9}$ defines a map

$$Z : R^{p+1} \times 0 \times R^{q+1+k} \times [0, \tfrac{2}{9}] \to R^{p+1} \times R \times R^{q+1+k} \times [0, \tfrac{2}{9}].$$

Since ζ_t is transverse regular along Im φ except for $t = \frac{1}{9}$ at $0 \times 0 \times 0$, Z is transverse regular along $(\text{Im } \varphi) \times [0, \frac{2}{9}]$ except possibly at $0 \times 0 \times (0, \frac{1}{9})$, but it is easy to check that Z is transverse regular there too. Thus, the reverse isotopy defined by $(x, t', y) \to (x, t' - t, y)$ restricted to Im φ defines a map

$$Z' : (\text{Im } \varphi) \times [0, \tfrac{2}{9}] \to R^{p+1} \times R \times R^{q+1+k} \times [0, \tfrac{2}{9}]$$

that is transverse regular along $R^{p+1} \times 0 \times R^{q+1+k} \times [0, \frac{2}{9}]$.

6. The effect of the bundle attaching map

Returning now to $f_1(D^{p+1}) \cup f_2(D^{p+1})$, we define an embedding

$$g : \mathfrak{D}^{p+2} \to X$$

such that $g \mid D^{p+1} = f_1$ and $g \circ A = f_2$. Say $(x, t) \in \mathcal{S}^{p+1} = A(D^{p+1})$ and $z \in R$ is small. Then define g by

$$g(\exp(z\tilde{U}(x, t))) = \exp(z\tilde{u}(f_2 \circ A^{-1}(x, t))).$$

Say $x' \in D^{p+1}$ and $t \in R$ is small. Define g by

$$g(x', t) = \exp(t\tilde{v}(f_1(x'))) \qquad \text{for } 0 \leq t \leq \tfrac{2}{9}.$$

These two versions of g have intersecting domains, but using $(*)$ it is easy to see that the two versions agree on the intersection. Thus we have obtained an embedding of a neighborhood in $\mathfrak{D}^{p+2}$ of $D^{p+1} \times 0 \cup \mathcal{S}^{p+1}$ into X. Using Poincaré duality, it is a straightforward argument that $X - M \cup N$ is $(n + 1)$-connected so, using again the Whitney embedding theorems, g may be extended to an embedding $g : \mathfrak{D}^{p+2} \to X$ such that

$$g(\mathfrak{D}^{p+2}) \cap (M \cup N) = f_1(D^{p+1}) \cup f_2(D^{p+1}) = g(D^{p+1} \times 0) \cup g(\mathcal{S}^{p+1}).$$

So far only the sphere attaching map of the spherical modification L has entered the constructions. The bundle attaching map will enter in such a way that the fact that the spherical modification preserves the canonical framing of $\nu(K)$ implies that a certain bundle ξ over $f_1(D^{p+1}) \cup f_2(D^{p+1})$ is trivial and so extends over $g(\mathfrak{D}^{p+2})$. Over $f_2(D^{p+1})$, the bundle ξ is defined to be $\nu(f_2(D^{p+1}):N)$. Over $f_1(D^{p+1})$, the bundle ξ is a little more complicated. The bundle attaching map $f' : S^p \times \varepsilon^{q+1} \to \tau(K)$ defines a field $\mathfrak{F}$ of $(q + 1)$-frames in $\nu(f(S^p):K)$. Since $\pi_p(V_{q+1,r-p-1}) = 0$, the field $\mathfrak{F}$ extends to a field, also called $\mathfrak{F}$ of $(q + 1)$-frames in $\nu(f_1(D^{p+1}):M)$. Let $v^{\perp}$ be the sub-bundle of $\nu(M:X) \mid f_1(D^{p+1})$ consisting of vectors normal to the field v. Then we define $\xi \mid f_1(D^{p+1})$ to be span $(\mathfrak{F}) \oplus v^{\perp}$. Over $f(S^p)$, these two bundles agree and so define a bundle ξ over $f_1(D^{p+1}) \cup f_2(D^{p+1})$. For dimensional reasons ξ will be trivial if and only if $\xi \oplus (\tau(g(\mathfrak{D}^{p+2})) \mid f_1(D^{p+1}) \cup f_2(D^{p+1}))$ is trivial since $\tau(g(\mathfrak{D}^{p+2}))$ is trivial. Thus it suffices to compute the characteristic map of this new bundle and to see that it is trivial.

