EXTENDED BASES FOR BANACH SPACES

BY

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1. Introduction

Let E be a Banach space over the real or complex field. A sequence $\{x_i\}$ in E is said to be a (weak) basis for E if for every x in E there corresponds a unique scalar sequence $\{\alpha_i\}$ such that $x = \sum_{i=1}^{\infty} \alpha_i x_i$, the convergence being in the norm (weak) topology of E. A basis with continuous coefficient functionals α_i is called a Schauder basis. In [2], Arsove and Edwards introduced the concept of an *extended* Schauder basis, where, by discarding the requirement of countability, they carried out the expansion according to a given directed set. Without proof they have given the following theorem: Every weak extended Schauder basis for E is an extended Schauder basis for E. As already indicated in [2], it is usually assumed that the expansions converge unconditionally, and so we obtain a slightly stronger definition of an extended basis: A family $\{x_\lambda\}$ $(\lambda \in \Lambda)$ is said to be a (weak) extended unconditional basis, or, in short, a *(weak) extended basis* for E if to each x in E there is a unique scalar family $\{\alpha_\lambda\}$ such that $x = \lim_\sigma \sum_{\lambda \in \sigma} \alpha_\lambda x_\lambda$ in the norm (weak) topology of E, where the σ's are finite subsets of Λ, directed by inclusion, and where $\lim_\sigma y_\sigma$ denotes the limit of a net $\{y_\sigma\}$ in E. If, according to Bessaga and Pelczynski [3], an absolute basis for E denotes a total set in E in which every sequence of distinct elements forms an unconditional basic sequence, it turns out that the extended bases for E and the absolute bases for E are the same. Moreover, it is easy to see that in a separable space the concepts of extended and unconditional bases are the same. But the example of a (non-separable) Hilbert space shows that there exist extended bases which are not bases. However, it is known [3] that there exist Banach spaces, e.g. l_∞, which have no absolute and hence no extended bases; a negative fact which is not yet cleared for bases. Since l_∞ is an $\mathfrak{N}_p$-space $(p > 1)$ there are $\mathfrak{N}_p$-spaces without extended bases (for the definition see [8]).

It is encouraging that many results from the theory of bases in separable spaces have their analogues for extended bases. Indeed, as shown in this note, one can establish necessary and sufficient conditions for E to have an extended basis, a theorem which formally resembles the theorem of Nikol'skiĭ which applies to bases for separable Banach spaces. It is also shown that the weak extended bases coincide with the extended bases and that the extended bases coincide with the extended Schauder bases. Using the natural extensions of the notions of shrinking or boundedly complete bases one obtains a

Received October 8, 1968.

generalization of the theorem of James to the non-separable case: A Banach space E with an extended basis is reflexive if and only if that basis is both shrinking and boundedly complete. A further theorem is shown which penetrates into the structure of E: If E has a boundedly complete extended basis then E is weakly sequentially complete.

2. Weak extended bases

Let Λ be an index set of arbitrary cardinality and let Σ be the set of all finite subsets (including the void set) of Λ. Then a family $\{x_\lambda\}$ in E (where $\lambda \in \Lambda$) is said to be *summable* to x in E if $\lim_\sigma \sum_{\lambda\in\sigma} x_\lambda$ exists in the norm topology of E. Weak and weak* summability are defined similarly. Let now S be the set of all scalar families $\{\beta_\lambda\}$ with $|\beta_\lambda| \leq 1$, $\lambda \in \Lambda$.

LEMMA 1. *If $\{x_\lambda\}$ is weakly summable in E, then*

$$\sup \{\| \textstyle\sum_{\lambda\in\sigma} \beta_\lambda x_\lambda \| \mid \{\beta_\lambda\} \in S, \sigma \in \Sigma\} < \infty .$$

Proof. By definition, $\lim_\sigma \sum_{\lambda\in\sigma} f(x_\lambda)$ exists for every f in E'. Hence from summability theory of scalar families [11] it is known that

$$\sup_\sigma \textstyle\sum_{\lambda\in\sigma} |f(x_\lambda)| < \infty, \qquad f \in E'.$$

