ON THE RELATION BETWEEN MEASURES AND CONVEX ANALYTIC FUNCTIONS

BY

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In this paper we prove two results on convex analytic functions. The author has recently learned that the first of these (Theorem 1) is known [1]. In order to prove these theorems we shall present a reduction of the problems to some measure-theoretic statements, which will then be proved. We shall use a relation between analytic functions and measures on the unit circle, which we shall repeatedly exploit. Although this relation has been known for a long time, to the author's knowledge this is rather little used in studying convex and star-like functions. We shall relate (essentially) the second coefficient of an analytic function to the center of mass of the associated mass distribution, and then shall study how the mass center is transformed if we transform the measure. The measure-theoretic results can be generalized to higher dimensions. For simplicity we do not state Theorem 2 here in its full generality, although the proof given is complete.

THEOREM 1. *Let D be a bounded convex domain in the plane. Let $h(w)$ = the interior mapping radius of D relative to the point w. Then h takes its maximum at a unique point w_0 of D.*

THEOREM 2. *Let D be a bounded convex domain, with smooth (C_1) boundary. Let $w_0 \in D$. Then for any $|\alpha| < 1$, there exists a mapping function*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

of the unit disc onto D so that

1. $a_2/a_1 = \alpha$
2. $f'(z_0) > 0$, *where* $z_0 = f^{-1}(w_0)$.

Preliminaries. Let $f(z) = z + a_2 z^2 + \cdots$ be analytic in the unit disc. Then f is a convex function if and only if $\operatorname{Re}(1 + zf''/f') > 0$ for $|z| < 1$. Then by the familiar Herglotz representation for functions of positive real part,

$$(1) \qquad 1 + \frac{zf''}{f'} = \int_0^{2\pi} \frac{1 + e^{-i\theta}z}{1 - e^{-i\theta}z}\, d\mu(\theta), \quad d\mu \geq 0, \quad \int d\mu = 1.$$

On subtracting 1, dividing through by z, and then integrating with respect to

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[1] I would like to acknowledge the advice and encouragement of my teacher and friend, the late Charles Loewner.

z, one obtains the familiar Schwarz-Christoffel representation:[2]

$$(2)\quad \log f'(z) = -2\int_0^{2\pi} \log(1 - e^{-i\theta}z)\, d\mu(\theta), \quad d\mu \geq 0, \quad \int d\mu = 1.$$

DEFINITION. Let $f(z)$ be an analytic function in the unit disc which has an everywhere non-vanishing derivative (locally-schlicht). If

$$f(z) = z + a_2 z^2 + \cdots$$

we call f *normalized.* If f is convex and normalized, then by (2) there is a non-negative measure $d\mu$ of total mass 1 on the unit circle associated with f. We say f *corresponds* to $d\mu$.

DEFINITION. Let $d\mu$ be a non-negative measure on the unit circle of total mass 1. If $d\mu$ has no atomic points, or if every atomic point of $d\mu$ has mass $< \frac{1}{2}$, we call $d\mu$ an *admissible* measure.

LEMMA 1. *Let $f(z)$ be convex and normalized in the unit disc. Let f correspond to $d\mu$. Then f is bounded if and only if $d\mu$ is admissible.*

Proof. If $d\mu$ is admissible, then it concentrates no mass $\geq \frac{1}{2}$ at a point. Hence there is an $M < \frac{1}{2}$, and $\varepsilon > 0$ so that for any interval I of the unit circle of length $\leq 2\varepsilon$, $d\mu(I) \leq M < \frac{1}{2}$.
From (2),

$$\begin{aligned}
\log |f'(re^{it})| &\leq -2\int_0^{2\pi} \log |1 - e^{-i(\theta-t)}r|\, d\mu(\theta) \\
&\leq -2\int_{t-\varepsilon}^{t+\varepsilon} \log |1 - e^{-i(\theta-t)}r|\, d\mu(\theta) - 2\int_{t+\varepsilon}^{t-\varepsilon} \log |1 - e^{-i(\theta-t)}r|\, d\mu(\theta) \\
&\leq -2\int_{t-\varepsilon}^{t+\varepsilon} \log (1 - r)\, d\mu(\theta) - 2\int_{t+\varepsilon}^{t-\varepsilon} \log |\sin \varepsilon|\, d\mu(\theta) \\
&\leq -2M \log (1 - r) - 2 \log |\sin \varepsilon|.
\end{aligned}$$

