ON EQUICONTINUITY OF HARMONIC FUNCTIONS IN AXIOMATIC POTENTIAL THEORY

BY

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Recently there have been several results in the literature on equicontinuity of harmonic functions in axiomatic potential theory ([5], [9] and [6]). The purpose of this note is to place these results in the setting of compact and weakly compact operators in Banach spaces. Our note is close in spirit to [6]; our results are somewhat more general and our proofs particularly simple.

1. Preliminaries. If E is a Banach space, we denote by E' its dual. Whenever we speak of bounded, compact or relatively compact sets in E, this will always mean in the norm topology of E. Occasionally we shall use the terms strong, strongly: these will always refer to the norm topology of E, while the terms weak, weakly will refer to the weak topology $\sigma(E, E')$ of E.

For any set Z we denote by $M^\infty(Z)$ the algebra of all bounded real-valued functions on Z, endowed with the supremum norm,

$$f \to \| f \|_\infty = \sup_{z \in Z} | f(z) |.$$

Suppose now that Z is a locally compact space. We denote by $\mathcal{C}(Z)$ the algebra of all continuous real-valued functions on Z and by $\mathcal{C}^0(Z)$ the subalgebra consisting of the functions vanishing at infinity [3]; further we denote by $\mathcal{B}_0(Z)$, $\mathcal{B}(Z)$ and $\mathcal{U}(Z)$ the σ-algebra of all Baire subsets of Z (generated by the compact G_δ's), the σ-algebra of all Borel subsets of Z (generated by the closed sets), and the σ-algebra of all universally measurable subsets of Z, respectively. We shall sometimes consider the one-point compactification of Z, $Z^* = Z \cup \{\infty\}$ and we shall identify $\mathcal{C}^0(Z)$ with a closed subspace of $\mathcal{C}(Z^*)$.

Let now $(X, \mathfrak{I})$ be a measurable space, i.e. X a set and $\mathfrak{I}$ a σ-algebra of subsets of X. We denote by $M^\infty(X, \mathfrak{I})$ the algebra of all bounded real-valued $\mathfrak{I}$-measurable functions on X. Clearly $M^\infty(X, \mathfrak{I})$ is a closed vector space of the Banach space $M^\infty(X)$. We denote by $\mathfrak{M}(X, \mathfrak{I})$ the vector space of all real-valued countably additive measures on $\mathfrak{I}$, endowed with the "total variation" norm $\mu \to \| \mu \|$. In what follows we identify $\mathfrak{M}(X, \mathfrak{I})$ with a closed subspace of the dual space $(M^\infty(X, \mathfrak{I}))'$ via the formulas

$$\langle f, \mu \rangle = \int f \, d\mu, \quad f \in M^\infty(X, \mathfrak{I}), \mu \in \mathfrak{M}(X, \mathfrak{I}).$$

If X is a locally compact space, then clearly

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$$\mathcal{C}^0(X) \subset M^\infty(X, \mathcal{B}_0(X)) \subset M^\infty(X, \mathcal{B}(X)) \subset M^\infty(X, \mathfrak{U}(X)) \subset M^\infty(X),$$

and each of these is a closed vector space of the Banach space $M^\infty(X)$.

We shall now recall a theorem on compact and weakly compact operators in Banach spaces due to Thorp and Whitley. Let E be a Banach space, Z a set and $S : E \to M^\infty(Z)$ a *continuous linear mapping*. Then S can be represented in the form

$$(Sx)(z) = \langle x, \lambda_z \rangle, \qquad x \in E$$

where $\lambda_z \in E'$ for each $z \in Z$ and $\lambda : z \to \lambda_z$ is a *bounded* mapping of Z into E'. Let $\lambda(Z) = \{\lambda_z \mid z \in Z\}$. Then (see [10, p. 597]; see also [1] and [7]):

THEOREM (Thorp and Whitley). (I) *The operator* $S : E \to M^\infty(Z)$ *is compact if and only if* $\lambda(Z)$ *is relatively compact in* E'.