Let $\tilde{v}$ stand also for the obvious framing of span $(\tilde{v})$ and $\tilde{u}$ for the obvious framing of span $(\tilde{u})$; let $v^{\perp}$ stand also for the canonical framing of $v^{\perp}$ that extends over a cone of K in M. Let

$$fr : S^p \times \varepsilon^{p+1} \to \tau(S^p) \times \varepsilon$$

be the framing which makes the following diagram commute

$$\begin{array}{ccc} S^p \times \varepsilon^{p+1} & \xrightarrow{fr} & \tau(S^p) \times \varepsilon \\ \cap & & \cap \\ D^{p+1} \times \varepsilon^{p+1} & \longrightarrow & \tau(D^{p+1}) \end{array}$$

where $S^p \times \varepsilon \to \tau(D^{p+1})$ via the unit inward normal along S^p. Then,

$$\begin{array}{ccc} S^p \times \varepsilon^{q+1} \times \varepsilon^{m-r-1} \times \varepsilon \times \varepsilon^{p+1} & \xrightarrow{f' \oplus 1 \oplus ((f_* \times 1) \circ fr)} & (\nu(f(S^p):K) \times \varepsilon^{m-r-1}) \\ & & \oplus\ (\varepsilon \times \tau(f(S^p)) \times \varepsilon) \\ \Big\downarrow (1 \oplus v^{\perp}) \oplus (\tilde{v} \oplus 1 \oplus \tilde{u}) & & \Big\downarrow \\ \longrightarrow & \xi \mid f(S^p) \oplus (\tau(g(\mathfrak{D}^{p+2}))) \mid f(S^p) & \longleftarrow \end{array}$$

covers f and extends over D^{p+1} to cover f_1. On the other hand, the field $\tilde{u}$ is extended over $g(\mathfrak{S}^{p+1})$ as the unit inward normal of $g(\mathfrak{D}^{p+2})$ along $g(\mathfrak{S}^{p+1})$. Then, using the definition of ξ over $g(\mathfrak{S}^{p+1}) = f_2(D^{p+1})$,

$$\xi \mid f(S^p) \oplus \tau(g(\mathfrak{D}^{p+2})) \mid f(S^p) = \tau(N) \mid f(S^p) \oplus \operatorname{span}(\tilde{u}).$$

Since K is contractible to a point in N, $\tau(N) \mid K$ admits a canonical bundle homotopy class ρ of bundle maps $\tau(N) \mid K \to \varepsilon^s$. But

$$\operatorname{span}(\tilde{u}) \mid f(S^p) \xrightarrow{\pi\tilde{u}} \varepsilon$$

is also well defined and extends over $g(\mathfrak{S}^{p+1})$ so we have that

$$\xi \mid f(S^p) \oplus \tau(g(\mathfrak{D}^{p+2})) \mid f(S^p) \xrightarrow{\rho \oplus \pi(\tilde{u})} \varepsilon^{s+1}$$

is a bundle class of maps that extends over $f_2(D^{p+1})$. Thus the characteristic map of the bundle

$$\xi \oplus \tau(g(\mathfrak{D}^{p+2})) \mid f_1(D^{p+1}) \cup f_2(D^{p+1}))$$

may be identified with the bundle class

$$P = (\rho \oplus \pi(\tilde{u})) \circ ((1 \oplus v^{\perp}) \oplus (\tilde{v} \oplus 1 \oplus \tilde{u})) \circ (f' \oplus 1 \oplus 1 \oplus ((f_* \times 1) \circ fr))$$

of bundle maps $S^p \times \varepsilon^{s+1} \to \varepsilon^{s+1}$, and the bundle is trivial if and only if this class extends over D^{p+1}.