Exponentiating,

$$|f'(re^{it})| \leq \frac{1}{|\sin \varepsilon|^2} \frac{1}{(1 - r)^{2M}}.$$

But then

$$|f(z)| < \int_0^1 |f'(re^{it})|\, dr \leq \int_0^1 \frac{1}{|\sin \varepsilon|^2} \frac{1}{(1 - r)^{2M}}\, dr$$

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[2] This form (2) is perhaps less familar than

$$f' = \textstyle\prod_1^n [(1 - z/a_i)^{-\mu_i}]^2, \qquad \sum_1^n \mu_i = 1$$

which is the Schwarz-Christoffel representation for mappings onto polygons [2]. Note that the convexity of the polygon is equivalent to $\mu_i \geq 0$, $i = 1, \cdots, n$. Taking logarithms, and letting $n \to \infty$ (polygonal approximation to the convex domain D), we obtain (2).

$$= \frac{K}{1-2M} < \infty, \quad \text{as} \quad M < \frac{1}{2}.$$

Hence f is bounded.

If $d\mu$ is not admissible, without loss of generality assume $d\mu$ concentrates mass $\lambda \geq \frac{1}{2}$ at the point 1. Taking imaginary parts of (2),

$$|\arg f'(r)| \leq 2\int_0^{2\pi} |\arg(1 - e^{-i\theta}r)|\, d\mu(\theta) = 2\int_{0+}^{2\pi-} |\arg(1 - e^{-i\theta}r)|\, d\mu(\theta)$$

Now,

$$|\arg(1 - e^{-i\theta}r)| \leq |\arg(1 - e^{-i\theta})| = |(\pi - \theta)/2|, \quad 0 \leq \theta \leq 2\pi,$$

and hence

$$|\arg f'(r)| \leq 2\int_{0+}^{2\pi-} \left|\frac{\pi - \theta}{2}\right| d\mu(\theta).$$

Now the supremum of the values of the integrand is not achieved in the integration, as the range of integration does not include 0 or 2π. Hence we may conclude

$$2\int_{0+}^{2\pi-} \left|\frac{\pi - \theta}{2}\right| d\mu(\theta) < 2\,\frac{\pi}{2}\int_{0+}^{2\pi-} d\mu(\theta) = \pi(1 - \lambda).$$

This estimate for the integral is independent of the choice of r, so

$$|\arg f'(r)| \leq 2\int_{0+}^{2\pi-} \left|\frac{\pi - \theta}{2}\right| d\mu(\theta) = K < \pi(1 - \lambda) \leq \frac{\pi}{2}.$$

Following an argument similar to the reasoning in the bounded case, we obtain a lower bound for the modulus of f':

$$|f'(r)| \geq L/(1 - r)^2.$$

Now

$$|f(r)| \geq \operatorname{Re} f(r) = \int_0^r \operatorname{Re}\{f'(t)\}\, dt \geq \int_0^r |f'(t)| \cos \arg f'(t)\, dt$$

$$\geq \int_0^r \frac{L}{(1-t)^2} \cos K\, dt = L \cos K \int_0^r \frac{dt}{(1-t)^2} \to \infty \quad \text{as} \quad r$$

tends to 1, because $K < \pi/2$ and $\lambda \geq \frac{1}{2}$. Hence f is unbounded.

By Lemma 1, if we wish to study bounded convex functions we may confine our attention to admissible measures on the unit circle. We now state and prove the fundamental result on admissible measures, which will ultimately prove Theorem 1.

