A NOTE ON THE GROWTH OF FUNCTIONS IN H^p

BY

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1. Introduction

The main results of this paper are two theorems on the growth of holomorphic functions in D, the open unit disc. The theorems are:

THEOREM 1. *Let* $\varphi(r) > 0$ *for* $0 \leq r < 1$ *and* $\varphi(r) \to 0$ *as* $r \to 1^-$. *Then there exists* $f \in H^2$ *such that* $|f(r_n)| \geq C\, \varphi(r_n)\, (1 - r_n)^{-1/2}$ *for* C *a constant and* r_n *a sequence of real numbers such that* $0 < r_n < r_{n+1} < 1$ *and* $r_n \uparrow 1$. *(We assume* $\varphi(r)\,(1 - r)^{-1/2} \to \infty$ *as* $r \to 1^-$ *without loss of generality.)*

THEOREM 2. *There exists a function* $f(z)$ *such that*

(i) $f(z)$ *is holomorphic for* $|z| < 1$;

(ii) $|f(z)| = O\{(1 - |z|)^{-\alpha}\}, \alpha > 0$;

(iii) *There exists a sequence* $\{r_n\}_{n=1}^{\infty}$ *such that* $r_n < 1$ *for all* n, $r_n \uparrow 1$ *and a constant* C *such that* $|f(r_n e^{i\theta})| \geq C(1 - r_n)^{-\alpha}$ *for each* n.

We shall give the proofs of these theorems in Section 2. At present we shall apply them to obtain some information on the growth of a function in H^p. By H^p, $1 \leq p < \infty$, we mean the class of all holomorphic functions in D, such that

$$\| f \|_p = \sup_{r<1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p}$$

is finite. As is well known H^p is a Banach space with norm $\| \quad \|_p$. See [2] for the basic facts about H^p. In [1] it is shown that $f \in H^p$ implies $|f(re^{i\theta})| = o\{(1 - r)^{-1/p}\}$. We shall use Theorems 1 and 2 to conclude that this is the best possible result.

We begin by indicating a simpler proof than that found in [1] for the growth condition. Namely, in [3; p. 31] it is shown by a simple argument using infinite series that $f \in H^2$ implies $|f(re^{i\theta})| = o\{(1 - r)^{-1/2}\}$. Since $f \in H^p$ implies $f = I \cdot F$ where I is inner function and F is an outer function in H^p with $|f(z)| \leq |F(z)|$, we prove the result for F. But F having no zeros implies $F^{p/2}$ belongs to H^2 so that the desired growth condition for F follows from the continuity of taking roots.

Using Theorem 1 and the factorization of functions in H^p we conclude that the above growth condition is the best possible.

THEOREM 3. *For* $f \in H^p$ *we have that*

$$|f(re^{i\theta})| = o\{(1 - r)^{-1/p}\}$$

is the best possible result.

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Proof. Suppose there exists $\varphi(r) \to 0$ as $r \to 1^-$, $\varphi(r) > 0$ for $0 \leq r < 1$ such that $|f(z)| = O\{\varphi(|z|)(1 - |z|)^{-1/p}\}$ for all $f \in H^p$. Let $f \in H^2$. Then $f = I \cdot F$ where I is an inner function and F is an outer function in H^2. Then $F^{2/p} \in H^p$ implying that

$$|F^{2/p}(z)| \leq C\varphi(|z|)(1 - |z|)^{-1/p},$$

C is a constant. Thus taking roots we see that we have contradicted Theorem 1.

The final remark of this section is to note that Theorem 2 implies there exist holomorphic functions of arbitrarily slow growth in D which are not in H^p. This says that H^p is a proper subset of all functions holomorphic on D and satisfying $|f(z)| = o\{(1 - |z|)^{-1/p}\}$ which is already well known and may be observed by various other examples (i.e. lacunary series).

