PEAKED PARTITION SUBSPACES OF $C(X)$

BY

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1. Introduction

Throughout this paper, X denotes a compact metric space, and $C(X)$ the Banach space of real (or complex) continuous functions on X. Call $\Phi = \{\phi_1, \cdots, \phi_n\}$ a *peaked partition of unity* on X if it is a partition of unity (i.e. the ϕ_i are non-negative continuous functions on X such that $\sum_{i=1}^n \phi_i(x) = 1$ for all $x \in X$), and $\| \phi_i \| = 1$ for $i = 1, \cdots, n$. The linear subspace $[\Phi]$ of $C(X)$ spanned by such a Φ is called a *peaked partition subspace.* Our purpose in this paper is to prove the following theorem.

THEOREM 1.1. *There exists an increasing sequence $E_1 \subset E_2 \subset \cdots$ of peaked partition subspaces of $C(X)$ whose union is dense in $C(X)$.*

This theorem can be sharpened in two directions: First, as our proof will show, E_1 can be specified arbitrarily in advance. And second, as [2, Corollary 5.2] will show, each E_n can be chosen to be n-dimensional.

Theorem 1.1 seems to be quite useful, and some applications will be found in [2] and [3]. There is one application, however, which is so easy that it can be given right now, while at the same time bringing out the significance of peaked partition subspaces.

Suppose E is spanned by the peaked partition of unity $\Phi = \{\phi_1, \cdots, \phi_n\}$ on X. Pick $x_i \in X$ so that $\phi_i(x_i) = 1$ $(i = 1, \cdots, n)$; then $\phi_j(x_i) = 0$ if $j \neq i$. It follows easily that

$$\| \textstyle\sum_{i=1}^n \alpha_i \phi_i \| = \sup_{i=1}^n | \alpha_i | \tag{1}$$

for all scalars $\alpha_1, \cdots, \alpha_n$. This implies that the linear map $\pi : C(X) \to E$, defined by

$$\pi(f) = \textstyle\sum_{i=1}^n f(x_i)\phi_i, \qquad f \in C(X),$$

is a projection of norm one onto E. More generally, (1) implies (using the Hahn-Banach theorem) that E is a $\mathcal{P}_1$-space; that is, E admits a projection of norm one from *any* Banach space in which it is isometrically embedded. Theorem 1.1 therefore implies:

COROLLARY 1.2. *There exists an increasing sequence of finite-dimensional $\mathcal{P}_1$-subspaces of $C(X)$ whose union is dense in $C(X)$.*

Corollary 1.2 answers a question raised by J. Lindenstrauss [1, p. 29], who also asked whether Corollary 1.2 remains true for non-metrizable compact X if

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"increasing sequence" is replaced by "directed set". This more general question seems to remain open.

The proof of Theorem 1.1 will be given in Sections 2 and 3, the essential step being Lemma 2.2. Section 4 contains examples, as well as some remarks on how the proof of Theorem 1.1 becomes fairly trivial if the theorem is weakened in certain directions.

2. Three lemmas

We begin with a known lemma, and include the simple proof for completeness. (A similar result (with the same proof) is true for all locally finite open coverings of normal spaces.)

LEMMA 2.1. *Let $\{U_1, \cdots, U_n\}$ be an open covering of X, and let $x_i \in U_i$ with $x_i \neq x_j$ for $i \neq j$. Then there exists a peaked partition of unity $\{\phi_1, \cdots, \phi_n\}$ on X such that ϕ_i vanishes outside U_i and $\phi_i(x_i) = 1$ for $i = 1, \cdots, n$.*

Proof. If $V_i = U_i - \{x_j : j \neq i\}$, then $\{V_1, \cdots, V_n\}$ is also an open covering of X. Let $\{\phi_1, \cdots, \phi_n\}$ be *any* partition of unity on X such that ϕ_i vanishes outside V_i for all i.[3] Then $\phi_j(x_i) = 0$ for $j \neq i$, so $\phi_i(x_i) = 1$, and that completes the proof.

If $\Phi = \{\phi_1, \cdots, \phi_n\}$ is a partition of unity on X, if $\mathfrak{U}$ is a covering of X, and if $\varepsilon > 0$, we say that Φ is *ε-subordinated* to $\mathfrak{U}$ if there are $U_i \in \mathfrak{U}$ $(i = 1, \cdots, n)$ such that

$$\sum_{i \in I(x)} \phi_i(x) \leq \varepsilon$$

for all $x \in X$, where

$$I(x) = \{i \leq n : x \notin U_i\}.$$

Note that "0-subordinated" is what is usually called "subordinated".

