RETRACTS OF SEMICOMPLEXES

BY

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1. Introduction

The quasi-complexes (simple weak semicomplexes) of Lefschetz [13, p. 322] and the semicomplexes of F. Browder [4] are both characterized as special cases of weak semicomplexes, as was shown in [17] and announced in [16]. Moreover, we described there a class of structures called simple semicomplexes which is, in a sense, the intersection of the theories of Lefschetz and Browder. In addition to their fixed point applications in analysis, these various types of semicomplexes have been treated topologically in [3], [5], [6], [7], [10], [15], [16], [17] and [18].

In the more restrictive settings for fixed point theory—such as the convexoids of Leray and the Euclidean neighborhood retracts of Dold discussed in [2, p. 249], [8] and [9]—the use of retractions has proved most helpful. Thus it is natural to consider the question posed to the author by Browder as to which, if any, of the classes of semicomplexes are closed under retractions. We answer this in Section 2 of this paper by showing that a retract of a weak semicomplex is again a weak semicomplex and that, under a mild restriction, a similar statement is true for semicomplexes. Additionally, we note in Section 3 that corresponding results are probably false for simple (weak) semicomplexes unless a fairly strong condition called properness is imposed on the retraction involved.

In recent years several different proofs have been given, showing that a local fixed point index can be defined on compact metric absolute neighborhood retracts (ANR's). Of particular interest in our context are the demonstrations given in [4] that ANR's are semicomplexes and by the author in [17] that the standard HLC^* spaces of [14] (and hence, a fortiori, ANR's) are semicomplexes. All of these arguments seem to utilize some quite tedious ad hoc construction to link the properties of ANR's with the requirements of a fixed point theory. It has been our feeling that the state of the art should be such that one could observe immediately, from known results about a fixed point theory and from some standard characterization of ANR's, that the desired local fixed point index exists. Using the results on retractions and a product theorem from [17], we are able to give such a "one line" proof in Section 4 of this paper.

We conclude these introductory remarks by establishing several definitions and notational conventions. Only compact Hausdorff spaces are considered in this paper and, if X is such a space, $\Sigma(X)$ will denote the set of all finite covers of X by open sets. The star of a member α of $\Sigma(X)$ [19, p. 133] will be written as st (α) and, if $A \subseteq X$, we define $\alpha \cap A$ to be $\{U \cap A \mid U \in \alpha\} \in \Sigma(A)$.

Received November 20, 1968.

[1] This research was supported by a National Science Foundation grant.

If $\alpha \in \Sigma(X)$, then N_α will stand for the nerve of α and $C(N_\alpha)$ will stand for the chain complex of N_α with coefficients in the set Q of rational numbers. The support of a chain $c \in C(N_\alpha)$ is the union of all sets in α which appear as vertices of some simplex in c, and is written as sup (c). When $\alpha, \beta \in \Sigma(X)$ with β refining α (i.e., $\beta > \alpha$), $\pi_\alpha^\beta : C(N_\beta) \to C(N_\alpha)$ will denote any of the usual chain maps induced by a vertex transformation based on set inclusion. We will use rational coefficients for all homology and employ Čech theory for non-polyhedral spaces.

The definitions of semicomplex (SC), weak semicomplex (WSC), simple semicomplex (SSC) and simple weak semicomplex (SWSC) are all given in [16] and [17], where the following relations are established between these structures and the Lefschetz fixed point theorem is proved for WSC's.

$$\begin{tikzcd}[column sep=small] & & \text{WSC} & \\ \text{quasi-complex} \arrow[r, Leftrightarrow] & \text{SWSC} \arrow[ur, Rightarrow] & & \text{SC} \arrow[ul, Rightarrow] \\ & & \text{SSC} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & \end{tikzcd}$$

For the convenience of the reader, these definitions, as well as those of SC-morphism, equivalence of SC's and fully reducible categories of SC's are reproduced in Section 5 of this paper.

Finally, recall that a retraction of a space X onto a subspace A is a map $r : X \to A$ such that the inclusion map $i : A \to X$ followed by r is the identity on A.

2. Main theorems

To avoid notational confusion we will treat WSC's and SC's separately. However, the underlying notions are identical.

THEOREM (2.1). *If $S(X)$ is a weak semicomplex and $r : X \to A$ is a retraction, then there exists a weak semicomplex $S(A)$ on A.*

When dealing with an SC, $S(X) = \{X, \mathcal{I}, \Omega, \alpha_0, C\}$, instead of a WSC we are forced to add the hypothesis that $\mathcal{I}$ is well ordered by refinement. This is an annoying technical condition but not a serious restriction since, for example, it is clear that any SC on a metric space can be assumed to satisfy this requirement.

THEOREM (2.2). *If $S(X) = \{X, \mathcal{I}, \Omega, \alpha_0, C\}$ is a semicomplex, $\mathcal{I}$ is well ordered by refinement and $r : X \to A$ is a retraction, then there exists a semicomplex $S(A)$ on A. Moreover, if $S(X)$ is simple, then $S(A)$ can be taken so that $r : S(X) \to S(A)$ is an SC-morphism.*

Since the proof of Theorem (2.2) is more complex, we will give that proof in detail and then indicate the modifications necessary to prove Theorem (2.1).

Proof of Theorem (2.2). We will use the notation of Definition (5.3) of this paper without further comment. Let $\mathcal{G}$ be the collection of all members of $\Sigma(A)$ which consist of connected open sets. Since A is closed in X, r is a closed map and A inherits local connectedness from X. Thus $\mathcal{G}$ is cofinal in $\Sigma(A)$.

If $\mu \in \mathcal{G}$, let $\lambda(\mu) = \min_{\lambda \in \mathfrak{s}} \{\lambda \mid \text{st } (\lambda \cap A) > \mu$ and λ refines r^{-1} of some double star refinement of $\mu\}$. Since $\mathfrak{s}$ is well ordered this yields an order preserving function from $\mathcal{G}$ into $\mathfrak{s}$. We now let $\beta_0(\mu) = \alpha_0(\lambda(\mu)) \cap A$ and note that $\beta_0 : \mathcal{G} \rightarrow \Psi = \text{image } \beta_0$ is an order preserving function such that $\beta_0(\mu) > \mu$ for all $\mu \in \mathcal{G}$.

