A CHARACTERIZATION OF NON-SEGREGATED ALGEBRAS

BY

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1. Introduction

In [8] the author and R. W. Richardson show that for an algebra, (associative or Lie), all deformations are essentially deformations of the Jacobson radical. Since the Hochschild groups play a central role in deformation theory, one would expect a somewhat similar situation to prevail in regard to extensions of an algebra by a square zero ideal. Obviously, if the ideal is square zero and the extension is finite-dimensional, then the ideal must be contained in the radical of the extension. One can then view the Hochschild extensions as enlargements of the radical. This is, in a sense, just the opposite of what happens with deformations, i.e., in deformations the radical gets smaller, if anything.

The object of this paper is to show just which parts of the radical can be enlarged. The theorem, and its corollaries, of Section 2 along with a theorem of J. P. Jans [6] give a complete description of the situation for extensions by a simple module.

In what follows, A will denote a finite-dimension algebra over a field K with subalgebra S and Jacobson radical N such that $A = S + N$, K-direct, and S is K-separable. In view of the Wedderburn-Malcev theorem this is the case for a wide variety of algebras. By an A-module we will mean a two-sided A-module satisfying

$$a(mb) = (am)b \quad \text{for all} \quad a, \quad b \in A; \quad m \in M.$$

Following Hochschild [2], $C^n(A, M)$ will denote

$$\mathrm{Hom}_k(A_1 \otimes_k A_2 \otimes_k \cdots \otimes_k A_n, M)$$

where each $A_1 = A$. δ will denote the usual Hochschild coboundary operator and $H^n(A, M)$ the corresponding Hochschild groups.

2. Characterization theorem

DEFINITION. Let $f \in C^2(A, M)$ and $\delta f = 0$. Form the K-vector space $A + M$. We define the algebra B_f to have underlying vector space $A + M$ and multiplication given by

$$(a + m_1)(b + m_2) = ab + am_2 + m_1 b + f(a, b).$$

That B_f is an associative algebra is insured by $\delta f = 0$. Notice, also, that M appears as an ideal in B_f and $M^2 = 0$; hence M is contained in the radical of B_f. It is also well known that any extension of A with square zero kernel is of the above type [2].

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DEFINITION. An algebra A is segregated if for every algebra B and homomorphism σ of B onto A with $(\ker \sigma)^2 = 0$ there exists an algebra homomorphism $\pi : A \to B$ such that $\sigma \circ \pi = 1_A$.

Hochschild [2] showed A is segregated if and only if $H^2(A, M) = 0$ for all two-sided A-modules M. We will use both descriptions freely throughout this paper.

THEOREM 1 (characterization). *Let $A = S + N$. Then there exists a simple two-sided module M and $f \in C^2(A, M)$ with $\delta f = 0$ such that $M \subset N_f^2$, where N_f is the radical of B_f, if and only if A is not segregated.*

Proof. Suppose such M and f exist. Then we claim f represents a nontrivial element of $H^2(A, M)$. Suppose not. Then $f - \delta g = 0$ for some $g \in C^1(A, M)$, and $B_f \cong B_0$. But in B_0, $M \not\subset N_0^2$ which leads to a contradiction.

We now prove the converse by proving a more general result.

THEOREM. *If $A = S + N$ and $H^2(A, M) \neq 0$, then $M \cap N_f^2 \neq 0$ if f is not equivalent to zero in $H^2(A, M)$.*

Proof. Let $0 \neq F \in H^2(A, M)$. By Lemma 10.1 of [4] we can pick a representative f of F such that $f(s \otimes a) = F(a \otimes s) = 0$ for all $a \in A$, $s \in S$. Let W be the S-complement of N^2 in A. Now if $M \cap N_f^2 = 0$, for any $a_1, b_1 \in N$, $i = 1, 2, \cdots, k$ such that $\sum_{i=1}^k a_i b_i = 0$, we have $\sum_{i=1}^k f(a_i \otimes b_i) = 0$. Thus for $a_i, b_i \in N$, $i = 1, 2, \cdots, k$ setting

