FINITE SEQUENCES OF GROUP ELEMENTS AND THEIR RELATION TO THE EXISTENCE, ORDER, AND INDEX OF SUBGROUPS

BY

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In [1] there appears the following problem due to Leo Moser:

(M) Let G be a group of order n. Then for every sequence $g_1, \cdots, g_n$ of n not necessarily distinct elements of G, there is a consecutive subsequence $g_s, g_{s+1}, \cdots, g_t$ $(1 \leq s \leq t \leq n)$ such that $g_s g_{s+1} \cdots g_t = 1$, the identity of G.

In this paper we extend and generalize this property of groups. A by-product of our results is a new criterion for a subset of a group to be a subgroup. The authors would like to thank the referee for his helpful suggestions and his proof of Theorem 2 which we present here.

Let G be a group (not necessarily finite) and K a subgroup of G. An *n-sequence* of G is a sequence $g_1, \cdots, g_n$ of n not necessarily distinct elements of G. A *segment* of an n-sequence $g_1, \cdots, g_n$ of G is a subsequence of the form $g_s, g_{s+1}, \cdots, g_t$ $(1 \leq s \leq t \leq n)$. An n-sequence $g_1, \cdots, g_n$ is said to be K-reducible if some segment $g_s, g_{s+1}, \cdots, g_t$ has its ordered product $g_s g_{s+1} \cdots g_t$ in K. Otherwise the n-sequence $g_1, \cdots, g_n$ is said to be *K-irreducible.* A non-empty subset H of G is called *K-n-reducible* if n is the least integer such that every n-sequence $h_1, \cdots, h_n$ of H is K-reducible.

In what follows we shall use the term segment to mean both the sequence $g_s, g_{s+1}, \cdots, g_t$ and its ordered product $g_s g_{s+1} \cdots g_t$. Which meaning is employed should be clear from the context.

THEOREM 1. *Let G be a group and K a subgroup of G. Then $[G:K] = n$ if and only if G is K-n-reducible.*

Proof. It suffices to prove:

(a) if $[G:K] = n$ then G is K-r-reducible for some $r \leq n$;
(b) if G is K-n-reducible then $[G:K] \leq n$.

Suppose $[G:K] = n$. Let $g_1, \cdots, g_n$ be an arbitrary n-sequence of G. Put $b_i = g_1 \cdots g_i$ $(i = 1, \cdots, n)$. Suppose that for some $i, j = 1, \cdots, n$, $i \neq j$, we have $b_i^{-1} b_j$ in K. If $i < j$, then $b_i^{-1} b_j = g_{i+1} \cdots g_j$. If $i > j$, then $b_i^{-1} b_j = (g_{j+1} \cdots g_i)^{-1}$. In either case it follows that some segment of $g_1, \cdots, g_n$ is in K, whence $g_1, \cdots, g_n$ is K-reducible. Otherwise, for all $i \neq j$ we have that $b_i^{-1} b_j$ is not in K. This means that the right cosets $b_j K$ $(i = 1, \cdots, n)$ are all distinct. Hence one of them, say $b_j K$, is K. Thus the

Received October 24, 1966.

segment b_j is in K and again $g_1, \cdots, g_n$ is K-reducible. Hence G is K-r-reducible for some $r \leq n$.

To prove (b), let $g_1, \cdots, g_{n-1}$ be K-irreducible, $g_n = 1$ and let g be an arbitrary element of G. Then $g_1, \cdots, g_{n-1}, g^{-1}$ is K-reducible and therefore there exists i, $1 \leq i \leq n$, such that $g_i g_{i+1} \cdots g_n g^{-1}$ belongs to K. Consequently g belongs to $Kg_i \cdots g_n$, thus proving that $[G:K] \leq n$.

Reviewing the proof of Theorem 1 we can make the following observations, the first being a useful equivalent form of irreducibility.

(1) An n-sequence $g_1, \cdots, g_n$ of G is K-irreducible if and only if the initial (terminal) segments $g_1 \cdots g_i (g_i \cdots g_n)$ form a set of distinct representatives for n right (n left) cosets of K in G.

(2) If $[K:1] = k$ and $[G:K] = n$, then there are precisely $k^{n-1}(n-1)!$ K-irreducible $(n-1)$-sequences $g_1, \cdots, g_{n-1}$ of G.