Now suppose that $\mu \in \mathcal{G}$ and $\alpha, \beta \in \Psi$ with $\alpha > \beta > \beta_0 = \beta_0(\mu)$. Let $\alpha_0 = \alpha_0(\lambda(\mu))$ and take $i_{\alpha_0}^{\beta_0} : N_{\beta_0} \rightarrow N\alpha_0$ to be the simplicial map induced by inclusion. For simplicity of notation, denote the corresponding chain map by the same symbol. Let $\alpha = \beta_0(\nu)$ and pick $\omega = \omega(\alpha, \beta) \in \Omega$ such that ω is a common refinement of $r^{-1}(\alpha)$, $\alpha_0(\lambda(\mu))$ and $\alpha_0(\lambda(\nu))$. We now define

$$d_\alpha^\beta = r_\alpha^\omega c_\omega^{\alpha_0} i_{\alpha_0}^{\beta_0} \pi_{\beta_0}^\beta : C(N_\beta) \rightarrow C(N_\alpha),$$

where $c_\omega^{\alpha_0} \in C_{\lambda(\mu)}$ and r_α^ω is induced by set inclusion. Let D_μ be the set of all such chain maps and let D assign to each $\mu \in \mathcal{G}$ the set D_μ. Since chain maps in $C_{\lambda(\mu)}$ preserve the Kronecker index, so must those in D_μ. We will show that $S(A) = \{A, \mathcal{G}, \Psi, \beta_0, D\}$ is an SC.

Before verifying the axioms it is expedient to note that there is a star refinement μ_1 of μ such that if $\sigma \in N_\beta$, then there exists $U \in \mu_1$ with

$$\sup(\sigma) \cup \sup(d_\alpha^\beta(\sigma)) \subseteq U.$$

To see this, let $\rho = i_{\alpha_0}^{\beta_0} \pi_{\beta_0}^\beta(\sigma)$ and use the fact that $S(X)$ is an SC to pick $V \in \lambda(\mu)$ such that $\sup(\rho) \cup \sup(c_\omega^{\alpha_0}(\rho)) \subseteq V$. By the way $\lambda(\mu)$ was chosen, $r(\text{st } V)$ is contained in a member U of some star refinement of μ. It is easily checked that U has the required property.

Suppose $\alpha > \beta > \gamma > \beta_0 = \beta_0(\mu)$, $\alpha, \beta, \gamma \in \Psi$, d_α^β, $d_\alpha^\gamma \in D_\mu$ and $\alpha_0 = \alpha_0(\lambda(\mu))$. In defining these chain maps covers $\omega_1 = \omega(\alpha, \beta)$ and $\omega_2 = \omega(\alpha, \gamma)$ have been selected in Ω. We now let $\omega \in \Omega$ be some common refinement of these, and construct the following sequence of chain homotopies by using the SC axioms for $S(X)$ and the fact that different projections and inclusions induce chain homotopic chain maps:

$$d_\alpha^\gamma \pi_\gamma^\beta = r_\alpha^{\omega_2} c_{\omega_2}^{\alpha_0} i_{\alpha_0}^{\beta_0} \pi_{\beta_0}^\gamma \pi_\gamma^\beta \sim r_\alpha^{\omega_2} \pi_{\omega_2}^\omega c_\omega^{\alpha_0} i_{\alpha_0}^{\beta_0} \pi_{\beta_0}^\beta$$
$$\sim r_\alpha^{\omega_1} \pi_{\omega_1}^\omega c_\omega^{\alpha_0} i_{\alpha_0}^{\beta_0} \pi_{\beta_0}^\beta \sim r_\alpha^{\omega_1} c_{\omega_1}^{\alpha_0} i_{\alpha_0}^{\beta_0} \pi_{\beta_0}^\beta = d_\alpha^\beta.$$

Note that each of the chain homotopies used above has suitably small support in terms of the cover μ_1 or in terms of $\lambda(\mu)$—and therefore of μ_1 under the effect of chain maps induced by r. Consequently, the composite chain homotopy connecting $d_\alpha^\gamma \pi_\gamma^\beta$ and d_α^β has the condition required in axiom (i) for an SC. Axiom (ii) is verified by a similar argument which need not be given here.

Assume that $\alpha > \beta_0 = \beta_0(\lambda(\mu))$, $\alpha \in \Psi$, $\alpha_0 = \alpha_0(\lambda(\mu))$ and $\omega = \omega(\alpha, \alpha)$ so that $d^{\alpha}_{\alpha *} = r^{\omega}_{\alpha *} c^{\alpha_0}_{\omega *} i^{\beta_0}_{\alpha_0 *} \pi^{\alpha}_{\beta_0 *}$. Recalling that we are using Čech homology with rational coefficients, consider $H(X)$ as based on the directed set

$$\Omega' = \{\beta \in \Omega \mid \beta > \alpha_0\}$$

and $H(A)$ as based on $\Psi' = \{\gamma \in \Psi \mid \gamma > \beta_0\}$. If $\beta \in \Omega'$, let $p_\beta : H(X) \to H(N_\beta)$ be the projection homomorphism and denote its image by $H'(N_\beta)$. We will adopt a similar convention for $H(A)$. To establish axiom (iii) of an SC for $S(A)$ we must show that $d^{\alpha}_{\alpha *}$ is an idempotent endomorphism on $H(N_\alpha)$ whose image is $H'(N_\alpha)$. This is accomplished by proving that $d^{\alpha}_{\alpha *} \mid H'(N_\alpha)$ is the identity homomorphism and that image $d^{\alpha}_{\alpha *} = H'(N_\alpha)$.