Since the last expression is an upper bound for

$$\sup \{| \textstyle\sum_{\lambda\in\sigma} f(\beta_\lambda x_\lambda) | \mid \{\beta_\lambda\} \in S, \sigma \in \Sigma\},$$

the uniform boundedness principle implies $\sup \{\| \sum_{\lambda\in\sigma} \beta_\lambda x_\lambda \| \mid \{\beta_\lambda\} \in S, \sigma \in \Sigma\} < \infty$. We now call a family $\{x_\lambda\}$ in E a (*weak, weak**) *extended basis* for E if for each x in E there is a unique scalar family $\{\alpha_\lambda\}$ such that $\{\alpha_\lambda x_\lambda\}$ is (weakly, weakly*) summable to x. Sometimes use is made of the notation $\{x_\lambda, f_\lambda\}$ to indicate that $\{f_\lambda\}$, defined by $f_\lambda(x) = \alpha_\lambda$, $x \in E$, is the (unique) family of linear coefficient functionals of an extended basis $\{x_\lambda\}$. A (weak, weak*) extended basis $\{x_\lambda, f_\lambda)$ for E with (weakly, weakly*) continuous coefficient functionals f_λ is called a (*weak, weak**) *extended Schauder basis* for E.

LEMMA 2. *Let $\{x_\lambda, f_\lambda\}$ be a weak extended basis for E. Then the function $\| \quad \|' : E \to \mathbf{R}$, given by*

$$\| x \|' = \sup \{\| \textstyle\sum_{\lambda\in\sigma} \beta_\lambda f_\lambda(x) x_\lambda \| \mid \{\beta_\lambda\} \in S, \sigma \in \Sigma\}, \qquad x \in E,$$

defines an equivalent norm on E.

Proof. By Lemma 1, $\| x \|' < \infty$ for all $x \in E$. It is easy to see that $\| \quad \|'$ is a norm on E. The topology on E, induced by $\| \quad \|$ is weaker than that induced by $\| \quad \|'$, since

$$\| x \| \leq \sup_\sigma \| \textstyle\sum_{\lambda\in\sigma} f_\lambda(x) x_\lambda \| \leq \| x \|'.$$

One can show that E is complete in the metric $\| \quad \|'$, hence that $\| \quad \|$ and $\| \quad \|'$ are equivalent.

Let the sequence $\{y_n\}$ be $\|\ \ \|'$-Cauchy in E. Consequently, $\{y_n\}$ is $\|\ \ \|$-Cauchy and converges with n to some element y of E. Note that by uniqueness, each x_λ is non-zero. Each f_λ is continuous in the topology induced by $\|\ \ \|'$ on E, for $|f_\lambda(x)| \leq \|x\|'/\|x_\lambda\|$, $x \in E$. This implies that each $f_\lambda(y_n)$ converges with n, say to α_λ. By hypothesis there is for every $\varepsilon > 0$ an integer n such that $\|y_p - y_q\|' < \varepsilon/3$ for all $p, q \geq n$. Hence

$$\sup\{\|\textstyle\sum_{\lambda\epsilon\sigma} \beta_\lambda[f_\lambda(y_p) - f_\lambda(y_q)]x_\lambda\| \mid \{\beta_\lambda\} \in S, \sigma \in \Sigma\} < \varepsilon/3, \qquad p, q \geq n.$$

Taking the limit on q, one obtains

$$(1) \qquad \sup\{\|\textstyle\sum_{\lambda\epsilon\sigma} \beta_\lambda[f_\lambda(y_p) - \alpha_\lambda]x_\lambda\| \mid \{\beta_\lambda\} \in S, \sigma \in \Sigma\} \leq \varepsilon/3, \qquad p \geq n.$$

Now, there is a fixed index $m \geq n$ for which $\|y_m - y\| < \varepsilon/3$, and for each $f \in E'$ with $\|f\| = 1$ there is a $\tau \in \Sigma$, depending on ε and m, such that

$$|f[\textstyle\sum_{\lambda\epsilon\sigma} f_\lambda(y_m)x_\lambda - y_m]| < \varepsilon/3, \qquad \tau \subset \sigma \in \Sigma.$$

Therefore,

$$\begin{aligned} &|f[\textstyle\sum_{\lambda\epsilon\sigma} \alpha_\lambda x_\lambda - y]| \\ &\quad \leq \|\textstyle\sum_{\lambda\epsilon\sigma} [\alpha_\lambda - f_\lambda(y_m)]x_\lambda\| + |f(\textstyle\sum_{\lambda\epsilon\sigma} f_\lambda(y_m)x_\lambda - y_m]| + \|y_m - y\| \\ &\quad < \varepsilon, \qquad \sigma \supset \tau. \end{aligned}$$

Hence $\alpha_\lambda = f_\lambda(y)$ on Λ, and (1) implies that $\lim_p \|y_p - y\|' = 0$. Thus E is complete in the metric $\|\ \ \|'$ and by this argument the lemma is verified.