$$g\left(\sum_{i=1}^k a_i b_i\right) = \sum_{i=1}^k f(a_i \otimes b_i)$$

defines a function on N^2. Note that in fact g is both a left and a right S-homomorphism. Now, since W is an S-module, g can be extended to a function, still denoted by g, on A which is still both a left and right S-homomorphism by defining $g(y) = 0$ for $y \in W$. Now form $f + \delta g = h_0$. Notice that $h_0(s \otimes a) = h_0(a \otimes s) = 0$, and that $M \cap N_{h_0}^2 = 0$. Moreover, one easily verifies that the image of h_0 is contained in $NM + MN$. One now repeats the argument to h_0 and finds h_1 equivalent to h_0 where $h_1(s \otimes a) = h_1(a \otimes s) = 0$ and $M \cap N_{h_1}^2 = 0$ and the image of h_1 is contained in $N^2M + NMN + MN^2$. Continuing in this way and using the fact that N is nilpotent it follows that f is equivalent to zero, a contradiction. Hence $M \cap N_f^2 \neq 0$.

As an immediate corollary we have,

COROLLARY. *If $F \in H^2(A, M)$ and $F \neq 0$, then there exists a representative f of F such that for some $a_i, b_i \in N$, $i = 1, 2, \cdots, k$, $\sum_{i=1}^k a_i b_i = 0$; but $\sum_{i=1}^k f(a_i \otimes b_i) \neq 0$ and this f can be picked so that $f(s \otimes a) = f(a \otimes s) = 0$ for $a \in A$ and $s \in S$.*

COROLLARY. *If M is a simple A-module, then for $F \in H^2(A, M)$ and any representative f of F, $M \subseteq N_f^2$ if and only if $F \neq 0$.*

Proof. If f and g are representatives of $F \in H^2(A, M)$ then $f = g + \delta h$ for $h \in C^1(A, M)$. $B_f \cong B_g$, the isomorphism being $a + m \to a + m + h(a)$ where $a \in A$ and $m \in M$. The result follows easily.

COROLLARY. *If M is a simple A-module and A is generalized uniserial and f is a representative of F, $0 \neq F \in H^2(A, M)$, then B_f is generalized uniserial.*

Proof. This follows from a lemma by Nakayama [7] which states that A is generalized uniserial if and only if A/N^2 is generalized uniserial.

3. An application

We begin this section with a definition of the associated free ring, see [6].

DEFINITION. Let $A = S + N$, as above. Since S is K-separable N^2 has an S-complement P in N. Let $P^{(n)}$ denote P tensored over S with itself n times. Form the weak direct sum

$$F(S, P) = S + P + P^{(2)} + P^{(3)} + \cdots + P^{(n)} + \cdots .$$

Give $F(S, P)$ multiplication defined on the generators by

$$s(p_1 \otimes p_2 \cdots \otimes p_l) = sp_1 \otimes p_2 \otimes \cdots \otimes p_l$$

and

$$(p_1 \otimes p_2 \otimes \cdots \otimes p_l)s = p_1 \otimes p_2 \otimes \cdots \otimes p_l s$$

and

$$(p_1 \otimes p_2 \otimes \cdots \otimes p_l)(p_1' \otimes \cdots \otimes p_m') = p_1 \otimes p_2 \otimes \cdots \otimes p_1' \otimes \cdots \otimes p_m' .$$

It is easily verified using the associativity of tensor that this is a ring.

Now there is a canonical homomorphism of $F(S, P)$ onto A. This homomorphism is induced by the identity on S and P and sends $p_1 \otimes p_2 \cdots \otimes p$ to $p_1 p_2 p_3 \cdots p_l$.

We now have

COROLLARY. *Let M be a simple A-module and $f \in C^2(A, M)$, $\delta f = 0$, and f not equivalent to zero in $H^2(A, M)$. Then B_f and A have the same associated free ring.*

Proof. Clearly $P \cong N/N^2 \cong Nf/N_f^2$ the isomorphisms being S-isomorphisms.

This leads to an alternative proof of a theorem of J. P. Jans [6] which is free of all topological considerations.

THEOREM (Jans). *A is a segregated algebra if and only if $A \cong F(S, P)$ as rings.*

Proof. Consider the homomorphisms

$$\Theta : F(S, P) \to A \quad \text{and} \quad \theta : F(S, P) \to B_f$$

given by the corollary. Then clearly Ker $\Theta \supseteq$ Ker θ, hence $A \cong F(S, P)$ implies $H^2(A, M) = 0$ for all simple M. Therefore $H^2(A, M) = 0$ for all M by a standard induction on the composition length of M. So we have isomorphic implies segregated.