Let $a(\alpha) \in H'(N_\alpha)$ with $a(\alpha) = p_\alpha(a)$ where $a = \{a(\gamma)\} \in H(A)$. Let $i : A \to X$ be inclusion and set $i_*(a) = x = \{x(\beta)\} \in H(X)$. Considering how i_* is defined, we see that $x(\alpha_0) = i^{\beta_0}_{\alpha_0 *} \pi^{\alpha}_{\beta_0 *}(a(\alpha))$. Now $x(\alpha_0) \in H'(N_{\alpha_0})$ so $c^{\alpha_0}_{\omega *}(x(\alpha_0)) = x(\omega)$ because $c^{\alpha_0}_{\omega *} \mid H'(N_{\alpha_0})$ is an isomorphism whose inverse is $\pi^{\omega}_{\alpha_0 *}$. Since r is a retraction we have $r_*(x) = a$. Thus $a(\alpha) = r^{\omega}_{\alpha *}(x(\omega))$ and, combining this with our previous result, we have $a(\alpha) = d^{\alpha}_{\alpha *}(a(\alpha))$.

One consequence of the last paragraph is that image $d^{\alpha}_{\alpha *} \supseteq H'(N_\alpha)$. Conversely, suppose that $c(\alpha) = d^{\alpha}_{\alpha *}(b(\alpha))$ and let

$$y(\omega) = c^{\alpha_0}_{\omega *} i^{\beta_0}_{\alpha_0 *} \pi^{\alpha}_{\beta_0 *}(b(\alpha)).$$

Since $y(\omega) \in H'(N_\omega)$ we can extend this to $\{y(\beta)\} \in H(X)$. If $\alpha < \gamma \in \Psi'$, pick a common refinement $\delta \in \Omega'$ of $r^{-1}(\gamma)$ and ω, and let $c(\gamma) = r^{\delta}_{\gamma *}(y(\delta))$. Now

$$\begin{aligned}\pi^{\gamma}_{\alpha *}(c(\gamma)) = \pi^{\gamma}_{\alpha *} r^{\delta}_{\gamma *}(y(\delta)) = r^{\omega}_{\alpha *} \pi^{\delta}_{\omega *}(y(\delta)) = r^{\omega}_{\alpha *} \pi^{\delta}_{\omega *} c^{\omega}_{\delta *}(y(\omega))\\ = r^{\omega}_{\alpha *}(y(\omega)) = d^{\alpha}_{\alpha *}(b(\alpha)) = c(\alpha).\end{aligned}$$

If $\alpha > \gamma \in \Psi'$, let $c(\gamma) = \pi^{\alpha}_{\gamma *}(c(\alpha))$. Putting these together we have defined $c = \{c(\gamma)\} \in H(A)$ such that $p_\alpha(c) = c(\alpha)$. Thus $c(\alpha) \in H'(N_\alpha)$, image $d^{\alpha}_{\alpha *} = H'(N_\alpha)$ and as we noted earlier, axiom (iii) is verified for $S(A)$.

To check axiom (iv), let $\mu_1 > \mu_2$ with $\mu_1, \mu_2 \in \Psi$. β_0 was constructed to preserve order so that $\beta_0(1) = \beta_0(\mu_1) > \beta_0(\mu_2) = \beta_0(2)$. Suppose that $\alpha > \beta > \beta_0(1)$ and use axioms (i) and (ii) for $S(A)$ together with the SC axioms for $S(X)$ to write the following sequence of chain homotopies which connects $d(2)^{\beta}_{\alpha} \in D_{\mu_2}$ and $d(1)^{\beta}_{\alpha} \in D_{\mu_1}$. To simplify notation, let $\omega_i = \omega(\alpha, \beta)$ for μ_i and pick a common refinement ω of ω_1 and ω_2. Also, let $\lambda_i = \lambda(\mu_i)$ and denote members of C_{λ_i} by appending the symbol (i). Similarly, let $\alpha_0(\lambda(\mu_i)) = \alpha_0(i)$. Now

$$\begin{aligned} d(2)^{\beta}_{\alpha} &= r^{\omega_2}_{\alpha} c(2)^{\alpha_0(2)}_{\omega_2} i^{\beta_0(2)}_{\alpha_0(2)} \pi^{\beta}_{\beta_0(2)} \sim r^{\omega_2}_{\alpha} c(2)^{\alpha_0(2)}_{\omega_2} \pi^{\alpha_0(1)}_{\alpha_0(2)} i^{\beta_0(1)}_{\alpha_0(1)} \pi^{\beta}_{\beta_0(1)} \\ &\sim r^{\omega_2}_{\alpha} c(2)^{\alpha_0(1)}_{\omega_2} i^{\beta_0(1)}_{\alpha_0(1)} \pi^{\beta}_{\beta_0(1)} \sim r^{\omega_2}_{\alpha} c(1)^{\alpha_0(1)}_{\omega_2} i^{\beta_0(1)}_{\alpha_0(1)} \pi^{\beta}_{\beta_0(1)} \\ &\sim r^{\omega_2}_{\alpha} \pi^{\omega}_{\omega_2} c(1)^{\alpha_0(1)}_{\omega} i^{\beta_0(1)}_{\alpha_0(1)} \pi^{\beta}_{\beta_0(1)} \sim r^{\omega_1}_{\alpha} \pi^{\omega}_{\omega_1} c(1)^{\alpha_0(1)}_{\omega} i^{\beta_0(1)}_{\alpha_0(1)} \pi^{\beta}_{\beta_0(1)} \\ &\sim r^{\omega_1}_{\alpha} c(1)^{\alpha_0(1)}_{\omega_1} i^{\beta_0(1)}_{\alpha_0(1)} \pi^{\beta}_{\beta_0(1)} = d(1)^{\beta}_{\alpha}. \end{aligned}$$

As with earlier axioms, these chain homotopies are constructed from chain homotopies whose support with respect to μ_i and λ_i is such that the resulting chain homotopy of $d(2)^{\beta}_{\alpha}$ to $d(1)^{\beta}_{\alpha}$ has the property required in axiom (iv). This completes the proof that $S(A)$ is an SC.