The lemma, which is fundamental for subsequent investigations, can now be used to establish the following important result, which is known for (ordinary) bases for separable complete metric linear spaces [1, Theorem 2]:

THEOREM 3. *Every (weak) extended basis for E is a (weak) extended Schauder basis for E.*

Proof. Let $\{x_\lambda, f_\lambda\}$ be a weak extended basis for E. Since

$$|f_\lambda(x)| \leq \|x\|'/\|x_\lambda\|,$$

each f_λ is continuous in the topology induced by $\|\ \ \|'$ on E and so also in the norm topology of E. Thus, by the fact that a linear functional on E is weakly continuous if and only if it is continuous, we have the theorem.

3. Existence of extended bases

A double family $\{x_\lambda, f_\lambda\}$, $x_\lambda \in E$, $f_\lambda \in E'$, $\lambda \in \Lambda$ is called a *biorthogonal system* for E if $x_\lambda(f_\mu) = \delta_{\lambda\mu}$, $\lambda, \mu \in \Lambda$, where $\delta_{\lambda\mu}$ denotes the Kornecker symbol. Let $\{U_\sigma\}$ be the family of linear operators in E, defined by $U_\sigma x = \sum_{\lambda\epsilon\sigma} f_\lambda(x)x_\lambda$, $x \in E$, $\sigma \in \Sigma$. Ovbiously, the operators U_σ are continuous projections of E with the properties $U_\sigma U_\tau = U_{\sigma\cap\tau}$, $\sigma, \tau \in \Sigma$.

THEOREM 4. *Let $\{x_\lambda, f_\lambda\}$ be a biorthogonal system for E such that $\sup_\sigma |f(U_\sigma x)| < \infty$, $x \in E$, $f \in E'$. Then one has*

(i) $\{x_\lambda\}$ *is an extended basis for* $\overline{\text{sp}}\,\{x_\lambda\}$ *in* E, *and*
(ii) $\{f_\lambda\}$ *is an extended basis for* $\overline{\text{sp}}\,\{f_\lambda\}$ *in* E'.

Up to modifications caused by summability, the proof is similar to that for the case of ordinary bases given e.g. in [9, p. 31].

THEOREM 5. $\{x_\lambda\}$ *is a (weak) extended basis for* E *if and only if there is a family* $\{f_\lambda\}$ *in* E *such that* $\{x_\lambda, f_\lambda\}$ *is a biorthogonal system for* E *and* $\lim_\sigma \sum_{\lambda\epsilon\sigma} f_\lambda(x)x_\lambda = x$ *in the strong (weak) topology of* E *for each* x *in* E.

Proof. The sufficiency follows directly from the preceding theorem. Necessity: According to Theorem 3 each coefficient functional f_λ belongs to E' and the uniqueness of $f_\lambda(x)$ for each $x \epsilon E$ implies $f_\lambda(x_\mu) = \delta_{\lambda\mu}$.

COROLLARY 6. $\{x_\lambda\}$ *is an extended basis for* E *if and only if it is a weak extended basis for* E.

The following theorem generalizes Nikol'skiĭ's theorem [10] to extended bases:

THEOREM 7. *A total family* $\{x_\lambda\}$ *of non-zero elements in* E *is an extended basis for* E *if and only if there is a constant* $M \geq 1$ *such that*

$$\| \textstyle\sum_{\lambda\epsilon\sigma} \alpha_\lambda x_\lambda \| \leqq M \, \| \textstyle\sum_{\lambda\epsilon\tau} \alpha_\lambda x_\lambda \|$$

for all $\sigma, \tau \epsilon \Sigma$ *with* $\sigma \subset \tau$ *and arbitrary scalars* α_λ.

Again the proof can be obtained by modifying the proof of Nikol'skiĭ's theorem given in [9, p. 57] to the case of extended bases. The proof will then be based on Theorem 4 and 5.