Definition. Let $d\mu$ be an admissible measure. Let M be the group of all linear fractional transformations of the unit disc onto itself:

$$e^{i\theta} \frac{z - a}{1 - \bar{a}z} \quad (|a| < 1).$$

For $T \in M$, if $T^{-1}(0) = -T(0)$, we call T *rotation-free*. In this case, T has the form $T(z) = (z - a)/(1 - \bar{a}z)$. Let us denote by $d\mu_T$ the measure induced by T:

$$d\mu_T(T(z)) = d\mu(z) \quad \text{for} \quad |z| = 1.$$

Note that T is defined and 1-1 on the unit circle. In case $T(z) = (z - a)/(1 - \bar{a}z)$ (i.e. that T is rotation-free), we denote $d\mu_T$ by $d\mu_a$ as well.

LEMMA 2. *Let $d\mu$ be an admissible measure on the unit circle. Then there exists a unique rotation-free $T \in M$ so that $d\mu_T$ has 0 mass center (center of gravity).*

Proof. Define

$$\begin{aligned} C(a) &= \text{mass center of } d\mu_a \\ &= \int_0^{2\pi} e^{i\theta}\, d\mu_a(e^{i\theta}) = \int_0^{2\pi} e^{i\theta}\, d\mu\left(\frac{e^{i\theta} + a}{1 + ae^{i\theta}}\right) \qquad (3) \\ &= \int_0^{2\pi} \frac{e^{i\theta} - a}{1 - \bar{a}e^{i\theta}}\, d\mu(e^{i\theta}). \end{aligned}$$

C is evidently a continuous function on $|a| < 1$. If $d\mu$ has no atomic points, then it is easily seen that C is continuous up to the boundary, and that on the boundary, $C(e^{it}) = -e^{it}$. It follows by an elementary winding number argument that C must cover the disc; in particular the origin is in the image of C. If $d\mu$ has an atomic point e^{it} of mass $d < \frac{1}{2}$, a calculation shows that

$$\limsup_{a \to e^{it}} |C(z) - (-e^{it})| \leq 2d. \qquad (4)$$

As the total mass of $d\mu$ is 1, $\sup_t d\mu(e^{it}) = \lambda < \frac{1}{2}$. Thus (4) shows that for r sufficiently close to 1, the image of the circle of radius r under C is a curve which has winding number $= 1$ with respect to 0 and hence the origin is again covered.

But the existence of a zero of C is exactly the existence of the required transformation.

To show uniqueness, we proceed as follows: If there were two transformed measures, $d\mu_a$ and $d\mu_b$ with 0 mass center, let $dv = d\mu_a$. Then one verifies that $dv_T = d\mu_b$, where

$$T(z) = \left(\frac{1 - \bar{a}b}{1 - a\bar{b}}\right)\left(\frac{z + (b - a)/(1 - a\bar{b})}{1 + z(\bar{b} - \bar{a})/(1 - \bar{a}b)}\right).$$

Thus[3] $T \in M$.

If dv_T has 0 mass center, then so does dv_S for any $S = e^{i\theta}T$. Hence we may assume that T is rotation-free, and thus $dv_T = dv_c$, for some $|c| < 1$. If

[3] T is the linear fractional transformation resulting from the compositions of $w = (z - a)/(1 - \bar{a}z)$ and $w' = (w - b)/(1 - \bar{b}w)$.

$c \neq 0$, it is clear that the transformation $(z - c)/(1 - \bar{c}z)$ moves all points on the unit circle in the direction of $-c/|c|$, with the exception of the stationary points $\pm c/|c|$. As dv cannot concentrate all its mass at these two points (a violation of the regularity assumption), the mass center must move as well. The projection of this movement onto the diameter containing c is all in one direction, and hence the movement of the mass center must be in this direction as well. That is, the mass center of $dv_c \neq 0$ if $c \neq 0$. The above geometrical reasoning can be restated analytically as follows: We may assume after coordinate rotation that $c > 0$. Then