THEOREM 2. *Let G be a group, K a subgroup of G, and H a union of a finite number n of distinct right cosets of K in G. Then H is a subgroup of G if and only if H is K-n-reducible.*

Proof. If H is a subgroup of G, then H contains 1 and so contains K. Hence $[H:K] = n$ and so, by Theorem 1, H is K-n-reducible.

Conversely, suppose H is K-n-reducible. We will show that H is a subgroup of G. It suffices to prove that (a) $K \subset H$ and that if $h_1, \cdots, h_r$ is an arbitrary K-irreducible sequence of H then (b) there exist $h_{r+1}, \cdots, h_{n-1}$ elements of H such that $h_1, \cdots, h_{n-1}$ is K-irreducible, and (c) $(h_1 \cdots h_r)^{-1}$ is an element of H. Indeed, assume (a) and (c) and let $u, v \in H$; we will show that $u^{-1}v \in H$ ((b) is necessary only for the proof of (c)). If $u^{-1}v \in K$ then by (a), $u^{-1}v \in H$. If either u or v belongs to K, but not both, then as $KH = H$, either $u^{-1}v \in H$ or $v^{-1}u \in H$ and (c) implies that in both cases $u^{-1}v \in H$. Finally assume that none of $u, v, u^{-1}v$ belongs to K; then by (c), $v^{-1} \in H$ and as v^{-1}, u is a K-irreducible sequence of H, (c) again implies that $u^{-1}v \in H$.

Let $x_1, \cdots, x_n$ be a fixed set of representatives of the cosets of K belonging to H. To prove (a) and (b) let $h_1, \cdots, h_r, h_{r+1}, \cdots, h_t$ be a longest K-irreducible sequence of H with the initial sequence $h_1, \cdots, h_r$ and let $h_{t+1} = 1$. Then the sequences $h_1, \cdots, h_t, x_i$ are K-reducible for $i = 1, \cdots, n$ and for each i there exists $s(i)$ such that $h_{s(i)} \cdots h_{t+1} x_i \in K$. If $i \neq j$ then $s(i) \neq s(j)$ since $x_i^{-1}x_j$ does not belong to K. Therefore if no x_i belongs to K then $t \geq n$ and $t \geq n - 1$ if some x_i belongs to K. But by the assumptions $t \leq n - 1$; therefore $t = n - 1$ and some x_i belongs to K. This proves (a) and (b). Also it follows that $s(i)$ is 1-1.

To prove (c) notice that since H is K-n-reducible, the sequences $x_i, h_1, \cdots, h_{n-1}$ are K-reducible for $i = 1, \cdots, n$, and consequently for each i there exists $t(i)$ such that $x_i h_o \cdots h_{t(i)} \in K$, where $h_0 = 1$. Similarly to $s(i)$, also $t(i)$ is 1-1. Therefore there exists x_i such that

$$x_i h_1 \cdots h_r \in K \quad \text{or} \quad (h_1 \cdots h_r)^{-1} \in Kx_i \subset H.$$

The proof of the theorem is complete.

In the statement of Theorem 2 we may replace H by a union of n *left* cosets of K in G since a trivial modification of the proof of Theorem 2 will yield a proof of the new statement. One looks at initial segments of a K-irreducible sequence rather than terminal ones.

COROLLARY 2.1. *Let H be a subset of a group G consisting of a finite number n of elements. Then H is a subgroup of G if and only if n is the least integer such that for every n-sequence $h_1, \cdots, h_n$ of elements of H some consecutive subsequence $h_s, h_{s+1}, \cdots, h_t$ $(1 \leq s \leq t \leq n)$ has its ordered product $h_s h_{s+1} \cdots h_t = 1$, the identity of G.*

Proof. Reread Theorem 2 wth $K = 1$.

COROLLARY 2.2. *If G is a finite group of order n and $g \in G^{\#}$, then there exists a sequence $g = g_1, \cdots, g_{n-1}$ such that*

(a) $g_i g_{i+1} \cdots g_j \neq 1$ *for all* $1 \leq i \leq j \leq n - 1$,
(b) *if* $h \in G^{\#}$ *then there exists* j *such that* $g_1 \cdots g_j = h$.

Proof. For (a), see the proof of Theorem 2, with $K = 1$. (b) follows from (a) which implies that all the initial segments of the above sequence are distinct elements of $G^{\#}$.

REFERENCE

1. HANS ZASSENHAUS, *The theory of groups, 2nd ed.*, New York, Chelsea, 1958.

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