According to the fact that in every Hilbert space there exists a total orthonormal (not necessarily countable) set, such that the inequality of the above theorem is satisfied with $M = 1$, it is apparent that there is an extended basis for every Hilbert space. The question naturally arises if every $\mathfrak{N}_p$-space or even every Banach space has such a basis. The answer is negative, as we will show in a corollary of the next proposition.

THEOREM 8. *A total family* $\{x_\lambda\}$ *is an extended basis for* E *if and only if every sequence of different elements from* $\{x_\lambda\}$ *forms an unconditional basis for its closed span.*

Proof. By a twofold application of the preceding theorem it follows that every sequence of different elements from an extended basis $\{x_\lambda\}$ is an unconditional basic sequence. Conversely, let $\{x_\lambda\}$ be a total family in E such that each sequence of distinct x_λ's is an unconditional basic sequence. Then there is for every x in E a countable subset Λ_0 of Λ such that $x \epsilon \overline{\text{sp}}\,\{x_\lambda \mid \lambda \epsilon \Lambda_0\}$. Since $\{x_\lambda \mid \lambda \epsilon \Lambda_0\}$ is an unconditional basic sequence there is a scalar family $\{\alpha_\lambda \mid \lambda \epsilon \Lambda\}$ with $\alpha_\lambda = 0$ for $\lambda \notin \Lambda_0$, such that $\lim_\sigma \sum_{\lambda\epsilon\sigma} \alpha_\lambda x_\lambda = x$. We show that this family is unique: Let $\{\beta_\lambda \mid \lambda \epsilon \Lambda\}$ be another family such that

$\lim_\sigma \sum_{\lambda\epsilon\sigma} \beta_\lambda x_\lambda = x$ and let Λ_1 be the (obviously countable) set in Λ on which $\beta_\lambda \neq 0$. Then

$$\lim_{\sigma\subset\Lambda_0\cap\Lambda_1} \sum_{\lambda\epsilon\sigma} \beta_\lambda x_\lambda = x,$$

and since $\{x_\lambda \mid \lambda \epsilon \Lambda_0 \cup \Lambda_1\}$ is an unconditional basis for its closed span, one must have $\beta_\lambda = \alpha_\lambda$ on Λ.

In other words, every extended basis is an absolute basis, where an *absolute basis* for E is defined to be a total set in E such that every sequence of distinct elements from the set is an unconditional basic sequence. Now, because l_∞ has no absolute basis [3, p. 172], one gets immediately

COROLLARY 9. *The Banach space l_∞ does not have an extended basis.*

Since l_∞ is isometrically isomorphic to a space $C(S)$, S compact Hausdorff [6, p. 445], and since $C(S)$, as we know [8, p. 76], is an $\mathfrak{N}_p$-space for every $p > 1$, we obtain another result:

COROLLARY 10. *For each $p > 1$, there exists an $\mathfrak{N}_p$-space with no extended basis.*

4. Extended bases and reflexivity

Let J be the canonical map of E into E''.

THEOREM 11. *If $\{x_\lambda, f_\lambda\}$ is an extended basis for E, then $\{f_\lambda, Jx_\lambda\}$ is a weak* extended Schauder basis for E'. Conversely, if $\{f_\lambda, F_\lambda\}$ is a weak* extended Schauder basis for E', then $F_\lambda \epsilon J(E)$ and $\{J^{-1}F_\lambda, f_\lambda\}$ is an extended basis for E.*

Proof. If $\{x_\lambda, f_\lambda\}$ is an extended basis for E, it follows that

$$\lim_\sigma [f - \textstyle\sum_{\lambda\epsilon\sigma} Jx_\lambda(f)f_\lambda](x) = \lim_\sigma f[x - \sum_{\lambda\epsilon\sigma} f_\lambda(x)x_\lambda] = 0, \quad x \epsilon E, f \epsilon E'.$$

In order to prove the uniqueness of the coefficients $Jx_\lambda(f)$, we assume that $\lim_\sigma \sum_{\lambda\epsilon\sigma} \alpha_\lambda f_\lambda(x) = 0$ for all x in E. Then with $x = x_\mu$, $\mu \epsilon \Lambda$ one obtains $\alpha_\mu = 0$ on Λ and the first part of the theorem follows.