Now suppose Θ is not an isomorphism. Pick the smallest r such that $\sum_{m=r-1}^{\infty} p^{(m)} \subset \ker \theta$. Such an r exists since N is nilpotent. Let $Q = \sum_{m=r}^{\infty} p^{(m)}$, and form $F(S, P)/Q = B$. Since $Q \subset \ker \Theta$, Θ factors through B. Let $\psi : B \to A$ be the induced map. Now with the obvious identifications we can write $B = S + P + V_1 + V_2$ where $V_2 = \ker \psi$ and V_1 is an S-complement of V_2 in the radical squared of B. Let W be a maximal B-submodule of V_2. Let C be an S-complement of W in V_2. It follows that C is simple as an S-module, and hence can be viewed as a simple A-module. If we now consider the extension $\bar{\psi} : B/W \to A$ where $\bar{\psi}$ is the ring homomorphism induced by ψ, we see that C is contained in the square of the radical of B/W and so by the second corollary to the theorem of Section 2 we have that this extension yields a non-trivial element of $H^2(A, C)$.

4. $H^2(A, M)$ as a module

DEFINITION. For a two-sided A-module M denote by $E(M)$ the ring of all two-sided A-homomorphisms of M into itself. If M is simple of course $E(M)$ is then a division ring. It is easy to see that if $\Theta \in E(M)$, then $B_f \cong B_{\Theta \circ f}$ for any $f \in C^2(A, M)$ with $\delta f = 0$. Moreover we can show the following:

THEOREM. *Let M be simple as a 2-sided A-module. Then*

$$H^r = \{F \in H^2(A, M) \mid M \not\subset N_F^{r+1}\}$$

is an $E(M)$-submodule of $H^2(A, M)$ and we have

$$0 = H^1 \subset H^2 \subset \cdots \subset H^{l+1} = H^2(A, M)$$

where

$$N^{l-1} \neq 0 \quad \textit{but} \quad H^l = 0.$$

Proof. Let $F \in H^r$ and $G \in H^r$ and f a representative of F and g a representative of G where

$$f(s \otimes a) = g(s \otimes a) = f(a \otimes s) = g(a \otimes s) = 0$$

for all $s \in S$ and $a \in A$.

Suppose $F - G \neq H^r$. Then $M \cap H_{f-g}^p \neq 0$ where $p > r$. Then there exists $n_1, \cdots, n_p$ such that $n_1 n_2 \cdots n_p = 0$ but

$$f(n_1 \otimes n_2 n_3 \cdots n_p) - g(n_1 \otimes n_2 \cdots n_p) \neq 0.$$

Then $f(n_1 \otimes n_2 \cdots n_p) \neq 0$ or $g(n_1 \otimes n_2 \cdots n_p) \neq 0$ so either $F \in H^p$ or $G \in H^p$, a contradiction. That $.F \in H^r$ follows by the fact that $B_F \cong B_{.F}$, the isomorphism being $a + m \to a + (m)$. That $H^{l+1} = H^2(A, M)$ is clear.

One might hope that the H^r/H^{r-1} are simple as $E(M)$-modules; however, this in general is not true for consider the polynomial ring in two indeterminates x and y over the field K, factored by the ideal generated by x^2, y^2 and xy, and the simple module M which is one copy of the field Km.

DEFINITION.

$$\begin{aligned} f(z \otimes w) &= k_1 k_2 M \quad \text{if} \quad z = k_1 x, \quad w = k_2 w, \\ &= 0 \qquad\qquad \text{otherwise.} \\ g(z \otimes s) &= h_1 h_2 M, \quad \text{if} \quad z = k_1 x, \quad w = k_2 y, \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned}$$

It is readily checked that $\partial f = \partial g = 0$ but that neither f nor g are coboundaries. Here f, and $g \in H^2$ but clearly $B_f \neq B_g$.

Remark. This paper resulted from the study of deformations of algebras while a graduate student at the University of Washington under the advisorship of Professor J. P. Jans. I would like to take this opportunity to express my thanks for his guidance.

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