To see that P extends over D^{p+1}, one must express in similar form the fact that L preserves the canonical framing of $\nu(K:U)$. Observe that

$$\tau(N) \mid K = \tau(K) \oplus \nu(M:X) \mid K.$$

Since K may be contracted to a point in M, there is a canonical framing

$$K \times \varepsilon^{m-r} \xrightarrow{\partial} \nu(M:X) \mid K;$$

clearly $\partial \mid f(S^p) = v^\perp \oplus \tilde{v}$. But then

$$\tau(K) \times \varepsilon^{m-r} \xrightarrow{1 \oplus \partial} \tau(N) \mid K \xrightarrow{\rho} \varepsilon^s$$

defines a framing of the stable tangent bundle of K. If σ is the canonical framing of $\nu(K:U)$, the framing $\rho \circ (1 \oplus \partial)$ is dual to σ in the sense that $\sigma \oplus \rho \circ (1 \oplus \partial)$ is the unique framing of $\tau(U) \times \varepsilon^{m-r}$ restricted to K. It is known that L preserves σ if and only if L preserves $\rho \circ (1 \oplus \partial)$. Let $\gamma : \tau(K) \times \varepsilon \to \tau(L)$ be the bundle homotopy class of bundle maps determined by the inward normal of L along K. Let $F : D^{p+1} \to L$ be an embedding which extends f; F may be chosen so that the following diagram commutes,

$$\begin{array}{ccc} \tau(S^p) \times \varepsilon & \xrightarrow{f_* \times 1} & \tau(K) \times \varepsilon \\ \cap & & \downarrow \gamma \\ \tau(D^{p+1}) & \xrightarrow{F_*} & \tau(L). \end{array}$$

Now $\rho \circ (1 \oplus \partial) \circ (\gamma \times 1^{m-r-1})^{-1}$ is a framing of $(\tau(L) \times \varepsilon^{m-r-1}) \mid K$ which must extend over L for L to preserve $\rho \circ (1 \oplus \partial)$. But it extends over all of L if and only if it extends over $F(D^{p+1})$. And it extends over $F(D^{p+1})$ if and only if the bundle homotopy class Q of bundle maps given by

$$\begin{array}{ccc} S^p \times \varepsilon^{q+1} \times \varepsilon^{p+1} \times \varepsilon^{m-r-1} & \xrightarrow{f' \oplus ((f_* \times 1) \circ fr) \oplus 1} & \tau(K) \times \varepsilon \times \varepsilon^{m-r-1} \\ \downarrow Q & & \downarrow 1 \oplus \partial \\ \varepsilon^s & \xleftarrow{\rho} & \tau(N) \mid K \end{array}$$

extends over D^{p+1}. But in view of the fact that $v^\perp \oplus \tilde{v} = \partial$, the framing Q will extend over D^{p+1} if and only if the framing P does so. But Q does extend since L does preserve σ and hence $\rho \circ (1 \oplus \partial)$. So P extends, and the bundle

$$\xi \oplus (\tau(g(\mathfrak{D}^{p+2})) \mid f_1(D^{p+1}) \cup f_2(D^{p+1}))$$

must be trivial. Consequently, ξ is trivial.

Now ξ is a sub-bundle of $\nu(g(\mathfrak{D}^{p+2}):X) \mid (f_1(D^{p+1}) \cup f_2(D^{p+1}))$. It may be thought of as given by a mapping $\xi : S^{p+1} \to G_{s-p-1,m-p-2}$ where $G_{i,j}$ is the i-planes through the origin in R^j. But $G_{i,j}$ contains the l-skeleton of $G_{i,\infty}$ if $j \geq l$, and ξ followed by $G_{s-p-1,m-p-2}\ G_{s-p-1,\infty}$ is trivial since ξ is a trivial bundle. Thus, $m - p - 2 \geq p + 2$ implies that ξ is already homotopic to the trivial map, so the bundle ξ extends to a sub-bundle, still to be called ξ, of $\nu(g(\mathfrak{D}^{p+2}):X)$ over all of $g(\mathfrak{D}^{p+2})$.