On the other hand, let $\{f_\lambda, F_\lambda\}$ be a weak* extended Schauder basis for E'. Then the weak continuity of the F_λ's implies [12, p. 112] that $F_\lambda \epsilon J(E)$. Consequently,

$$f(x) = \lim_\sigma \textstyle\sum_{\lambda\epsilon\sigma} F_\lambda(f)f_\lambda(x) = \lim_\sigma f[\sum_{\lambda\epsilon\sigma} f_\lambda(x)J^{-1}F_\lambda], \qquad x \epsilon E, f \epsilon E'.$$

That $\{J^{-1}F_\lambda, f_\lambda\}$ is a weak extended basis and hence an extended basis for E now follows at once from the assumption

$$\lim_\sigma f(\textstyle\sum_{\lambda\epsilon\sigma} \alpha_\lambda J^{-1}F_\lambda) = 0, \qquad f \epsilon E',$$

which by $f_\mu(J^{-1}F_\lambda) = F_\lambda(f_\mu) = \delta_{\lambda\mu}$ shows that $\alpha_\mu = 0$, $\mu \epsilon \Lambda$.

We say, in analogy to the case where Λ is the set of positive integers, that an extended basis $\{x_\lambda\}$ for E, with associated family of expansion operators $\{U_\sigma \mid \sigma \epsilon \Sigma\}$, is *shrinking* if and only if

$$\lim_\sigma \sup \{|f(x - U_\sigma x)| \mid \|x\| \leq 1\} = 0$$

for each $f \in E'$. This is equivalent to saying that $\lim_\sigma \|f\|_\sigma = 0, f \in E'$, where $\|f\|_\sigma$ is the norm of the restriction of f to the subspace $(I - U_\sigma)(E)$ and I is the identity of E (this is a consequence of the uniform boundedness principle).

THEOREM 12. *Let $\{x_\lambda, f_\lambda\}$ be an extended basis for E. Then the following statements are equivalent*:

(i) *$\{x_\lambda\}$ is shrinking.*
(ii) *$\{f_\lambda\}$ is an extended basis for E'.*
(iii) *$\{f_\lambda\}$ is total in E'.*

Proof. (i) $\Rightarrow$ (ii). If $\{x_\lambda, f_\lambda\}$ is shrinking, then for all $f \in E'$,

$$0 = \lim_\sigma \sup \{|f(x - U_\sigma x)| \mid \|x\| \leq 1\}$$
$$= \lim_\sigma \sup \{|[f - \textstyle\sum_{\lambda\in\sigma} Jx_\lambda(f)f_\lambda](x))| \mid \|x\| \leqq 1\}.$$

Since $Jx_\lambda(f_\mu) = f_\mu(x_\lambda) = \delta_{\lambda\mu}$ it follows from Theorem 5 that $\{f_\lambda, Jx_\lambda\}$ is an extended basis for E'.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). $\{f_\lambda, Jx_\lambda\}$ is biorthogonal system for E'. Since $\sup_\sigma |f(U_\sigma x) < \infty$, $x \in E, f \in E'$ we infer from Theorem 4 that $\{f_\lambda\}$ is an extended basis for E', say with corresponding biorthogonal family $\{F_\lambda\}$ in E'' (which exists by Theorem 5). But

$$Jx_\lambda(f) = Jx_\lambda[\lim_\sigma \textstyle\sum_{\mu\in\sigma} F_\mu(f)f_\mu] = F_\lambda(f), \qquad f \in E'.$$

Thus

$$0 = \lim_\sigma \|f - \textstyle\sum_{\lambda\in\sigma} Jx_\lambda(f)f_\lambda\| = \lim_\sigma \sup \{|f(x - U_\sigma x)| \mid \|x\| \leq 1\},$$

which shows (i).

An extended basis for X is said to be *boundedly complete* if $\{\sum_{\lambda\in\sigma} \alpha_\lambda x_\lambda\}$ converges for each scalar family $\{\alpha_\lambda\}$ with $\sup_\sigma \|\sum_{\lambda\in\sigma} \alpha_\lambda x_\lambda\| < \infty$.