(II) *The operator* $S : E \to M^\infty(Z)$ *is weakly compact if and only if* $\lambda(Z)$ *is relatively weakly compact in* E'.

2. We shall state here the main theorem; as will be seen, this is an easy consequence of the Thorp-Whitley theorem and yields all the known results on weak compactness and compactness of operators arising in axiomatic potential theory in connection with equicontinuity of harmonic functions (Sections 3 and 4).

Throughout this section it is assumed that $(X, \mathfrak{I})$ *and* $(Y, \mathfrak{F})$ are *measurable spaces* and $S : M^\infty(X, \mathfrak{I}) \to M^\infty(Y, \mathfrak{F})$ is a *continuous linear mapping*. For each $y \in Y$ the equations

$$(Sf)(y) = \langle f, \lambda_y \rangle, \qquad f \in M^\infty(X, \mathfrak{I})$$

uniquely determine λ_y as an element of $(M^\infty(X, \mathfrak{I}))'$ and $\lambda : y \to \lambda_y$ is a *bounded* mapping of Y into $(M^\infty(X, \mathfrak{I}))'$. We shall say that S is *represented* by the family $(\lambda_y)_{y \in Y}$ and write $\lambda(Y) = \{\lambda_y \mid y \in Y\}$. It is clear that S maps bounded pointwise convergent sequences into pointwise convergent sequences if and only if S is represented by measures, that is $\lambda(Y) \subset \mathfrak{M}(X, \mathfrak{I})$.

Below we shall make constant use of the Eberlein-Smulian theorem [7, p. 430].

THEOREM 1. *Suppose that* $\lambda(Y) \subset \mathfrak{M}(X, \mathfrak{I})$. *Then the following assertions are equivalent*:

(1) *If* $(f_n) \subset M^\infty(X, \mathfrak{I})$ *is a bounded decreasing sequence with pointwise limit* f, *then* (Sf_n) *converges strongly to* Sf.

(2) *If* $(f_n) \subset M^\infty(X, \mathfrak{I})$ *is a bounded pointwise convergent sequence with pointwise limit* f, *then* (Sf_n) *converges strongly to* Sf.

(3) *There is* $\mu \in \mathfrak{M}_+(X, \mathfrak{I})$ *such that every* λ_y *is absolutely continuous with respect to* μ. *Furthermore, if we denote by* G_y (a version of) *the Radon-Nikodym derivative of* λ_y *with respect to* μ, *i.e.*

$$(Sf)(y) = \langle f, \lambda_y \rangle = \int f(x) G_y(x)\, d\mu(x), \qquad f \in M^\infty(X, \mathfrak{I})$$

then the set $\{G_y \mid y \in Y\} \subset \mathfrak{L}^1(X, \mathfrak{I}, \mu)$ *is uniformly integrable.*

(4) $\lambda(Y)$ *is relatively weakly compact in* $(M^\infty(X, \mathfrak{J}))'$.
(5) $S : M^\infty(X, \mathfrak{J}) \to M^\infty(Y, \mathfrak{F})$ *is weakly compact.*

Proof. (5) $\Leftrightarrow$ (4) by the Thorp-Whitley theorem.

(4) $\Rightarrow$ (3). Since $\mathfrak{M}(X, \mathfrak{J})$ is a closed subspace of $(M^\infty(X, \mathfrak{J}))'$, it follows that $\lambda(Y)$ as a subset of the Banach space $\mathfrak{M}(X, \mathfrak{J})$ is relatively weakly compact and thus weakly sequentially compact. There is then $\mu \in \mathfrak{M}_+(X, \mathfrak{J})$ such that $\lim_{\mu(E)\to 0}|\lambda_y(E)| = 0$ uniformly for $y \in Y$ [7, p. 306–307]. This shows that every λ_y is absolutely continuous with respect to μ and that the absolute continuity is uniform for $y \in Y$. By a well-known theorem of measure theory, this implies that the set

$$\{G_y \mid y \in Y\} \subset \mathfrak{L}^1(X, \mathfrak{J}, \mu)$$

is uniformly integrable (since it is also bounded).