7. Construction of the isotopy

Let $(\delta_1, \cdots, \delta_{s-p-1})$ be a field of $(s - p - 1)$-frames in the extended ξ such that $(\delta_1, \cdots, \delta_{q+1})$ restricted to $f_1(D^{p+1})$ is $\mathfrak{F}$. Let

$$w : (f_1(D^{p+1}), f(S^p)) \to (O(s - n - 1), 1)$$

be any map. It may be used to change the field $(\delta_1, \cdots, \delta_{s-p-1})$ by operating on the last $s - n - 1$ entries. The changed field may be extended over $g(\mathfrak{D}^{p+2})$; such an extension will be denoted by $\delta^w = (\delta_1^w, \cdots, \delta_{s-p-1}^w)$. Set $k = s - p - 1 - q - 1 = s - n - 1$ and embed $\mathfrak{D}^{p+2} \times D^{q+1+k}$ in X by

$$G^w(x, t_1, \cdots, t_{q+1+k}) = \exp\left(\frac{t_1\, \delta_1^w(g(x)) + \cdots + t_{q+1+k}\, \delta_{q+1+k}^w\,(g(x))}{B}\right),$$

where $B > 0$ is large enough to make G^w an embedding. Clearly the extension $(\delta_1, \cdots, \delta_{q+1})$ could be chosen over a neighborhood of $f_1(D^{p+1})$ in $g(\mathfrak{D}^{p+2})$ so that $x_i(G^w(y, t, z)) = x_i(G^w(y, 0, z))$ for $i \neq r + 1$ and $y \in D^{p+1}$, $t \in [0, \frac{2}{9}]$, and $z \in D^{q+1}$, and so that $x_{r+1}(G^w(y, t, z)) = t$ under the same conditions.

Now, there is an isotopy $\eta_t^w : N \to X$ modulo boundaries so that

$$(N - G^w(\mathfrak{S}^{p+1} \times D^{q+1+k})) \cup G^w(\operatorname{Im}\varphi) = \eta_1(N)$$

and there is an isotopy $\zeta_t : M \to X$ modulo boundaries extending the isotopy given by

$$(x_1, \cdots, x_r, 0, \cdots, 0) \to (x_1, \cdots, x_r, t, 0, \cdots, 0)$$

in terms of coordinates in U; we may assume that

$$\zeta_t(M) \cap U = \{y \mid x_1(y), \cdots, x_r(y), 0, \cdots, 0) \in x(M)\}$$

for $t \in [0, 1]$ and $\zeta_t(M)$ meets $\eta_1^w(N)$ only inside U.

The isotopy ζ_t for $0 \leq t \leq \frac{2}{9}$ defines a map

$$Z : M \times [0, \tfrac{2}{9}] \to X \times [0, \tfrac{2}{9}]$$

which, according to Section 4 is transverse regular along $\eta_1^w(N) \times [0, \frac{2}{9}]$. Let

$$L_1^w = Z(M \times [0, \tfrac{2}{9}]) \cap (\eta_1^w(N) \times [0, \tfrac{2}{9}])$$

by $G^{w\prime}(x, t, y) = (G^w(x, t, y), t)$. Then

$$L_1^w = ((K - G^w(S^p \times D^{q+1})) \times [0, \tfrac{2}{9}]) \\ \cup (G^{w\prime}((\operatorname{Im}\varphi) \cap (D^{p+1} \times [0, \tfrac{2}{9}] \times D^{q+1+k}))).$$

The natural projection $X \times [0, \frac{2}{9}] \to [0, \frac{2}{9}]$ restricted to L_1^w defines a function which, according to Section 4, has a single critical point, nondegenerate of index $p + 1$, at $G^{w\prime}(0, \frac{1}{9}, 0)$. Thus, L_1^w is a spherical modification of K. The map $f : S^p \to K$ may be chosen to be the sphere attaching map of L_1^w. Let $D^{p+1} \times R \xrightarrow{pr} R$ be the natural projection. Then using Section 4, we define a map $f_3 : D^{p+1} \to L_1^w$, which extends f, by setting

$$f_3(x) = (\eta_1 \circ f_2(x), pr \circ g^{-1} \circ \eta_1 \circ f_2(x)).$$

Clearly f_3 is an embedding extending f. Using f_3, η, and ξ, it is straightforward to see that the bundle attaching map of L_1^w is f'. Consequently, L_1^w is diffeomorphic to L.