Suppose now that $S(\mathrm{X})$ is simple (see Definition (5.4)). Using the notation of Definition (5.5) we will show that $r : S(X) \to S(A)$ is an SC-morphism.

If $\lambda \in \mathscr{I}$ and $\mu \in \mathcal{J}$, let $\lambda_1 = \lambda_1(\lambda, \mu) \in \mathscr{I}$ be any common refinement of $\lambda(\mu)$ and λ and let $\mu_1 = \mu_1(\lambda, \mu) = \mu \in \mathcal{J}$. Since members of C_{λ_1} are chain homotopic to the corresponding members of $C_{\lambda(\mu)}$, we will suppress the distinction in the following computations.

Suppose $\varphi, \psi \in \Sigma(x)$ and $\chi, \omega \in \Sigma(A)$ are given. Let $\delta \in \Psi$ be a common refinement of $\beta_0 = \beta_0(\mu_1)$ and ω; let $\beta \in \Omega$ be a common refinement of $r^{-1}(\delta)$, $\alpha_0(\lambda_1)$ and ψ; let $\gamma \in \Psi$ be a common refinement of $\beta \cap A$, δ and χ; and let $\alpha \in \Omega$ be a common refinement of $r^{-1}(\gamma)$, β, $\bar{\omega} = \omega(\gamma, \delta)$ and φ. We will now show that $d^{\delta}_{\gamma} r^{\beta}_{\delta}$ is chain homotopic to $r^{\alpha}_{\gamma} c^{\beta}_{\alpha}$.

The first step is to use the axioms for the SC, $S(A)$, and the SSC, $S(X)$ to construct a sequence of chain homotopies. As usual, α_0 stands for $\alpha_0(\lambda(\mu))$.

$$
\begin{aligned}
d^{\delta}_{\gamma} r^{\beta}_{\delta} = r^{\bar{\omega}}_{\gamma} c^{\alpha_0}_{\bar{\omega}} i^{\beta_0}_{\alpha_0} \pi^{\delta}_{\beta_0} r^{\beta}_{\delta} &\sim r^{\bar{\omega}}_{\gamma} \pi^{\alpha}_{\bar{\omega}} c^{\alpha_0}_{\alpha} i^{\beta_0}_{\alpha_0} \pi^{\delta}_{\beta_0} r^{\beta}_{\delta} \sim r^{\alpha}_{\gamma} c^{\alpha_0}_{\alpha} i^{\beta_0}_{\alpha_0} \pi^{\delta}_{\beta_0} r^{\beta}_{\delta} \\
&\sim r^{\alpha}_{\gamma} c^{\alpha_0}_{\alpha} i^{\beta_0}_{\alpha_0} \pi^{\delta}_{\beta_0} r^{\beta}_{\delta} c^{\beta}_{\beta} \sim r^{\alpha}_{\gamma} c^{\alpha_0}_{\alpha} i^{\beta_0}_{\alpha_0} \pi^{\delta}_{\beta_0} r^{\beta}_{\delta} \pi^{\alpha}_{\beta} c^{\beta}_{\alpha} \sim r^{\alpha}_{\gamma} c^{\alpha_0}_{\alpha} i^{\beta_0}_{\alpha_0} \pi^{\delta}_{\beta_0} \pi^{\gamma}_{\delta} r^{\alpha}_{\gamma} c^{\beta}_{\alpha} \\
&\sim r^{\alpha}_{\gamma} c^{\alpha_0}_{\alpha} \pi^{\beta}_{\alpha_0} i^{\gamma}_{\beta} r^{\alpha}_{\gamma} c^{\beta}_{\alpha} \sim r^{\alpha}_{\gamma} c^{\beta}_{\alpha} i^{\gamma}_{\beta} r^{\alpha}_{\gamma} c^{\beta}_{\alpha} .
\end{aligned}
$$

Denote this last chain map by e.

We want to show that $e \sim r^{\alpha}_{\gamma} c^{\beta}_{\alpha}$. This is complicated by the fact that $r^{\alpha}_{\gamma} c^{\beta}_{\alpha} i^{\gamma}_{\beta}$ need not be chain homotopic to the identity chain map on $C(N_{\gamma})$. The key tool at our disposal is that $r_* i_* = 1 : H(A) \to H(A)$. This is used most efficiently if we consider the relation of homology rather than chain homotopy. As was shown in [17, Lemma (II, 4.2)] these two notions coincide in the present context. Thus $e \sim r^{\alpha}_{\gamma} c^{\beta}_{\alpha}$ if and only if $e_* = r^{\alpha}_{\gamma *} c^{\beta}_{\alpha *}$.

Let $x(\beta) \in H(N_{\beta})$ and note that since $S(X)$ is simple, $p_{\beta}(H(X)) = H(N_{\beta})$. Hence there exists $x = \{x(\varphi)\} \in H(X)$ such that $p_{\beta}(x) = x(\beta)$ and $c^{\beta}_{\alpha *}(x(\beta)) = x(\alpha)$. Let $r_*(x) = a = \{a(\psi)\} \in H(A)$, which implies that $a(\gamma) = r^{\alpha \dagger}_{\gamma}(x(\alpha))$. Now $p_{\alpha} i_*(a) = c^{\beta}_{\alpha *} p_{\beta} i_*(a) = c^{\beta}_{\alpha *} i^{\gamma}_{\beta *}(a(\gamma))$ and $r_* i_*(a) = a$. Therefore,

$$
\begin{aligned}
r^{\alpha}_{\gamma *} c^{\beta}_{\alpha *}(x(\beta)) &= a(\gamma) = p_{\gamma}(a) = p_{\gamma} r_* i_*(a) = r^{\alpha}_{\gamma *} p_{\alpha} i_*(a) \\
&= r^{\alpha}_{\gamma *} c^{\beta}_{\alpha *} i^{\gamma}_{\beta *}(a(\gamma)) = r^{\alpha}_{\gamma *} c^{\beta}_{\alpha *} i^{\gamma}_{\beta *} r^{\alpha}_{\gamma *} c^{\beta}_{\alpha *}(x(\beta)) = e_*(x(\beta)).
\end{aligned}
$$

As we observed above, this is sufficient to give the required chain homotopy connecting $d^{\delta}_{\gamma} r^{\beta}_{\delta}$ and $r^{\alpha}_{\gamma} c^{\beta}_{\alpha}$.