(3) $\Rightarrow$ (2) is immediate.

Since (2) $\Rightarrow$ (1) obviously, it remains to show (1) $\Rightarrow$ (4). Note that if $(E_n) \subset \mathfrak{J}$ is a decreasing sequence with $\bigcap_n E_n = \emptyset$, then $\lim_n|\lambda_y(E_n)| = 0$ uniformly for $y \in Y$. Since the countable additivity of λ_y is uniform with respect to $y \in Y$ and since the set $\lambda(Y)$ is also bounded in norm, it follows that $\lambda(Y)$ as a subset of the Banach space $\mathfrak{M}(X, \mathfrak{J})$ is weakly sequentially compact [7, p. 305] and thus relatively weakly compact. As $\mathfrak{M}(X, \mathfrak{J})$ is a closed subspace of $(M^\infty(X, \mathfrak{J}))'$, we conclude that $\lambda(Y)$ as a subset of $(M^\infty(X, \mathfrak{J}))'$ is relatively weakly compact. This completes the proof of the theorem.

Remarks. (1) We could have also derived Theorem 1 (at least part of the implications) from the general representation theorem of weakly compact operators from $\mathcal{C}(Z)$ to E (Z a compact space, E a Banach space) given in terms of vector-valued measures [7, p. 493] or by using a profound result of Grothendieck ($S : \mathcal{C}(Z) \to E$ is weakly compact if and only if it maps weakly convergent sequences into strongly convergent sequences; see [8]). We preferred however to avoid vector-valued measures; on the other hand, for the purposes of measure theory, it is more natural to consider pointwise convergence than weak convergence for sequences of functions belonging to $M^\infty(X, \mathfrak{J})$. That is why we used the Thorp-Whitley theorem instead.

(2) Under the conditions of Theorem 1, S maps relatively weakly compact subsets of $M^\infty(X, \mathfrak{J})$ into relatively compact subsets of $M^\infty(Y, \mathfrak{F})$ use the Eberlein-Smulian theorem (see also [7, p. 494] and [8]).

Let now $(Z, \mathcal{G})$ be *another measurable space* and $T : M^\infty(Y, \mathfrak{F}) \to M^\infty(Z, \mathcal{G})$ a *continuous linear mapping*. Suppose that T is *represented* by the family $(\mu_z)_{z\in Z}$. From Theorem 1 and Remark (2) above we obtain (see also [7, p. 494]):

THEOREM 2. *Suppose that* $\lambda(Y) \subset \mathfrak{M}(X, \mathfrak{J})$, $\mu(Z) \subset \mathfrak{M}(Y, \mathfrak{F})$ *and that* S *and* T *are weakly compact. Then* $T \circ S : M^\infty(X, \mathfrak{J}) \to M^\infty(Z, \mathcal{G})$ *is compact.*

3. Throughout this section we assume that $(X, \mathfrak{J})$ is a *measurable space,* Y is a *locally compact space,* and $S : M^\infty(X, \mathfrak{J}) \to \mathcal{C}^0(Y)$ a *positive linear* *mapping* (hence continuous). Suppose that S is *represented* by the family $(\lambda_y)_{y \in Y}$.

THEOREM 3. *Suppose that* $\lambda(Y) \subset \mathfrak{M}(X, \mathfrak{J})$. *Then*
(i) $S : M^\infty(X, \mathfrak{J}) \to \mathcal{C}^0(Y)$ *is weakly compact;*
(ii) *for any closed vector space satisfying the inclusions*

$$\mathcal{C}^0(Y) \subset E \subset M^\infty (Y),$$

S regarded as an operator of $M^\infty(X, \mathfrak{J})$ into E is weakly compact.

Proof. It is enough to note that S satisfies statement (1) of Theorem 1 (by Dini's theorem) and to use the equivalence (1) $\Leftrightarrow$ (5).

Remark. In connection with Theorem 3 see also Theorem VI.7.1 in [7, p. 490].