Thus, if we reparametrize ζ so t runs over $[0, 1]$ we have that

$(\zeta, M, \eta_1^w(N), X)$ realizes the spherical modification L. Let $\mathfrak{z}_t : X \to X$ be an isotopy modulo boundaries restricting to ζ_t. Let $\psi_t = \mathfrak{z}_t^{-1} \mid N$. We may assume both $t \to \psi_t$ and $t \to \eta_t^w$ to be constant near 0 and 1. Define $\lambda_t^w : N \to X$ by $\lambda_t^w = \eta_{2t}^w$ for $\zeta \leq t \leq \frac{1}{2}$ and $\lambda_t^w = \psi_{2t-1}$ for $\frac{1}{2} \leq t \leq 1$. Then λ_t^w is an isotopy modulo boundaries such that (λ_t^w, N, M, X) realize L. Moreover, $\lambda_t^w = 1$ on $K - G^w(S^p \times D^{q+1})$; let

$$\Lambda^w : N \times I \to X \times I$$

be the map defined by λ_t^w and let $L^w = \Lambda^w(N \times I) \cap (M \times I)$.

So far, only the spherical modification L of the framed spherical modification (L, Σ) has entered the constructions. Next we wish to choose

$$w : (f_1(D^{p+1}), f(S^p)) \to (O(r - n - 1), 1)$$

so that λ_t^w realizes (L, Σ). Let $\pi_w : L^w \to I$ be the restriction of $X \times I \to I$ to L^w; let $\mathfrak{m}(\pi_w)$ be the mapping cylinder of π_w. We begin by finding a map

$$\mathfrak{m} : \mathfrak{m}(\pi_w) \to \Lambda^w(N \times I)$$

extending $L^w \subset \Lambda^w(N \times I)$ and maps of cones $C(K) \to N$ and $C(K') \to N$. For this, pick a smooth mapping C_1 of the cone

$$C(S^p \times D^{q+1+k} \cup D^{p+1} \times S^{q+k}) \to D^{p+1} \times D^{q+1+k}$$

extending the inclusion and sending the vertex to 0×0. This map defines a smooth family of maps of cones

$$C_t' : C(\lambda_t^w \circ G^w \circ (A \times 1)(S^p \times D^{q+1+k} \cup D^{p+1} \times S^{q+k})) \to \lambda_t^w(N).$$

All these have the same restriction on $K - G^w(S^p \times D^{q+1})$ and this restriction extends to a map $C'' : C(K - G^w(S^p \times D^{q+1})) \to N$. Let

$$C_t = C_t' \cup \lambda_t^w \circ C'';$$

it is a smooth family of maps

$$C_t : C(\lambda_t^w \cap M) \to \lambda_t^w(N)$$

which defines $\mathfrak{m} : \mathfrak{m}(\pi_w) \to \Lambda^w(N \times I)$ as desired.

Let ξ' be the bundle which is the sum of the orthogonal complement $u^\perp$ of u in $\nu(N{:}X) | f_2(D^{p+1})$ and the orthogonal complement $(\text{span } \mathfrak{F})^\perp$ of span $\mathfrak{F}$ in $v(f_1(D^{p+1}) : M)$. Then ξ' is the orthogonal complement of

$$\xi \oplus (\tau(g(\mathfrak{D}^{p+2})) | (f_1(D^{p+1}) \cup f_2(D^{p+1}))$$

and extends over $g(\mathfrak{D}^{p+2})$ as such. Then

$$\xi \oplus \xi' \oplus \tau(g(\mathfrak{D}^{p+2})) = \tau(X) | g(\mathfrak{D}^{p+2})$$

which has a canonical framing. Let δ'' be any framing of $\tau(g(\mathfrak{D}^{p+2}))$ and let δ' be a framing of ξ' such that $\delta \oplus \delta' \oplus \delta''$ is the canonical framing of

$$\xi \oplus \xi' \oplus \tau(g(\mathfrak{D}^{p+2})).$$

Now, for any map

$$(f_1(D^{p+1}), f(S^p)) \xrightarrow{w} (O(s - n - 1(, 1)$$

there is a map

$$w' : (f_1(D^{p+1}), f(S^p)) \to (O(m - s - 1), 1)$$

unique up to homotopy mod $f(S^p)$, such that

$$w \oplus w' : (f_1(D^{p+1}), f(S^p)) \to O(m - n - 2)$$

is trivial; every class $(f_1(D^{p+1}), f(S^p)) \to (O(m - s - 1), 1)$ contains a w' for some w unique up to homotopy mod $f(S^p)$. Given w, find w' and let it operate on δ' to obtain $\delta^{w'}$. Then extend $\delta^{w'}$ to all of ξ' and call it still $\delta^{w'}$. Then $\delta^w \oplus \delta^{w'} \oplus \delta''$ is the canonical framing.