$$\begin{aligned}
\operatorname{Re}\,(\text{mass center of } dv_c) &= \operatorname{Re} \int_0^{2\pi} e^{i\theta}\, dv_c(e^{i\theta}) \\
&= \operatorname{Re} \int_0^{2\pi} \frac{e^{i\theta} - c}{1 - \bar{c}e^{i\theta}}\, dv(e^{i\theta}) \\
&= \int_0^{2\pi} \operatorname{Re} \left\{ \frac{e^{i\theta} - c}{1 - \bar{c}e^{i\theta}} \right\} dv(e^{i\theta}) \\
&< \int_0^{2\pi} \operatorname{Re}\,\{e^{i\theta}\}\, dv(e^{i\theta}) \\
&= \operatorname{Re}\,(\text{mass center of } dv) = 0.
\end{aligned}$$

Thus the mass center of $dv_c \neq 0$.

This completes the uniqueness portion of Lemma 2.

The hypotheses of the following lemma can be considerably weakened, but we prove only what we need.

LEMMA 3. *Let f be convex and bounded. Then*

$$|f'(z)|(1 - z\bar{z}) \to 0 \quad \text{as} \quad |z| \to 1.$$

Proof. We use Koebe's "$\frac{1}{4}$-Theorem": Let f be univalent in the unit disc and map onto a domain D. Then the distance from $f(0)$ to the boundary of D is greater than or equal to $\frac{1}{4}|f'(0)|$. That is,

$$\rho(f(0), \partial D) \geq \tfrac{1}{4}|f'(0)|.$$

Now, let f satisfy the hypotheses, and suppose that

$$h_a(z) = f((z + a)/(1 + \bar{a}z)), \quad \text{where} \quad |a| < 1.$$

Then h_a is certainly univalent, and has the same range as f, call it D. By the $\frac{1}{4}$-Theorem,

$$\rho(f(a), \partial D) = (h_a(0), \partial D) \geq \tfrac{1}{4}|h_a'(0)| = \tfrac{1}{4}|f'(a)|(1 - a\bar{a}).$$

But as $|a| \to 1$, $\rho(f(a), \partial D) \to 0$ since D is bounded and schlicht. Thus

$$|f'(a)|(1 - a\bar{a}) \to 0 \quad \text{as} \quad |a| \to 1.$$

LEMMA 4. *The function $|f'(z)|(1 - z\bar{z})$, for f analytic, is an absolute in-*

variant of the group M. That is, if $T \epsilon M$, and $z_1 = T(z)$, and f_1 is defined by

$$f_1(z_1) = f(z),$$

then

$$|f_1'(z_1)|(1 - z_1 \bar{z}_1) = |f'(z)|(1 - z\bar{z}).$$

Proof. This follows immediately from the fact that $|df|$ is a conformal invariant, and $|dz|/(1 - z\bar{z})$ is an invariant of the group M. Their quotient is the object in question. We now proceed to complete the proof of Theorem 1.

Let D be a bounded convex domain. Let $w_1 \epsilon D$, and let f map the unit disc onto D, with $f(0) = w_1$. Then a simple calculation shows that for any $w \epsilon D$, $h(w) = \{$interior mapping radius of D relative to $w\} = |f'(a)|(1 - a\bar{a})$, where $a = f^{-1}(w)$. As f^{-1} is a 1-1 function, we are reduced to showing that $|f'(a)|(1 - a\bar{a})$ takes a unique maximum in the unit disc. We shall do this by showing that $|f'(a)|(1 - a\bar{a})$ has exactly one stationary point in the unit disc, and since this function is positive on the disc and vanishes on the boundary, it must then take an absolute maximum at the stationary point, and can take it nowhere else.