THEOREM 13. *If $\{x_\lambda, f_\lambda\}$ is an extended basis for E which is shrinking, then $\{f_\lambda, Jx_\lambda\}$ is a boundedly complete extended basis for E'.*

Proof. As a corollary to the preceding theorem one gets the assertion that $\{f_\lambda, Jx_\lambda\}$ is an extended basis for E'. Let $\{\alpha_\lambda\}$ be such that

$$\sup_\sigma \|\textstyle\sum_{\lambda\in\sigma} \alpha_\lambda f_\lambda\| < \infty.$$

Since $\overline{\text{sp}}\ \{x_\lambda\} = E$ and $\alpha_\mu = \lim_\sigma \sum_{\lambda\in\sigma} \alpha_\lambda f_\lambda(x_\mu)$, $\mu \in \Lambda$, one can invoke the Banach-Steinhaus theorem to infer that $g(x) = \lim \sum_{\lambda\in\sigma} \alpha_\lambda f_\lambda(x)$ exists for each x in E, and that $g \in E'$. Consequently, $g(x_\mu) = \alpha_\mu$ on Λ and so

$$g = \lim_\sigma \textstyle\sum_{\lambda\in\sigma} Jx_\lambda(g)f_\lambda = \lim_\sigma \sum_{\lambda\in\sigma} \alpha_\lambda f_\lambda$$

which shows that $\{f_\lambda, Jx_\lambda\}$ is boundedly complete.

THEOREM 14. *Let $\{x_\lambda, f_\lambda\}$ be an extended basis for E such that $\overline{\text{sp}}\ \{f_\lambda\} = E'$. Then E is reflexive if and only if that basis is boundedly complete.*

Proof. If E is reflexive $\overline{\text{sp}}\ \{Jx_\lambda\} = E''$. A twofold application of Theorm 12 implies that $\{f_\lambda, Jx_\lambda\}$ is an extended basis for E' which is shrinking. Thus $\{Jx_\lambda\}$ is a boundedly complete extended basis for E'' and, since J is onto, $\{x_\lambda\}$ is also boundedly complete. Conversely let U and V be the unit balls in E and E'' respectively, and let $F \in E''$ be arbitrary. Since $J(U)$ is weakly* dense in V, there is a sequence $\{y_n\}$ in U with $F(f) = \lim_n Jy_n(f)$, $f \in E'$. Therefore,

$$\| \textstyle\sum_{\lambda\in\sigma} F(f_\lambda)x_\lambda \| = \| \lim_n \sum_{\lambda\in\sigma} Jy_n(f_\lambda)x_\lambda \|$$
$$= \lim_n \| \textstyle\sum_{\lambda\in\sigma} U_\sigma\, y_n \| \le \sup_\sigma \| U_\sigma \| < \infty .$$

If $\{x_\lambda, f_\lambda\}$ is boundedly complete, there must be an $x \in E$ such that

$$F(f_\mu) = \lim_\sigma \textstyle\sum_{\lambda\in\sigma} F(f_\lambda)Jx_\lambda(f_\mu) = Jx(f_\mu)$$

for all $\mu \in \Lambda$. Thus, due to $\overline{\text{sp}}\ \{f_\lambda\} = E'$, one has $F = Jx$ as we wished to prove.

The next theorem is the analogue of the James theorem [7] which is valid for separable Banach spaces.

THEOREM 15. *A Banach space E with an extended basis is reflexive if and only if that basis is both shrinking and boundedly complete.*

Proof. If $\{x_\lambda, f_\lambda\}$ is a shrinking and boundedly complete extended basis for E, then $\overline{\text{sp}}\ \{f_\lambda\} = E'$ and, by the last theorem, E must be reflexive. For the argument in the other direction, let E be reflexive and have an extended basis $\{x_\lambda, f_\lambda\}$. Then $\{f_\lambda, Jx_\lambda\}$ is a biorthogonal system for E' and for each $x \in E$ and $f \in E'$ one obtains

$$Jx(f) = f(x) = \lim_\sigma \textstyle\sum_{\lambda\in\sigma} f_\lambda(x)f(x_\lambda) = \lim_\sigma Jx[\sum_{\lambda\in\sigma} Jx_\lambda(f)f_\lambda].$$

Due to Theorem 5 and Corollary 6, $\{f_\lambda, Jx_\lambda\}$ thus is an extended basis for E', so $\{x_\lambda\}$ has to be shrinking. By the preceding theorem it finally follows that $\{x_\lambda\}$ is boundedly complete and we are done.