We are now ready for Lemma 2.2. As Example 4.2 will show, "ε-subordinated" cannot be replaced by "subordinated".

LEMMA 2.2. *Let $\Phi = \{\phi_1, \cdots, \phi_n\}$ be a peaked partition of unity on X, let $\mathfrak{U}$ be an open covering of X, and let $\varepsilon > 0$. Then there exists a peaked partition of unity $\Psi = \{\psi_1, \cdots, \psi_m\}$ on X which is ε-subordinated to $\mathfrak{U}$ and with $[\Phi] \subset [\Psi]$.*

Proof. Pick $x_1, \cdots, x_n$ in X such that $\phi_i(x_i) = 1$ for $i = 1, \cdots, n$. We will obtain $\{\psi_1, \cdots, \psi_m\}$, and $z_1, \cdots, z_m$ in X with $\psi_j(z_j) = 1$ for $j = 1, \cdots, m$, so that $m \geq n$ and $z_j = x_j$ for $1 \leq j \leq n$. Let λ denote $1 - \varepsilon$; we may clearly suppose that $\epsilon < 1$, so that $0 < \lambda < 1$.

For each $z \in X$, pick $U_z \in \mathfrak{U}$ such that $z \in U_z$, let

$$(2) \qquad \begin{aligned} U_{z,i} &= U_z, && \text{if} \quad \phi_i(z) = 0, \\ U_{z,i} &= U_z \cap \{x \in X : \phi_i(x) > \lambda\phi_i(z)\}, && \text{if} \quad \phi_i(z) > 0, \end{aligned}$$

[3] For instance, let $\phi_i(x) = d(x, X - V_i)[\sum_{j=1}^{n} d(x, X - V_j)]^{-1}$, where d is a metric on X.

and let

$$U(z) = \bigcap_{i=1}^{n} U_{z,i}.$$

Then $U(z)$ is an open set containing z, so we can find $z_1, \cdots, z_m$ in X such that the $U(z_j)$ with $j = 1, \cdots, m$ cover X; clearly we can take $m \geq n$ and $z_j = x_j$ for $1 \leq j \leq n$.

By Lemma 2.1 there exists a peaked partition of unity $\Psi^* = \{\psi_1^*, \cdots, \psi_m^*\}$ on X such that ψ_j^* vanishes outside $U(z_j)$ and $\psi_j^*(z_j) = 1$ for $j = 1, \cdots, m$. For each $i = 1, \cdots, n$, let

$$\phi_i^*(x) = \sum_{j=1}^{m} \phi_i(z_j)\psi_j^*(x). \tag{3}$$

(In other words, ϕ_i^* is the image of ϕ_i under the projection from $C(X)$ onto $[\Psi^*]$ generated by the ψ_j^* and the z_j.) Let us show that, for $i = 1, \cdots, n$,

$$\phi_i^*(z_j) = \phi_i(z_j), \qquad j = 1, \cdots, m, \tag{4}$$

and

$$\phi_i(x) \geq \lambda\phi_i^*(x), \qquad x \in X. \tag{5}$$

The truth of (4) follows from the fact that $\psi_k^*(z_j) = \delta_{k,j}$. To prove (5), note that if $\psi_j^*(x) > 0$ (so that $x \in U(z_j)$), then

$$\phi_i(x) \geq \lambda\phi_i(z_j);$$

this is clear if $\phi_i(z_j) = 0$, and follows from the definition of $U(z_j)$ if $\phi_i(z_j) > 0$. Hence

$$\sum_{j=1}^{m} \phi_i(x)\psi_j^*(x) \geq \lambda \sum_{j=1}^{m} \phi_i(z_j)\psi_j^*(x),$$

and that is equivalent to (5).

For all $x \in X$ and $i = 1, \cdots, n$, let

$$\gamma_i(x) = \sup\{\gamma \leq 1 : \phi_i(x) \geq \gamma\phi_i^*(x)\},$$

$$\gamma(x) = \min\{\gamma_i(x) : i = 1, \cdots, n\}.$$

Then

$$\gamma(x) \geq \lambda \tag{6}$$

for all $x \in X$ by (5),

$$\gamma(z_j) = 1 \tag{7}$$

for $j = 1, \cdots, m$ by (4), and

$$\phi_i(x) \geq \gamma(x)\phi_i^*(x) \tag{8}$$

for all $x \in X$ and $i = 1, \cdots, n$ by definition of $\gamma(x)$.