Suppose b is a stationary point of $|f'(z)|(1 - z\bar{z})$. Then if

$$z_1 = (z + b)/(1 + \bar{b}z) \text{ and } f_1(z_1) = f(z),$$

then f_1 is convex, and evidently by Lemma 4, $|f'(z_1)|(1 - z_1 \bar{z}_1)$ has a stationary point at 0. Suppose f corresponds to $d\mu$. The representation (2) holds for non-normalized convex functions as well, modulo some additive constants, and one verifies from the geometry that f_1 corresponds to the measure $d\mu_b$. By expanding (2) in a power series about 0, we see that $a_2 = 0$ if and only if the mass center of $d\mu$ is 0. If $g(z) = z + b_2 z^2 + \cdots$, then $|g'(z)|(1 - z\bar{z})$ has a stationary point at 0 if and only if b_2 is 0, as is seen by an elementary calculation. Applying the last two observations to the function f_1 above, we see that if b is a stationary point of f, then $d\mu_b$ must have 0 mass center. Thus (by Lemma 2) $|f'(z)|(1 - z\bar{z})$ has exactly one stationary point in the unit disc. This completes the proof.

The theorem is not necessarily true if we remove the assumption of convexity, even if the domain is starlike. To see this consider a domain D_t, which consists of the unit disc slit along the real axis from -1 to $-t$, and from t to 1. For t sufficiently small, 0 cannot be the point for which the unique maximum of the inner radius is achieved, as 0 lies too close to the boundary. On the other hand, D_t is carried onto itself by a rotation of 180° about the origin, and every point of D_t other than the origin moves to a new point. It is obvious then that if the maximum is achieved at some point different from 0, it is therefore achieved at two points at least.

If the domain is convex, but unbounded, the theorem is definitely false. In this case we distinguish two possibilities: If the domain is a strip, say $|\operatorname{Im} w| < \pi/2$, then there are a continuum of maxima, since the domain is invariant under translations by any real α: $w' = w + \alpha$.

If the domain D, and hence the mapping function f, is unbounded, the corresponding measure $d\mu$ is not admissible measure by Lemma 1. Consideration of (2) shows that f maps onto a strip if and only if $d\mu$ is a two point measure with equal masses $(\frac{1}{2})$. This is the case we just considered. On the other hand, $d\mu(e^{i\theta}) \geq \frac{1}{2}$ for some θ, and if f is not a strip map, it is not difficult to see that no transform of $d\mu$ can have 0 mass center, as one point will always have mass $\geq \frac{1}{2}$. Evidently, then, the function $|f'(z)|(1 - z\bar{z})$ can have no stationary point in the unit disc. Hence no maximum can be achieved.

A second proof of Theorem 1 has been found by Loewner, and I should like to present it here. Although this proof is shorter, the emphasis here has been on the former proof in order to give a development of some possibly useful relations between analytic functions and measure-theoretic approaches. Furthermore, these methods can be generalized to higher dimensions, to give potential theoretic results.

LEMMA (Loewner). *Let f be convex and bounded in the unit disc. Then*

$$K(z) = \log|f'(z)| + \log(1 - z\bar{z})$$

is a strictly convex function of hyperbolic arclength along geodesics of the Poincaré model of non-Euclidean geometry in the unit disc.

Proof. We have by Lemma 4 that $|f'(z)|(1 - z\bar{z})$ is invariant with respect to the group M, so certainly $K(z)$ is. That is, if $T \in M$, and $z^* = T(z)$, and $f^*(z^*) = f(z)$, then $K^*(z^*) = K(z)$, where K^* has the obvious meaning. Now, if s is hyperbolic arclength, by the invariance of K and s with respect to a fixed transform in M,

$$\frac{\partial^2}{\partial s_*^2} K^*(z^*) = \frac{\partial^2}{\partial s^2} K(z), \tag{5}$$

where differentiation on the right hand side is with respect to hyperbolic arclength along any geodesic through z, and differentiation on the left hand side is with respect to arclength on the corresponding geodesic through z^*. By (5) it is then enough to show

$$\frac{\partial^2}{\partial s^2} K^*(0) < 0. \tag{6}$$

To do this, we may assume that after rotation, the geodesic through 0 is the diameter which lies on the real axis. The metric ρ of the Poincaré model satisfies $\rho(0, x) = \frac{1}{2}\log((1 + x)/(1 - x))$, and a computation shows then that along the real axis,