The condition given in Definition (5.5) on the support of this chain homotopy must now be considered. As before, this is simply a matter of checking that the corresponding condition holds for each of the chain homotopies used. However, the technical difficulties of this task are exacerbated by the fact that some of these chain homotopies arose from the condition of two chain maps being homologous. Thus one must keep track of the size of supports on simplexes as chain homotopies are converted to homologies and back to chain homotopies. This is extremely tedious and relies heavily on the fact that $S(X)$ is simple and hence that we have information about the chain homotopy connecting maps in C_{λ_1} with the corresponding identities. As the proof requires only patience rather than ingenuity, there is no need to give it here. This completes the demonstration that $r : S(X) \to S(A)$ is an SC-morphism and establishes Theorem (2.2).

We can now indicate the required alterations of this proof for the less complicated case of WSC's.

Proof of Theorem (2.1). Let $S(X) = \{X, \Omega, C\}$ and adopt the notation of Definition (5.1). If $\mu \in \Sigma(A)$, let μ_1 be a star refinement of μ and pick $\lambda(\mu) \in \Sigma(X)$ such that $\lambda(\mu) > r^{-1}(\mu_1)$ and $\lambda(\mu) \cap A > \mu_1$. Let $\beta_0 = \beta_0(\mu) = \alpha_0(\lambda(\mu)) \cap A$ and set

$$\Psi_\mu = \{\alpha \cap A \mid \alpha \in \Omega_{\lambda(\mu)}\}.$$

If $\alpha' = \alpha \cap A$ and $\beta' = \beta \cap A$ are in Ψ_μ with $\alpha' > \beta'$, pick $\omega = \omega(\alpha', \beta') \in \Omega_{\lambda(\mu)}$ such that $\omega > r^{-1}(\alpha')$ and $\omega > \alpha$. Set $\alpha_0 = \alpha_0(\lambda(\mu))$ and define a chain map

$$d_{\alpha'}^{\beta'} = r_{\alpha'}^{\omega}\, c_{\omega}^{\alpha_0} i_{\alpha_0}^{\beta_0}\, \pi_{\beta_0}^{\beta'} : C(N_{\beta'}) \to C(N_{\alpha'})$$

where $c_\omega^{\alpha_0} \in C_{\lambda(\mu)}$ and $i_{\alpha_0}^{\beta_0}$ is induced by the inclusion map of A into X. Let D_μ be the collection of all such chain maps and let $S(A) = \{A, \Psi, D\}$. The same methods used in the proof of Theorem (2.2) can now be applied to show that $S(A)$ is a WSC.

We will use the following consequence of the proof of Theorem (2.2) in Section 4 of this paper.

COROLLARY (2.3). *Under the hypotheses of Theorem* (2.2), *if chain maps in C send chains with integral coefficients into chains with integral coefficients, then chain maps in D also have that property.*

Remark (2.4). Since the inclusion map $i : A \to X$ is basic to the notion of retraction, it is natural to inquire whether this is an SC-morphism from $S(A)$ into $S(X)$. It seems possible to give an affirmative answer to this question only when $S(A)$ and $S(X)$ are both simple. However, there is no new information gained in that case, since all maps between SSC's are known to be SC-morphisms [17, Proposition (V, 1.4)].

If a retraction $r : X \to A$ and an SC-structure on X are used to induce an SC-structure on A by the process of Theorem (2.2), we will indicate this notationally by using the same letter for the original and the induced structure. e.g., $S(X)$ induces $S(A)$ and $T(X)$ induces $T(A)$. This construction is natural since it respects the relation of equivalence ($\approx$) of SC's given in [17, Chapter III, Section 3].

PROPOSITION (2.5). *If $S(X)$ and $T(X)$ are equivalent semicomplexes which induce $S(A)$ and $T(A)$ via a retraction $r : X \to A$, then $S(A) \approx T(A)$.*

Proof. Let

$$S(X) = \{X, \mathcal{S}, \Omega, \alpha_0, C\}, \qquad T(X) = \{X, \mathcal{S}', \Omega', \alpha_0', C'\}$$

and suppose

$$S(A) = \{A, \mathcal{G}, \Psi, \beta_0, D\} \quad \text{and} \quad T(A) = \{A, \mathcal{G}', \Psi', \beta_0', D'\}$$

are formed as in Theorem (2.2). Let J be the identity map on X which is an SC-morphism of $S(X)$ into $T(X)$ and denote by I the identity map on A which we will show is an SC-morphism of $S(A)$ into $T(A)$.

If $\mu \in \Psi$ and $\mu' \in \Psi'$, pick a common refinement $\bar{\mu} \in \Psi$. Using the notation of Definition (5.5) for the SC-morphism J, we have

$$\lambda_1 = \lambda_1(\lambda(\bar{\mu}), \lambda'(\mu')) \in \Phi, \qquad \lambda_1' = \lambda_1'(\lambda(\bar{\mu}), \lambda'(\mu')) \in \Phi',$$

$$\delta = \delta(\lambda'(\mu'), \lambda'(\mu')) \in \Omega' \quad \text{and} \quad \beta = \beta(\delta, \delta) \in \Omega.$$

Let $\mu_1 = \mu_1(\mu, \mu') \in \Psi$ be a refinement of $\bar{\mu}$ such that $\beta_0(\mu_1) > \beta \cap A$ and let $\mu_1' = \mu_1'(\mu, \mu') \in \Psi'$ be a refinement of μ' such that $\beta_0(\mu_1') > \beta \cap A$.