It is very instructive to see that Corollary 9[1] can now also be derived from some fundamental properties of the space l_∞ : (i) Weak and weak* convergence of sequences are equivalent in $(l_\infty)'$ [4, p. 109] and (ii) every weakly compact linear transformation from l_∞ into a Banach space maps weakly convergent sequences into strongly convergent sequences [6, p. 445–494] (based on these properties, Dean [5] proved that l_∞ does not have a Schauder decomposition). Now, it can be shown that (i) implies that every weak* extended basis for $(l_\infty)'$ is an extended basis for $(l_\infty)'$. Thus, in view of Theorems 11 and 14, an extended basis for l_∞ cannot be boundedly complete. Suppose now that $\{x_\lambda\}$ is such a basis, then it is possible to establish a sequence $\{\sigma_i\}$ of mutually dis-

[1] The author realized Theorem 8 only near the end of his work on this paper and it provided a more direct proof of Corollary 9.

joint sets in Σ and a sequence of unit vectors $\{y_i\}$ with $y_i \in \text{sp}\,\{x_\lambda \mid \lambda \in \sigma_i\}$ which converges weakly to zero. To this sequence one can find a bounded biorthogonal sequence $\{g_i\}$ in $(l_\infty)'$ such that $Z = \overline{\text{sp}}\,\{g_i\}$ in $(l)'$ is reflexive. The evaluation map T of l_∞ into Z' is then weakly compact [6, p. 483]. Hence by (ii), $\{Ty_i\}$ converges strongly in Z', and the limit must be zero. This contradicts the fact that $\{Ty_i\}$ is bounded away from zero, which shows that l_∞ does not have an extended basis.

5. Connections with the structure of the space

LEMMA 16. *Let $\{x_\lambda\}$ be an extended basis for E and let $\{\alpha_\lambda\}$ and $\{\beta_\lambda\}$ be scalar families such that $|\alpha_\lambda| \leq |\beta_\lambda|$ for all $\lambda \in \Lambda$. Then for each σ in Σ,*

$$\| \textstyle\sum_{\lambda\in\sigma} \alpha_\lambda x_\lambda \|' \leq \| \textstyle\sum_{\lambda\in\sigma} \beta_\lambda x_\lambda \|'$$

where $\| \quad \|'$ is the norm on E given in Lemma 2.

Proof. Without loss of generality one may assume that $\beta_\lambda \neq 0$ on σ. The lemma now follows directly from the estimate

$$\begin{aligned} \| \textstyle\sum_{\lambda\in\sigma} \alpha_\lambda x_\lambda \|' &= \sup \{ \| \textstyle\sum_{\lambda\in\sigma\cap\tau} \gamma_\lambda \alpha_\lambda x_\lambda \| \mid \{\gamma_\lambda\} \in S,\ \tau \in \Sigma \} \\ &= \sup \{ \| \textstyle\sum_{\lambda\in\sigma\cap\tau} \gamma_\lambda (\alpha_\lambda/\beta_\lambda)\beta_\lambda x_\lambda \| \mid \{\gamma_\lambda\} \in S,\ \tau \in \Sigma \} \\ &\leq \{ \| \textstyle\sum_{\lambda\in\sigma\cap\tau} \gamma_\lambda \beta_\lambda x_\lambda \| \mid \{\gamma_\lambda\} \in S,\ \tau \in \Sigma \} \\ &= \| \textstyle\sum_{\lambda\in\sigma} \beta_\lambda x_\lambda \|'. \end{aligned}$$

THEOREM 17. *If E has a boundedly complete extended basis, then E is weakly sequentially complete.*

Proof. Let $\{x_\lambda, f_\lambda\}$ be an extended basis for E and $\{y_j\}$ any weakly convergent sequence in E. Then there is a constant $M > 0$ with $\sup_j \| y_j \| \leq M$. Defining $\alpha_\lambda = \lim_j f_\lambda(y_j)$, one gets

$$\sup_\sigma \| \textstyle\sum_{\lambda\in\sigma} \alpha_\lambda x_\lambda \| = \sup_\sigma \| \lim_j \textstyle\sum_{\lambda\in\sigma} f_\lambda(y_j) x_\lambda \| \leq M \sup_\sigma \| U_\sigma \| < \infty.$$

Since $\{x_\lambda, f_\lambda\}$ is boundedly complete, there is a $y \in E$ such that $\lim_\sigma \sum_{\lambda\in\sigma} \alpha_\lambda x_\lambda = y$, and $\alpha_\lambda = f_\lambda(y)$ on Λ.