Extend the framing

$$\delta^{w'} \oplus \tilde{u} \quad \text{of} \quad \nu(N:X) | g(\mathcal{S}^{p+1})$$

to a framing over $G^w(\mathcal{S}^{p+1} \times D^{q+1+k})$; the extension may be prescribed to be the same for all w over $G^w(S^p \times D^{q+1+k})$ since all $w = 1$ over $f(S^p)$. Let $\bar{\lambda}_t^w : X \to X$ be an isotopy covering λ_t^w. Then $\lambda^w_{t\,*}(\delta^{w'} \oplus \tilde{u})$ defines a framing of

$$\nu(\lambda_t^w(N):X) | \lambda_t^w \circ G^w(\mathcal{S}^{p+1} \times D^{q+1+k}).$$

The canonical framing σ of $\nu(K:M)$ restricted to $K - G^w(S^p \times D^{q+1})$ extends each of these framings to

$$\lambda_t^w \circ G^w(\mathcal{S}^{p+1} \times D^{q+1+k}) \cup (K - G^w(S^p \times D^{q+1})).$$

These framings in turn define a framing of

$$\nu(\Lambda^w(N \times I):X \times I) | \Lambda^w (G^w(\mathcal{S}^{p+1} \times D^{q+1+k}) \times I) \cup (\Lambda^w(N \times I) \cap (M \times I))$$

which is already extended over the map m above. Thus, this framing restricted to L^w is canonical; we will call it Σ^w.

Thus we have a correspondence $w \to L^w$ from maps

$$(f_1(D^{p+1}), f(S^p)) \to (O(s - n - 1), 1)$$

to framings of $v(L)$ extending the canonical framing of $v(K)$. If this map is onto, the proof of Proposition 3 is complete. In fact, the map is one-to-one and onto. To see this, let

$$\begin{aligned} L_d = \{\text{pt.} \mid \text{pt.} &= \exp(-t\tilde{u}(z)), \quad 0 \le t \le d, \quad z \in K \quad \text{or} \\ \text{pt.} &= G^w(x, 0, y) \quad \text{with} \quad x \in D^{p+1}, \quad y \in D^{q+1}, \\ &\qquad\qquad 0 \le h(|y|)\beta(|x|) \le \tfrac{2}{9}\} \subset M \end{aligned}$$

for sufficiently small $d > 0$. Notice that L_d is independent of w and that $\nu(L_d:M)$ is an extension of ξ' over the collar $\{\text{pt.} \mid \text{pt.} = \exp(-t\tilde{u}(z)), z \in K, 0 \le t \le d\}$. Thus all the framings of $\nu(L_d:M)$ that restrict to σ on $\nu(K)$ are given by the $\delta^{w'}$'s. On the other hand, the framing Σ^w on $\nu(L^w)$ is clearly equivalent via an isotopy to the framing $\delta^{w'} \times 1$ on $\xi' = \nu(L_d: U)$, and the proof of Proposition 3 is complete.

BIBLIOGRAPHY

1. P. E. CONNER AND E. E. FLOYD, *Differentiable periodic maps*, Erg. Mat., 1964.
2. R. LASHOF, *Poincaré duality and cobordism*, Trans. Amer. Math. Soc., vol. 109 (1963), pp. 257–277.
3. J. MILNOR, *Groups of homotopy spheres*, Ann. of Math., vol. 77 (1963), pp. 504–537.
4. ———, *The h-cobordism theorem*, Princeton, 1965.
5. R. WELLS, *Cobordism groups of immersions*, to appear.

UNIVERSITY OF CHICAGO
CHICAGO, ILLINOIS