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let n_1 be an integer such that $\varphi(1 - 1/2n_1) < 1$. Let $r_1 = 1 - 1/2n_1$. Choose n_2 an integer such that $n_2 > 2n_1$ and $\varphi(1 - 1/2n_2) < 1/2$. Let $r_2 = 1 - 1/2n_2$. Continue by induction getting a sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that $2n_k < n_{k+1}$ and $\varphi(r_k) < 1/k$ where $r_k = 1 - 1/2n_k$.

Let

$$f(z) = \sum_{k=1}^{\infty} \sum_{n=n_k+1}^{2n_k} (\varphi(r_k)/\sqrt{n_k})z^n.$$

Note that f is holomorphic for $|z| < 1$ since each Taylor coefficient, a_n, of f is less than $M/\sqrt{n}$, where M is some constant. Also, we note that

$$\begin{aligned} |f(r_k)| = f(r_k) &\geq \sum_{n=n_k+1}^{2n_k} (\varphi(r_k)/\sqrt{n_k})(r_k)^n \\ &\geq (\varphi(r_k)/\sqrt{n_k})(1 - 1/2n_k)^{2n_k}(n_k) \\ &\geq (\sqrt{n_k}/4)\varphi(r_k) = (\sqrt{2}/8)\varphi(r_k)(1 - r_k)^{-1/2} \end{aligned}$$

for all k.

Finally, we see that $f \in H^2$ since

$$\sum_{k=1}^{\infty} \sum_{n=n_k+1}^{2n_k} (\varphi(r_k)/\sqrt{n_k})^2 = \sum_{k=1}^{\infty} \varphi^2(r_k) < \sum_{k=1}^{\infty} 1/k^2 < \infty.$$

Proof of Theorem 2. (The author would like to thank Professor P. Lappan for the assistance he rendered here.)

We need the following well known lemma.

LEMMA 1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be holomorphic for* $|z| < 1$. *If there exists a constant* C *such that* $\sum_{n=0}^{N} |a_n| \leq C(N + 1)^{\alpha}$ *for all* N *then* $|f(re^{i\theta})| = O\{(1 - r)^{-\alpha}\}$ *where* $\alpha > 0$.

Proof. Fix z such that $|z| < 1$; then

$$\begin{aligned}(1 - |z|)\,|f(z)| &\leq (1 - |z|) \textstyle\sum_{n=0}^{\infty} |a_n|\,|z|^n \\ &\leq C \textstyle\sum_{n=0}^{\infty} (N + 1)^{\alpha}|z|^n \\ &\leq K(1 - |z|)^{-\alpha+1}\end{aligned}$$

since $(1 - |z|)^{-\beta} \sim \sum_{n=0}^{\infty} (n + 1)^{\beta-1}|z|^n$ for $\beta > -1$.

Proof of Theorem 3. Let $r_1 = \frac{1}{2}$, $n_1 = 2$ and $P_1(z) = n_1^{\alpha} z^{n1}$. Noting that for r fixed $(r < 1)$, $n^{\alpha} r^n \to 0$ as $n \to \infty$ and also that $(1 - 1/n)^n \downarrow e^{-1}$ $(n > 2)$, we choose n_2 such that $n_2^{\alpha} > 8(n_1^{\alpha} - 1)$, $n_2 > n_1$ and $r_1^{n2} < \frac{1}{2}$. Let $r_2 = 1 - 1/n_2$; then $(1 - r_2)^{-\alpha} = n_2^{\alpha}$. Let $P_2(z) = P_1(z) + n_2^{\alpha} z^{n2}$. Note that for $|z| = r_2$ we have

$$|P_2(z)| \geq n_2^{\alpha} (1 - 1/n_2)^{n2} - n_1^{\alpha} \geq (n_2^{\alpha}/4) - (n_1^{\alpha}/8) + 1 \geq (n_1^{\alpha}/8) + 1.$$

Suppose $n_1 < n_2 < \cdots < n_{k-1}$ have been choosen (set $r_i = 1 - 1/n_i$ for $i = 1, \cdots, k - 1$) satisfying

(i) $n_i^{\alpha} r_j^{n_i} \leq 2^{j-i}$ for $i > j$;
(ii) $n_i^{\alpha} \leq 8(n_{i-1}^{\alpha} + \cdots + n_1^{\alpha} - 1)$ for $i \leq k - 1$; and
(iii) $|P_i(z)| \geq (n_i^{\alpha}/8) + 1$ for $|z| = r_i$ where $P_i(z) = n_1^{\alpha} z_1 + \cdots + n_i^{\alpha} z^{n_i}$, $i \leq k - 1$.