Let us show that each function γ_i—and hence γ—is continuous. This is clear on the open set $\{x \in X : \phi_i^*(x) > 0\}$, for there

$$\gamma_i(x) = \min(1, \phi_i(x)/\phi_i^*(x)).$$

So suppose that $\phi_i^*(y) = 0$, and let us show that γ_i is continuous at y. Since $\gamma_i(y) = 1$, it will suffice to show that $\gamma_i(x) = 1$ for all x in some neighborhood W_i of y. If $\phi_i(y) > 0$ this is clear, so suppose that $\phi_i(y) = 0$. Let

$$J_i = \{j \leq m : \phi_i(z_j) > 0\}, \tag{9}$$

$$W_i = \{x \in X : \phi_i(x) < \lambda\phi_i(z_j) \text{ for all } j \in J_i\}. \tag{10}$$

Then W_i is open, and $y \in W_i$ since $\phi_i(y) = 0$. Now suppose that $x \in W_i$, and let us show that $\phi_i^*(x) = 0$, which implies that $\gamma_i(x) = 1$. By (3), it suffices to show that, if $\phi_i(z_j) > 0$ for some $j \leq m$, then $\psi_j^*(x) = 0$. But if $\phi_i(z_j) > 0$, then $j \in J_i$ by (9), so (10) and (2) imply that $x \notin U_{z_j,i}$; but $U_{z_j,i} \supset U(z_j)$, and hence $\psi_j^*(x) = 0$. That completes the proof that γ_i is continuous.

We are now ready to define $\Psi = (\psi_1, \cdots, \psi_j\}$. Let

$$\psi_j(x) = \gamma(x)\psi_j^*(x) + (\phi^j(x) - \gamma(x)\phi_j^*(x)), \quad 1 \leq j \leq n,$$

$$\psi_j(x) = \gamma(x)\psi_j^*(x), \qquad n < j \leq m.$$

Then, for all $j = 1, \cdots, m$, ψ_j is clearly continuous,

$$\psi_j(x) \geq \gamma(x)\psi_j^*(x) \geq 0 \tag{11}$$

by (6) and (8), and $\psi_j(z_j) = 1$ by (4) and (7). Moreover, by (3),

$$\textstyle\sum_{j=1}^m \phi_i(z_j)\psi_j(x) = \gamma(x)\phi_i^*(x) + \phi_i(x) - \gamma(x)\phi_i^*(x) = \phi_i(x),$$

and hence $\phi_i \in [\Psi]$ for $i = 1, \cdots, n$. This equation also implies that, for all $x \in X$,

$$\begin{aligned}\textstyle\sum_{j=1}^m \psi_j(x) &= \textstyle\sum_{j=1}^m (\sum_{i=1}^n \phi_i(z_j))\psi_j(x)\\ &= \textstyle\sum_{i=1}^n(\sum_{j=1}^m \phi_i(z_j)\psi_j(x))\\ &= \textstyle\sum_{i=1}^n \phi_i(x) = 1,\end{aligned}$$

so Ψ is a partition of unity on X. To see, finally, that Ψ is ε-subordinated to $\mathfrak{U}$, pick $U_j \in \mathfrak{U}$ so that $U(z_j) \subset U_j$ for $j = 1, \cdots, m$. Let $x \in X$, and let $J(x) = \{j \leq m : x \notin U_j\}$ and $K(x) = \{j \leq m : x \in U_j\}$. Then, by (11),

$$\begin{aligned}\textstyle\sum_{j\in J(x)}\psi_j(x) &= 1 - \textstyle\sum_{j\in K(x)}\psi_j(x) \leq 1 - \gamma(x)\sum_{j\in K(x)} \psi_j^*(x)\\ &= 1 - \gamma(x)\textstyle\sum_{j=1}^m \psi_j^*(x) = 1 - \gamma(x) \leq 1 - \lambda = \varepsilon,\end{aligned}$$

and that completes the proof.