Since $\mu_1 > \bar{\mu}$ and $\lambda_1 > \lambda(\bar{\mu})$, chain maps in $C_{\lambda(\mu_1)}$ are chain homotopic to the corresponding members of C_{λ_1} or $C_{\lambda(\bar{\mu})}$. Additionally, these chain homotopies are of suitably small support with respect to μ' under chain maps induced by r. Thus we suppress any notational distinction between these chain maps. Members of $C_{\lambda'(\mu_1')}$, $C_{\lambda_1'}$ and $C_{\lambda'(\mu')}$ will be subject to a similar convention.

To prove that $I : S(A) \to T(A)$ is an SC-morphism, it is sufficient to show that if $\bar{\alpha}, \bar{\beta} \in \Psi$ and $\bar{\gamma}, \bar{\delta} \in \Psi'$ with $\bar{\alpha} > \bar{\gamma} > \bar{\beta} > \bar{\delta}$, and $\bar{\bar{\delta}}$ is a common refinement of $\beta_0(\mu_1)$ and $\beta_0'(\mu_1')$, then $I^{\bar{\alpha}}_{\bar{\gamma}} d^{\bar{\beta}}_{\bar{\alpha}} \sim d'^{\bar{\bar{\delta}}}_{\bar{\gamma}} I^{\bar{\beta}}_{\bar{\bar{\delta}}}$ with a chain homotopy of the required type. Taking $\bar{\alpha}$, $\bar{\gamma}$, $\bar{\beta}$ and $\bar{\bar{\delta}}$ as above, use J as an SC-morphism to find $\gamma \in \Omega'$ which is a common refinement of β, $r^{-1}(\bar{\alpha})$ and $\omega'(\bar{\gamma}, \bar{\bar{\delta}})$. Also find $\alpha \in \Omega$ which refines γ and $\omega(\bar{\alpha}, \bar{\beta})$. Hence $J^{\alpha}_{\gamma} c^{\beta}_{\alpha} \sim c'^{\delta}_{\gamma} J^{\beta}_{\delta}$. Some of the relevant chain complexes and maps are shown in Figure (2.6). In the interest of clarity, the name of a cover is used to stand for the chain complex of its nerve. e.g., α replaces $C(N_\alpha)$.

Letting $\alpha_0 = \alpha_0(\lambda(\mu_1))$, $\alpha_0' = \alpha_0'(\lambda'(\mu_1'))$, $\beta_0 = \beta_0(\mu_1)$ and $\beta_0' = \beta_0'(\mu_1')$, we have

$$\begin{aligned} I^{\bar{\alpha}}_{\bar{\gamma}} d^{\bar{\beta}}_{\bar{\alpha}} &= I^{\bar{\alpha}}_{\bar{\gamma}} r^{\omega}_{\bar{\alpha}} c^{\alpha_0}_{\omega} i^{\beta_0}_{\alpha_0} \pi^{\bar{\beta}}_{\beta_0} \sim I^{\bar{\alpha}}_{\bar{\gamma}} r^{\alpha}_{\bar{\alpha}} c^{\alpha_0}_{\alpha} i^{\beta_0}_{\alpha_0} \pi^{\bar{\beta}}_{\beta_0} \sim I^{\bar{\alpha}}_{\bar{\gamma}} r^{\alpha}_{\bar{\alpha}} c^{\beta}_{\alpha} i^{\bar{\beta}}_{\beta} \\ &\sim r^{\gamma}_{\bar{\gamma}} J^{\alpha}_{\gamma} c^{\beta}_{\alpha} i^{\bar{\beta}}_{\beta} \sim r^{\gamma}_{\bar{\gamma}} c'^{\delta}_{\gamma} J^{\beta}_{\delta} i^{\bar{\beta}}_{\beta} \sim r^{\omega'}_{\bar{\gamma}} c'^{\delta}_{\omega'} J^{\beta}_{\delta} i^{\bar{\beta}}_{\beta} \sim r^{\omega'}_{\bar{\gamma}} c'^{\alpha_0'}_{\omega'} i^{\beta_0'}_{\alpha_0'} \pi^{\delta}_{\beta_0'} I^{\bar{\beta}}_{\bar{\delta}} = d'^{\bar{\delta}}_{\bar{\gamma}} I^{\bar{\beta}}_{\bar{\delta}}. \end{aligned}$$

As before, the verification that the chain homotopy given above has the required type of support on simplexes of $N_{\bar{\beta}}$ is tedious but straightforward and will be omitted.

Since I is an SC-morphism from $S(A)$ into $T(A)$, these are equivalent SC's.

Proposition (2.5) can be used to weaken the conditions given in Theorem (2.2) under which a retraction $r : S(X) \to S(A)$ is known to be an SC-morphism.