Then choose n_k such that $n_k > n_{k-1}$, $n_k^{\alpha} r_i^{n_k} \leq 2^{i-k}$ for $i < k$ and

$$n_k^{\alpha} > 8(n_{k-1}^{\alpha} + \cdots + n_1^{\alpha} - 1).$$

Let $P_k(z) = P_{k-1}(z) + n_k^{\alpha} z^{nk}$ and $r_k = 1 - 1/n_k$. Note that for $|z| = r_k$,

$$\begin{aligned}|P_k(z)| &\geq n_k^{\alpha} r_k^{nk} - |P_{k-1}(z)| \\ &\geq (n_k^{\alpha}/4) - (n_{k-1}^{\alpha} + \cdots + n_1^{\alpha} - 1) \\ &\geq (n_k^{\alpha}/8) + 1.\end{aligned}$$

Let $f(z) = \sum_{k=1}^{\infty} n_k^{\alpha} z^{nk}$. We have that $f(z)$ is holomorphic for $|z| < 1$ since $\limsup (n_k^{\alpha})^{1/n_k} = 1$. For $|z| = r_i$, we have

$$\begin{aligned}|f(z)| &\geq n_i^{\alpha} r_i^{\alpha} - \sum_{j=1}^{i-1} n_j^{\alpha} - \sum_{j=i+1}^{\infty} n_j^{\alpha} r^{n_j} \\ &\geq (n_i^{\alpha}/4) - (n_i^{\alpha}/8) + 1 - \sum_{j=i+1}^{\infty} 2^{i-j} = (n_i^{\alpha}/8).\end{aligned}$$

Thus $f\,|(r_k e^{i\theta})| \geq (\frac{1}{8})\,(1 - r_k)^{-\alpha}$ for each k. Also we note that $r_k \uparrow 1$, showing that condition (iii) is satisfied.

To obtain (ii) let n be a positive integer and choose n_p such that $n_p \leq n < n_{p+1}$. Then

$$\begin{aligned}\textstyle\sum_{i=0}^{n} |a_i| = \sum_{k=1}^{p} |a_{n_k}| &= n_p^{\alpha} + (n_1^{\alpha} + \cdots + n_{p-1}^{\alpha}) \\ &\leq (\tfrac{9}{8})n_p^{\alpha} \leq (\tfrac{9}{8})\,(n + 1)^{\alpha}.\end{aligned}$$

Thus $|f(re^{i\theta})| = O\{(1 - r)^{-\alpha}\}$ by Lemma 1.

3. Concluding remarks

In this section we conclude the paper with two observations. The first concerns Theorem 1.

THEOREM 1′. *Theorem* 1 *is true for* p *such* $0 < p < \infty$. (*Replace* $\frac{1}{2}$ *by* $1/p$ *in all appropriate places.*)

Proof. Suppose $\varphi(r) > 0$ for $0 \leq r < 1$ and $\varphi(r) \to 0$ as $r \to 1^-$ then $\psi(r) = \varphi^{p/2}(r)$ has the same properties. Letting F be the outer part of the $f \in H^2$ with the property that

$$|f(r_n)| \geq C\psi(r_n)(1 - r_n)^{-1/2},$$

we have that $G(z) = F^{2/p}(z) \in H^p$ and

$$|G(r_n)| = |F(r_n)|^{2/p} \geq C'\varphi(r_n)(1 - r_n)^{-1/p}$$

on the sequence $\{r_n\}_{n=1}^{\infty}$ where $C' = C^{2/p}$.

The second observation is that the result on the growth condition for H^p also holds for $0 < p < 1$ since the factorization theory of H^p is also valid for $0 < p < 1$ [4; Pg. 338].

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