LEMMA 2.3. *Let $f \in C(X)$ and $\varepsilon > 0$. Then there exists an $\alpha > 0$ and an open covering $\mathfrak{U}$ of X such that, if Φ is a partition of unity on X which is α-subordinated to $\mathfrak{U}$, then $\| f - f' \| < \varepsilon$ for some $f' \in [\Phi]$.*

Proof. If $f = 0$, there is nothing to prove. If not, let $\alpha = \varepsilon/(4\| f \|)$. For each scalar y with $| y | \leq \| f \|$, let

$$U(y) = \{x \in X : | f(x) - y | < \tfrac{1}{2}\varepsilon\},$$

and let $\mathfrak{U}$ be the collection of all such $U(y)$. Then $\mathfrak{U}$ is certainly an open covering of X.

Suppose now that $\Phi = \{\phi_1, \cdots, \phi_n\}$ is α-subordinated to $\mathfrak{U}$. Then there exist scalars $y_1, \cdots, y_n$, with $|y_i| \leq \|f\|$ for all i, so that

$$\sum_{i \in I(x)} \phi_i(x) < \alpha$$

for all x, where $I(x) = \{i \leq n : x \notin U(y_i)\}$. Let

$$f' = \sum_{1=1}^{n} y_i \phi_i, \tag{12}$$

and let us check that $\|f - f'\| < \varepsilon$. If $x \in X$, then

$$\begin{aligned} |f(x) - f'(x)| &= |\textstyle\sum_{i=1}^{n} \phi_i(x)(f(x) - y_i)| \\ &\leq \textstyle\sum_{i=1}^{n} \phi_i(x)|f(x) - y_i| \\ &= (\textstyle\sum_{i \notin I(x)} + \sum_{i \in I(x)})(\phi_i(x)|f(x) - y_i|) \\ &\leq \sup_{i \notin I(x)} |f(x) - y_i| + \alpha \sup_{i=1}^{n} |f(x) - y_i| \\ &\leq \tfrac{1}{2}\varepsilon + 2\alpha\|f\| = \varepsilon. \end{aligned}$$

Hence $\|f - f'\| < \varepsilon$, and that completes the proof.

In conclusion, it should be observed that, if $\Phi = \{\phi_1, \cdots, \phi_n\}$ in Lemma 2.3 is actually a *peaked* partition of unity, and if $\phi_i(x_i) = 1$ for $i = 1, \cdots, n$, then formula (12) for f' can be modified to

$$f'(x) = \sum_{i=1}^{n} f(x_i)\phi_i(x).$$

In other words, f' is the image of f under the projection from $C(X)$ onto $[\Phi]$ determined by the ϕ_i and the x_i.

3. Proof of Theorem 1.1

Since X is compact metric, it has a sequence $\mathfrak{U}_n$ $(n = 1, 2, \cdots)$ of finite open coverings so that each element of $\mathfrak{U}_n$ has diameter $<1/n$. It follows that, if $\mathfrak{U}$ is any open covering of X, then $\mathfrak{U}_n$ is a refinement of $\mathfrak{U}$ for all sufficiently large n.

Let us now construct the peaked partitions of unity Φ_n which span the required E_n. Let Φ_1 be arbitrary; for instance, we can take $\Phi_1 = \{1\}$. Now apply Lemmas 2.2 and 2.3 inductively to construct peaked partitions of unity Φ_n such that $[\Phi_n] \subset [\Phi_{n+1}]$ for all n, and Φ_n is $1/n$-subordinated to $\mathfrak{U}_n$ for $n > 1$. Let $E_n = [\Phi_n]$.

To see that $\bigcup_{n=1}^{\infty} E_n$ is dense in $C(X)$, let $f \in C(X)$ and $\varepsilon > 0$. Pick $\alpha > 0$ and an open covering $\mathfrak{U}$ of X as in Lemma 2.3, and then pick $n > 1$ large enough so that $1/n < \alpha$ and $\mathfrak{U}_n$ is a refinement of $\mathfrak{U}$. Then Φ_n is α-subordinated to $\mathfrak{U}$, so by Lemma 2.3 there is an $f' \in [\Phi_n]$ with $\|f - f'\| < \varepsilon$. That completes the proof.

4. Concluding examples and remarks

We begin with a very simple lemma which is needed in the proofs of Example 4.2 and Lemma 4.4.

LEMMA 4.1. *If* $\Phi = \{\phi_1, \cdots, \phi_n\}$ *is a peaked partition of unity on* X, *if* $\phi_i(x_i) = 1$ *for* $i = 1, \cdots, n$, *and if* $f \in [\Phi]$, *then*

$$f(x) = \sum_{i=1}^{n} f(x_i)\phi_i(x)$$

for every $x \in X$.