We prove the theorem by contradiction. Let $\{y_j\}$ have no weak limit in E. In particular y then is not a weak limit of $\{y_j\}$ in E. Thus there is an $f \in E'$ of norm one, an $\varepsilon > 0$ and a subsequence $\{y_j'\}$ of $\{y_j\}$ such that $z_j = y - y_j'$, $\text{Re}\, f(z_j) > \varepsilon$ for all j. By hypothesis, there is a sequence $\{\sigma_j\}$ in Σ with $\sigma_1 \subset \sigma_2 \subset \cdots$ and an increasing sequence $\{n_j\}$ of integers such that σ_0 is the void set, $n_1 = 1$, $\| z_{n_j} - U_{\sigma_j} z_{n_j} \| < \delta$ and $\| U_{\sigma_j} z_{n_{j+1}} \| < \delta$, where $\delta > 0$ shall be fixed later on. Using Lemmas 2 and 16, one obtains for any scalar sequence $\{\beta_j\}$,

$$\begin{aligned} &\| \textstyle\sum_{j=1}^n \beta_j z_{n_j} \| \\ &\geq \| \textstyle\sum_{j=1}^n \beta_j U_{\sigma_j - \sigma_{j-1}} z_{n_j} \| - \| \textstyle\sum_{j=1}^n \beta_j (z_{n_j} - U_{\sigma_j} z_{n_j}) \| - \| \textstyle\sum_{j=1}^n \beta_j U_{\sigma_{j-1}} z_{n_j} \| \\ &> K \| \textstyle\sum_{j=1}^n |\beta_j| \, U_{\sigma_j - \sigma_{j-1}} z_{n_j} \| - 2\delta \textstyle\sum_{j=1}^n |\beta_j| \end{aligned}$$

$$\geq K \sum_{j=1}^{n} |\beta_j| [\operatorname{Re} f(z_n) - \| z_{n_j} - U_{\sigma_j} z_{n_j} \| - \| U_{\sigma_{j-1}} z_{n_j} \|] - 2\delta \sum_{j=1}^{n} |\beta_j|$$
$$> [K\varepsilon - 2(K+1)\delta] \sum_{j=1}^{n} |\beta_j|,$$

where K is a constant in $(0, 1]$. But

$$\| \sum_{j=1}^{n} \beta_j z_{n_j} \| \leq (\| y \| + M) \sum_{j=1}^{n} |\beta_j|.$$

Choosing $\delta < \varepsilon K/2(K+1)$ it is clear that the linear transformation

$$T : l_1 \to E,$$

defined by $T\{\beta_j\} = \lim_n \sum_{j \leq n} \beta_j z_{n_j}$, $\{\beta_j\} \in l_1$, is bounded. Due to

$$\| T\{\beta_j\} \| \geq [K\varepsilon - 2(K+1)\delta] \| \{\beta_j\} \|,$$

T is a topological isomorphism and so $T(l_1)$ is weakly sequentially complete since l_1 is. $\{z_{n_j}\}$ is in $T(l_1) \subset E$ (one has $z_{n_j} = T\{\delta_{ji}\}$), and by hypothesis and the Hahn-Banach theorem, $\lim_j g(z_{n_j})$ exists for all $g \in T(l_1)'$. Hence $\{z_{n_j}\}$ converges to a $z \in T(l_1)$ in the weak topology of $T(l_1)$ and

$$\begin{aligned} z &= \lim_\sigma \sum_{\lambda\in\sigma} f_\lambda(z) x_\lambda \\ &= \lim_\sigma \sum_{\lambda\in\sigma} x_\lambda \lim_j f_\lambda(z_{n_j}) \\ &= \lim_\sigma \sum_{\lambda\in\sigma} x_\lambda [f_\lambda(y) - \lim_j f_\lambda(y'_{n_j})] \\ &= \lim_\sigma \sum_{\lambda\in\sigma} x_\lambda [\alpha_\lambda - \lim_j f_\lambda(y_j)] = 0. \end{aligned}$$

Since now $\{y'_{n_j}\}$ and hence $\{y_j\}$ converge weakly to y (which contradicts our assumption), E is weakly sequentially complete.

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