4. Suppose now that X, Y, Z are *locally compact spaces,*

$$S : M^\infty(X, \mathfrak{U}(X)) \to \mathcal{C}^0(Y) \quad \textit{and} \quad T : M^\infty(Y, \mathfrak{U}(Y)) \to \mathcal{C}^0(Z)$$

are *positive linear mappings.* Assume that S is *represented* by the family $(\lambda_y)_{y \in Y}$ and T is *represented* by the family $(\mu_z)_{z \in Z}$. Using the Arzela-Ascoli theorem [7, p. 266] we obtain from Theorem 3 and Theorem 2:

THEOREM 4 (C. Constantinescu). *Suppose that λ_y is a Radon measure on X for each $y \in Y$ and μ_z is a Radon measure on Y for each $z \in Z$. Then the operator*

$$T \circ S : M^\infty(X, \mathfrak{U}(X)) \to \mathcal{C}^0(Z)$$

is compact and thus maps bounded subsets of $M^\infty(X, \mathfrak{U}(X))$ into bounded equicontinuous subsets of $\mathcal{C}^0(Z)$.

Remarks. (1) Theorem 4 obviously remains true if in the above statements we replace everywhere $\mathfrak{U}(X)$, $\mathfrak{U}(Y)$ by $\mathfrak{B}(X)$, $\mathfrak{B}(Y)$ or by $\mathfrak{B}_0(X)$, $\mathfrak{B}_0(Y)$ respectively.

(2) In probabilistic language, the condition $S(M^\infty(X, \mathfrak{B}(X))) \subset \mathcal{C}^0(Y)$ means that the kernel $\lambda : y \to \lambda_y$ is strongly-Feller.

5. We shall make here several remarks which will lead to an elementary, short, self-contained proof of Theorem 3 (independent of the Thorp-Whitley theorem and of Theorem 1 above, or of Theorem VI.7.1 in [7, p. 490]).

Let $(X, \mathfrak{J})$ be a *measurable space.*

Remarks. (1) Let $A \subset \mathfrak{M}(X, \mathfrak{J})$ be a set with the property that there is $\nu \in \mathfrak{M}_+(X, \mathfrak{J})$ such that every $\mu \in A$ is absolutely continuous with respect to ν. Then, for each $\rho \in (M^\infty(X, \mathfrak{J}))''$, there is $u_\rho \in M^\infty(X, \mathfrak{J})$ such that $\langle \rho, \mu \rangle = \langle u_\rho , \mu \rangle$ for all $\mu \in A$; in particular,

$$\sigma((M^\infty(X, \mathfrak{J}))', M^\infty(X, \mathfrak{J})) \quad \text{and} \quad \sigma((M^\infty(X, \mathfrak{J}))', (M^\infty(X, \mathfrak{J}))'')$$

induce the same topology on A (note that, for fixed $\rho \in (M^\infty(X, \mathfrak{J}))''$, $g \to \langle \rho, g. \nu \rangle$ is a continuous linear functional on $\mathcal{L}^1(X, \mathfrak{J}, \nu)$).

(2) If $A \subset \mathfrak{M}(X, \mathfrak{I})$ is any countably infinite set, then A satisfies the assumption in Remark (1) and thus the conclusion of Remark (1) applies (note that if $A = \{\mu_1, \mu_2, \cdots, \mu_n, \cdots\}$, then every μ_n is absolutely continuous with respect to the measure ν on $\mathfrak{I}$ defined by $\nu = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\mu_n|}{1 + |\mu_n|(X)}$).

(3) Let E be a Banach space, $A \subset E'$ a $\sigma(E', E)$-compact set. Then A is weakly compact if and only if on every countably infinite subset of A, $\sigma(E', E)$ and $\sigma(E', E'')$ induce the same topology (apply the Eberlein-Smulian theorem).