Proof. By assumption, there are scalars $\alpha_1, \cdots, \alpha_n$ such that $f = \sum_{i=1}^{n} \alpha_i \phi_i$. Evaluating this equation at x_i yields $f(x_i) = \alpha_i$, and that proves the lemma.

Our first example shows that Lemma 2.2 becomes false if "ε-subordinated" is replaced by "subordinated".

EXAMPLE 4.2. *If* X *is the interval* $[0, 1]$, *there exists a peaked partition of unity* $\Phi = \{\phi_1, \phi_2, \phi_3\}$ *on* X *and an open cover* $\mathfrak{U} = \{U, V\}$ *of* X, *such that there is no peaked partition of unity* Ψ *on* X *which is subordinated to* $\mathfrak{U}$ *and such that* $[\Phi] \subset [\Psi]$.

Proof. Let T be the triangle in $\mathbf{R}^3$ spanned by the vertices $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, and $v_3 = (0, 0, 1)$. Let $\phi : X \to T$ be any continuous function which maps $[0, \frac{1}{2}]$ homeomorphically onto a circular arc $C \subset T$, and such that $\phi(X) \supset \{v_1, v_2, v_3\}$. For $i = 1, 2, 3$, let

$$\phi_i(x) = (\phi(x))_i, \qquad x \in X.$$

Then $\Phi = \{\phi_1, \phi_2, \phi_3\}$ is clearly a peaked partition of unity on X. Let $U = [0, \frac{1}{2})$, $V = (\frac{1}{3}, 1]$, and $\mathfrak{U} = \{U, V\}$.

Let $\Psi = \{\psi_1, \cdots, \psi_m\}$ be a peaked partition of unity on X which is subordinated to $\mathfrak{U}$. For all $j = 1, \cdots, m$, pick $z_j \in X$ so that $\psi_j(z_j) = 1$. Let

$$J = \{j \leq m : \psi_j \text{ vanishes outside } V\},$$
$$K = \{j \leq m : j \notin J\}.$$

Suppose now that $[\Phi] \subset [\Psi]$. Then

$$\phi_i(x) = \sum_{j=1}^{m} \psi_j(x)\phi_i(z_j), \qquad x \in X$$

for $i = 1, 2, 3$ by Lemma 4.1, and hence

$$\phi(x) = \sum_{j=1}^{m} \psi_j(x)\phi(z_j), \qquad x \in X.$$

Consider $A = X - V$. If $x \in A$, then $x \notin V$, so $\psi_j(x) > 0$ implies that $j \in K$. Hence

$$\phi(x) = \sum_{j \in K} \psi_j(x)\phi(z_j), \qquad x \in A. \tag{13}$$

But if $j \in K$, then ψ_j vanishes outside U, so $z_j \in U$, and hence $\phi(z_j) \in C$. But now (13) implies that $\phi(A)$, which is a sub-arc of C, lies in the convex hull of the finite subset $\{\phi(z_j) : j \in K\}$ of C, and that is impossible. This contradiction completes the proof.

The proof of Theorem 1.1 could have been considerably simplified if, whenever E_1 and E_2 were peaked partition subspaces of $C(X)$, there existed a peaked partition subspace E_3 of $C(X)$ containing both E_1 and E_2. The following example shows that this is false. We use real scalars for convenience.

EXAMPLE 4.3. *Let* $X = [0, 2\pi]$, *and let* $\phi_1(x) = \frac{1}{2}(1 + \sin x)$, $\phi_2(x) = \frac{1}{2}(1 - \sin x)$, $\psi_1(x) = \frac{1}{2}(1 + \cos x)$, *and* $\psi_2(x) = \frac{1}{2}(1 - \cos x)$ *for* $x \in X$. *Then* $\Phi = \{\phi_1, \phi_2\}$ *and* $\Psi = \{\psi_1, \psi_2\}$ *are peaked partitions of unity on* X, *but there is no peaked partition subspace* E *of* X *which contains both* $[\Phi]$ *and* $[\Psi]$.

Proof. That Φ and Ψ are peaked partitions of unity is clear. Suppose that E is a subspace of $C(X)$ containing both $[\Phi]$ and $[\Psi]$. Let $f(x) = \sin x$ and $g(x) = \cos x$ for $x \in X$, and let $F = [\{f, g\}]$. Since $f \in [\Phi]$ and $g \in [\Psi]$, we have $F \subset E$. Using Schwarz's inequality, it is easily checked that

$$\| \alpha f + \beta g \| = \alpha^2 + \beta^2$$

for any scalar α and β, so that the unit sphere of F is a circle. Now if E is a peaked partition subspace of $C(X)$, then formula (1) in the introduction implies that the unit sphere of E is the surface of a cube (n-dimensional), so that the unit sphere of any subspace of E is a polyhedron. But a circle is not a polyhedron, and this contradiction completes the proof.