COROLLARY (2.7). *If $S(X)$ is a semicomplex which is equivalent to a*

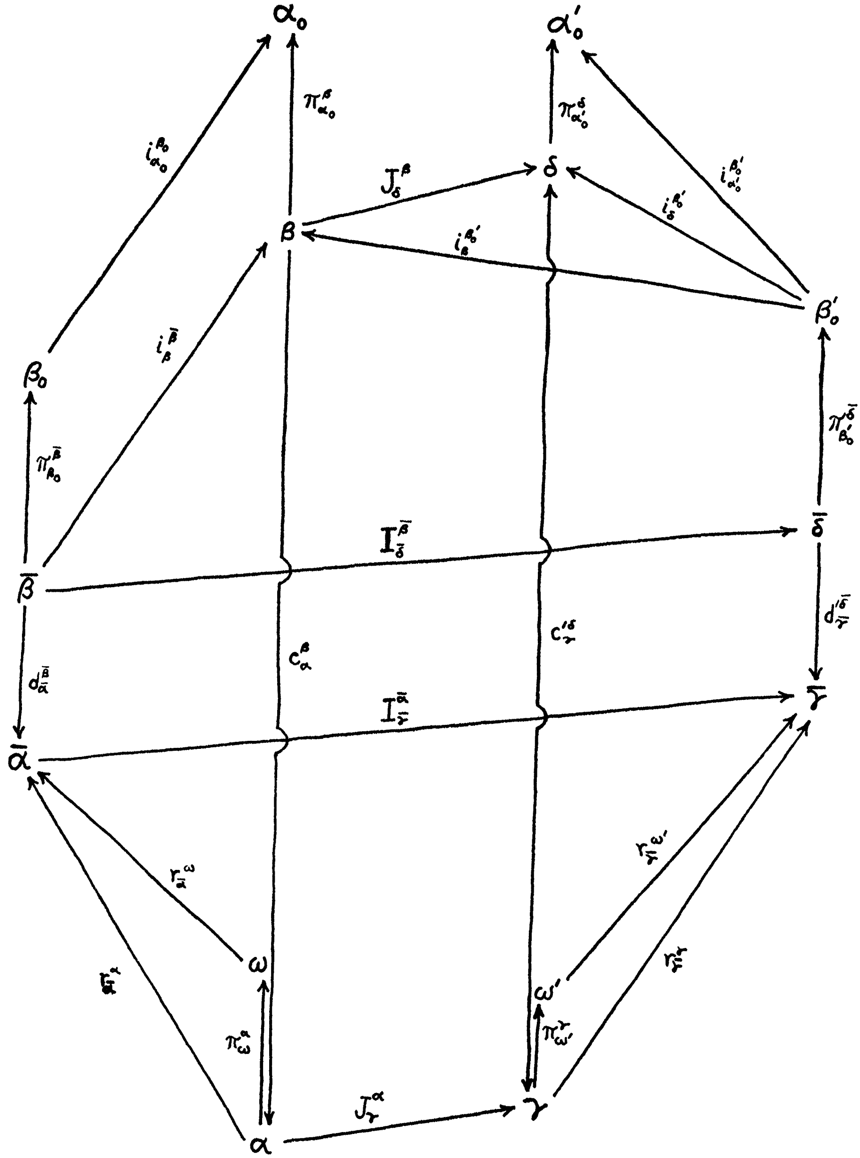

FIGURE (2.6)

simple semicomplex, $r : X \to A$ is a retraction and $S(X)$ induces $S(A)$, then r is an SC-morphism from $S(X)$ into $S(A)$.

Proof. Let $S(X) \approx T(X)$ where $T(X)$ is an SSC. By Proposition (2.5), $1_A : T(A) \to S(A)$ is an SC-morphism and, by Theorem (2.2), $r : T(X) \to T(A)$ is an SC-morphism. Thus, as seen from the following diagram, $r : S(X) \to S(A)$ is the composition of SC-morphisms and is also an SC-morphism [17, Proposition (III, 1.6)]:

$$\begin{tikzcd} S(X) \arrow[r, "1_X"] \arrow[d, "r"] & T(X) \arrow[d, "r"] \\ S(A) & T(A) \arrow[l, "1_A"'] \end{tikzcd}.$$

3. Proper retractions

It might seem natural to suppose that the construction given in Section 2 of this paper would produce a simple semicomplex $S(A)$ when applied to a simple semicomplex $S(X)$. Unfortunately, although no counterexample is known, it appears unlikely that retractions preserve simplicity in this fashion. However, we can give sufficient conditions on a retraction $r : X \to A$ to guarantee that $S(A)$ is simple.

DEFINITION (3.1). A retraction $r : X \to A$ is proper with respect to a set of covers $\Sigma' \subseteq \Sigma(X)$ if, for each $\lambda \in \Sigma'$, $r^{-1}(\lambda \cap A) < \lambda$.

PROPOSITION (3.2). *If $S(X) = \{X, \Omega, C\}$ is a simple weak semicomplex and $r : X \to A$ is a retraction which is proper with respect to Ω_λ for each $\lambda \in \Sigma(X)$, then $S(A)$ is also a simple weak semicomplex.*

PROOF. We will use the notation given in the proof of Theorem (2.1) where $S(A) = \{A, \Psi, D\}$ was defined and in Definition (5.2) of an SWSC. Suppose $\mu \in \Sigma(A)$ and $\alpha' = \alpha \cap A \in \Psi_\mu$ where $\alpha \in \Omega_{\lambda(\mu)}$. Note that $\alpha > \alpha_0 = \alpha_0(\lambda(\mu))$ and, since r is proper with respect to $\Omega_{\lambda(\mu)}$, we have $r^{-1}(\alpha') < \alpha$. Letting $\omega = \omega(\alpha', \alpha')$ and $\beta_0 = \beta_0(\mu)$, we can write

$$\begin{aligned} d_{\alpha'}^{\alpha'} = r_{\alpha'}^{\omega} c_{\omega}^{\alpha_0} i_{\alpha_0}^{\cdot\beta_0} \pi_{\beta_0}^{\alpha'} \sim r_{\alpha'}^{\omega} c_{\omega}^{\alpha_0} \pi_{\alpha_0}^{\alpha} i_{\alpha}^{\cdot\alpha'} \sim r_{\alpha'}^{\omega} c_{\omega}^{\alpha} i_{\alpha}^{\cdot\alpha'} \\ \sim r_{\alpha'}^{\alpha} \pi_{\alpha}^{\omega} c_{\omega}^{\alpha} i_{\alpha}^{\cdot\alpha'} \sim r_{\alpha'}^{\alpha} c_{\alpha}^{\alpha} i_{\alpha}^{\cdot\alpha'} \sim r_{\alpha'}^{\alpha} i_{\alpha}^{\cdot\alpha'}. \end{aligned}$$

Suppose $U' = U \cap A \in \alpha'$ and $U \in \alpha$. Since $r^{-1}(\alpha') < \alpha$ there exists $V' \in \alpha'$ such that $U \subseteq r^{-1}(V')$. Thus $U' \subseteq r(U) \subseteq V'$ and we can take $r_{\alpha'}^{\alpha} i_{\alpha}^{\cdot\alpha'}(U')$ to be V'. This shows that $r_{\alpha'}^{\alpha} i_{\alpha}^{\cdot\alpha'}$ is just one of the possible projections $\pi_{\alpha'}^{\alpha'}$. Since all projections induce chain homotopic chain maps,

$$r_{\alpha'}^{\alpha} i_{\alpha}^{\cdot\alpha'} \sim \pi_{\alpha'}^{\alpha'} \sim 1 : C(N_{\alpha'}) \to C(N_{\alpha'})$$

and $S(A)$ is seen to be simple.