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial s^2} \quad \text{at} \quad 0.$$

Using the definition of K^*, and (2), a straightforward calculation shows

$$K_{xx}^*(0) = 2\left\{\operatorname{Re}\int_0^{2\pi} e^{-2i\theta}\, d\mu(\theta) - 1\right\}.$$

The assumption of boundedness of f ensures that

$$\left| \int e^{-2i\theta}\, d\mu(\theta) \right| < 1,$$

so that

$$K^*_{xx}(0) < 0.$$

COROLLARY. *The function* $|f'(z)|(1 - z\bar{z})$ *has a unique maximum in the unit disc.*

Proof. If the function had two maxima, say at a and b, then so would $K(z)$. But then K could not be strictly convex along the geodesic joining a and b.

To prove Theorem 2, we proceed as follows: Given $w_0 \in D$, let g be the mapping function of the unit disc onto D so that $g(0) = w_0$ and $g'(0) > 0$. Now any other mapping function of D is given by

$$f(z) = g\left(e^{i\theta}\, \frac{z + a}{1 + \bar{a}z}\right).$$

Then $f^{-1}(w_0) = -a$, and $f'(-a) = e^{i\theta}g'(0)(1 - |a|^2)$. So $f'(-a) > 0$ if and only if $e^{i\theta} = 1$. To find the function having the required properties, it thus suffices to look among those f which can be written as

$$f(z) = g((z + a)/(1 + \bar{a}z)).$$

Although f and g are not normalized, equation (2) is still valid for them modulo some constants, and it is easily seen that if g corresponds to $d\mu$, then f corresponds to $d\mu_{-a}$. On the other hand, $\frac{1}{2}g''(0)/g'(0)$ = mass center of $d\mu$, and $\frac{1}{2}f''(0)/f'(0)$ = mass center of $d\mu_{-a}$. Hence the mass centers of the transformed measures give the totality of values a_2/a_1 which can be achieved, subject to the normalization conditions, 2.

Consideration of the proof of Lemma 2 then leads immediately to the following:

LEMMA. *Let* $d\mu$ *be a non-negative measure on the unit circle, of total mass* 1. *Let* $\rho = \sup_t d\mu(e^{it})$. *Then every point in the disc* $|w| < 1 - 2\rho$ *is the mass center of at least one transform of* $d\mu$, *say* $d\mu_a$.

Note now that the number ρ remains unchanged if we transform the measure from $d\mu$ to $d\mu_a$. Hence ρ depends only on the domain D, and not on the particular mapping function. ρ may be characterized in terms of D in several ways. Geometrically, $\rho\pi$ is the supremum of the exterior angles of the "corners" of D, as is easily seen by the Schwarz-Christoffel representation (2). Alternately, if f is any mapping function of the unit disc onto D,

$$\rho = \inf\{\lambda : |f'(z)|(1 - z\bar{z})^{2\lambda} \to 0 \quad \text{as} \quad |z| \to 1\}.$$

Collecting these observations, we have

THEOREM 2. *Let D be a bounded convex domain. Let ρ be the number associated with D (as above). Then for any $|\alpha| < 1 - 2\rho$, there exists a mapping function $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ of the unit disc onto D so that*

1. $a_2/a_1 = \alpha$
2. $f'(z_0) > 0$, *where* $z_0 = f^{-1}(w_0)$.

In particular, if the domain has a smooth (C^1) boundary, we can conclude that every value α in the unit disc can be achieved as a_2/a_1 for a mapping function f for which $f'(z_0) > 0$, $z_0 = f^{-1}(w_0)$. We cannot, in general, ensure that the values a_2/a_1 (with condition 2) will cover the disc (although they will always cover the disc almost everywhere), as is seen by considering the measure which concentrates mass $\frac{1}{4}$ at each of the fourth roots of unity.

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