The following lemma, which may have some independent interest, is used in Example 4.5 below.

LEMMA 4.4. *Let* $\{E_n\}$ *be as in Theorem* 1.1. *If* E_n *is spanned by the peaked partition of unity* $\Phi^n = \{\phi_1^n, \cdots, \phi_{k(n)}^n\}$, *and if* $\phi_i^n(x_i^n) = 1$, *then*

$$D = \{x_i^n : i = 1, \cdots, k(n); n = 1, 2, \cdots\}$$

is dense in X.

Proof. Suppose not. Then there is a non-empty open set $U \subset X$ which contains no x_i^n. Let $y \in U$, and pick $f \in C(X)$ so that $f(y) = 1$ and $f(X - U) = 0$. Let us show that there is no g in any E_n such that $\| f - g \| < \frac{1}{2}$, so that $\bigcup_{n=1}^{\infty} E_n$ cannot be dense in $C(X)$.

Suppose that there were such an n and g. Then

$$g(x) = \sum_{i=1}^{k(n)} g(x_i^n)\phi_i^n(x), \qquad x \in X \tag{14}$$

by Lemma 4.1. Now for $i = 1, \cdots, k(n)$ we have $x_i^n \notin U$, hence $f(x_i^n) = 0$, and therefore $g(x_i^n) < \frac{1}{2}$. By (14), it follows that $g(x) < \frac{1}{2}$ for all $x \in X$. But then $|f(y) - g(y)| > \frac{1}{2}$, hence $\| f - g \| > \frac{1}{2}$, and this contradiction completes the proof.

Our last example deals with differentiable functions on $X = [0, 1]$, but a similar result is true if X is any compact differentiable manifold, with or without boundary.

EXAMPLE 4.5. *If $X = [0, 1]$, then the spaces E_n in Theorem 1.1 cannot all consist of functions having continuous derivatives on $(0, 1)$.*

Proof. Let us first show that, if $\Phi = \{\phi_1, \cdots, \phi_k\}$ is a differentiable partition of unity on $(0, 1)$, and if $\phi_i(x) = 1$ for some i and some $x \in (0, 1)$, then $f'(x) = 0$ for all $f \in [\Phi]$: In fact, since ϕ_i has a maximum at x, $\phi_i'(x) = 0$. But if $j \neq i$, then $\phi_j(x) = 0$, so ϕ_j has a minimum at x, and hence $\phi_j'(x) = 0$. Hence $f'(x) = 0$ for all $f \in [\Phi]$.

Suppose now that the assertion of the example is false. Let $f \in E_n$ for some n. Then $f \in E_m$ for all $m \geq n$, so the previous paragraph and Lemma 4.4 imply that f' is 0 on a dense subset of $(0, 1)$. Since f' is assumed continuous, f' is 0 on all of $(0, 1)$, and hence f is constant. So $\bigcup_{n=1}^{\infty} E_n$ contains only constant functions and thus cannot be dense. This contradiction completes the proof.

We conclude this paper by showing how the proof of Theorem 1.1 becomes much simpler if the statement of the theorem is weakened in either of two directions.

First, suppose the requirement that the sequence E_n be *increasing* is dropped. Then Lemma 2.2 becomes superfluous, and can simply be replaced in the proof by Lemma 2.1. Moreover, Lemma 2.3 can be simplified, since we need only consider subordinated—rather than ε-subordinated—partitions of unity.

Second, suppose we dropped the word "peaked" from the statement. Now Lemma 2.1 becomes superfluous. Lemma 2.2 cannot now be eliminated, but with the word "peaked" removed it becomes rather trivial: In fact, if $P = \{p_1, \cdots, p_m\}$ is *any* partition of unity subordinated to $\mathfrak{U}$, let

$$\Psi = \{\phi_i p_j : i \leq n, j \leq m\}.$$

Then Ψ is also a partition of unity subordinated to $\mathfrak{U}$, and $[\Phi] \subset [\Psi]$. Note that here Ψ is actually subordinated—rather than merely ε-subordinated—to $\mathfrak{U}$, so Lemma 2.3 can again be simplified.

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