Let now Y *be a locally compact space* and $S : M^{\infty}(X, \mathfrak{I}) \to \mathcal{C}^0(Y)$ *a continuous linear mapping.* Let S be represented by the family $(\lambda_y)_{y \in Y}$; we assume that S is *represented by measures,* that is $\lambda(Y) = \{\lambda_y \mid y \in Y\} \subset \mathfrak{M}(X, \mathfrak{I})$. It is obvious that $\lambda : y \to \lambda_y$ is a continuous mapping of Y into $(M^{\infty}(X, \mathfrak{I}))'$ when endowed with the topology $\sigma(M^{\infty}(X, \mathfrak{I}))', M^{\infty}(X, \mathfrak{I}))$. Note that by extending the mapping $y \to \lambda_y$ (and the operator S accordingly) to $Y^* = Y \cup \{\infty\}$ with $\lambda_\infty = 0$, it remains continuous and thus the set $\lambda(Y^*) = \{\lambda_y \mid y \in Y^*\}$ is $\sigma((M^{\infty}(X, \mathfrak{I}))', M^{\infty}(X, \mathfrak{I}))$*-compact.* We shall show that the operator S is weakly compact. This of course implies that Theorem 3 and hence Theorem 4 hold for arbitrary continuous linear mappings (without any positivity assumption). Below, for $\mu, \nu \in \mathfrak{M}(X, \mathfrak{I})$, we write $\nu \prec \mu$ if ν is absolutely continuous with respect to μ.

Second proof of Theorem 3 (without assuming S positive). By Remarks (2) and (3) above, $\lambda(Y^*) \subset (M^{\infty}(X, \mathfrak{I}))'$ is weakly compact. Thus $\lambda(Y^*)$ is weakly compact in $\mathfrak{M}(X, \mathfrak{I})$ and there is $\mu \in \mathfrak{M}_+(X, \mathfrak{I})$ such that $\lambda_y \prec \mu$ for every $y \in Y^*$ [7, p. 306–307]. Consider the adjoint of S,

${}^tS : (\mathcal{C}(Y^*))'$ ($=$ the space of Radon measures on Y^*) $\to (M^{\infty}(X, \mathfrak{I}))'$.

Using the definition of the adjoint operator it is easily seen that for each $\theta \in (\mathcal{C}(Y^*))'$, ${}^tS(\theta)$ is countably additive, i.e. ${}^tS(\theta) \in \mathfrak{M}(X, \mathfrak{I})$, and that ${}^tS(\theta) \prec \mu$ (note that S maps bounded pointwise convergent sequences into bounded pointwise convergent sequences and that the relations $E \in \mathfrak{I}$, $\mu(E) = 0$ imply $S\varphi_E = 0$). Let now $L^1 = L^1(X, \mathfrak{I}, \mu)$, $L^{\infty} = L^{\infty}(X, \mathfrak{I}, \mu)$ and for each $f \in M^{\infty}(X, \mathfrak{I})$ let $\tilde{f}$ be the corresponding element (equivalence class) in L^{∞}; the equation $S\tilde{f} = Sf$ unambiguously defines $S\tilde{f}$. We deduce that $S : L^{\infty} \to \mathcal{C}(Y^*)$ is continuous if L^{∞} and $\mathcal{C}(Y^*)$ are endowed with the topologies $\sigma(L^{\infty}, L^1)$ and $\sigma(\mathcal{C}(Y^*), (\mathcal{C}(Y^*))')$ respectively (use the fact that for every $\theta \in (\mathcal{C}(Y^*))'$, ${}^tS(\theta) \prec \mu$). If now $H \subset M^{\infty}(X, \mathfrak{I})$ is bounded, then $\tilde{H} \subset L^{\infty}$ is $\sigma(L^{\infty}, L^1)$-relatively compact and hence $S(H) = S(\tilde{H}) \subset \mathcal{C}(Y^*)$ is relatively weakly compact. This completes the proof.

6. For the immediate application of Theorem 4 to equicontinuity of bounded sets of harmonic functions in axiomatic potential theory see [6]. Here the author supposes only two axioms verified: a weakened form of Axiom 2 of Brelot [4, p. 62] and a weakened form of Axiom (K_1'') of Bauer

[2, p. 16]. For the purpose of applying Theorem 4, the latter can be further weakened by using Remark (1) after Theorem 4.

Other interesting applications to Markov processes and potential theory will be given elsewhere.

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