PROPOSITION (3.3). *If $S(X) = \{X, \mathcal{I}, \Omega, \alpha_0, C\}$ is a simple semicomplex which induces $S(A)$ under a retraction $r : X \to A$ that is proper with respect to Ω, then $S(A)$ is a simple semicomplex.*

Proof. We will use notation from the proof of Theorem (2.2) where

$$S(A) = \{A, \mathcal{I}, \Psi, \beta_0, D\}$$

was defined and from Definition (5.4). Suppose $\mu \in \mathcal{G}$ and $d^\alpha_\alpha \in D_\mu$ with $\alpha = \beta_0(\nu) = \alpha_0(\lambda(\nu)) \cap A \in \Psi$. Since $\mathcal{G}$ is well ordered, either $\lambda(\nu) > \lambda(\mu)$ or $\lambda(\nu) < \lambda(\mu)$. To simplify the notation, let $\omega = \omega(\alpha, \alpha)$, $\alpha_0 = \alpha_0(\lambda(\mu))$, $\beta_0 = \beta_0(\mu)$ and $\alpha_1 = \alpha_0(\nu)$ so that $d^\alpha_\alpha = r^\omega_\alpha c^{\alpha_0}_\omega i^{\beta_0}_{\alpha_0} \pi^\alpha_{\beta_0}$.

First assume that $\lambda(\nu) > \lambda(\mu)$, which implies that $\alpha_1 > \alpha_0$. Now

$$d^\alpha_\alpha \sim r^\omega_\alpha c^{\alpha_0}_\omega \pi^{\alpha_1}_{\alpha_0} i^{\alpha}_{\alpha_1} \sim r^\omega_\alpha c^{\alpha_1}_\omega i^{\alpha}_{\alpha_1}$$

and, since $r^{-1}(\alpha) < \alpha_1$, this latter chain map is chain homotopic to

$$r^{\alpha_1}_\alpha \pi^\omega_{\alpha_1} c^{\alpha_1}_\omega i^{\alpha}_{\alpha_1} \sim r^{\alpha_1}_\alpha i^{\alpha}_{\alpha_1}.$$

As we saw in the proof of Proposition (3.2), this is chain homotopic to the identity chain map on $C(N_\alpha)$.

On the other hand, assume that $\lambda(\nu) < \lambda(\mu)$, which implies that $\alpha_1 < \alpha_0$ and $\alpha < \beta_0$. In this case,

$$d^\alpha_\alpha \sim r^{\alpha_1}_\alpha \pi^{\alpha_0}_{\alpha_1} \pi^\omega_{\alpha_0} c^{\alpha_0}_\omega i^{\beta_0}_{\alpha_0} \pi^\alpha_{\beta_0} \sim r^{\alpha_1}_\alpha i^{\alpha}_{\alpha_1} \sim 1{:}C(N_\alpha) \to C(N_\alpha).$$

In either case one can check directly that the chain homotopies constructed between d^α_α and the identity chain map have sufficiently small supports on generators of $C(N_\alpha)$ to satisfy axiom (v) for an SSC.

We are now in a position to analyze the failure of an arbitrary retraction to preserve simplicity.

Remark (3.4). To say that r is proper with respect to a cover $\lambda \in \Sigma(X)$ implies that r induces $r^\lambda_{\lambda'} : C(N_\lambda) \to C(N_{\lambda'})$, where $\lambda' = \lambda \cap A$. As we have seen, this is something like a retraction, since $r^\lambda_{\lambda'} i^{\lambda'}_\lambda$ is chain homotopic to the identity chain map.

Clearly there exist covers for which an arbitrary retraction, r, is proper. For example, this is the case with $r^{-1}(\mu) \in \Sigma(X)$ for any $\mu \in \Sigma(A)$. However, the collection of such covers need not be cofinal in $\Sigma(X)$.

The existence of an SWSC- or SSC-structure on X implies that there are arbitrarily fine homologically nice covers of X, i.e., covers $\alpha \in \Sigma(X)$ such that $H(N_\alpha) \simeq H(X)$. The difficulty in the general case is that r may not be proper with respect to these covers. Propositions (3.2) and (3.3) show that, when r is proper with respect to the homologically nice covers, these covers yield homologically nice covers for A. Hence simplicity is carried to A via r.

4. Applications to ANR's

As was observed in the introduction of this paper, prior definitions of either a local or a global fixed point index for mappings on ANR's have employed some type of special construction linking the ANR's to the fixed point theory used. However, the theory of SC's is now sufficiently developed to obviate any such process and thus allow one to apply the theory directly to ANR's.

THEOREM (4.1). *The class of all compact metric ANR's admits an integer-valued local fixed point index for continuous functions.*

Proof. It was shown in [17, Chapter V, Section 1] that all polyhedra, as well as the Hilbert cube I^ω, admit SSC-structures. Thus, by the product theorem for SSC's [17, Theorem (VI, 2.1)], all prisms (products of a polyhedron and I^ω) have SSC-structures. Since Borsuk shows in [1, p. 105] that any ANR is homeomorphic to a retract of a prism, we can apply the retraction Theorem (2.2) of this paper to conclude that all ANR's have SC-structures. As we noted in Section 2, the fact that prisms are metric spaces allows us to assume that their SSC-structures satisfy the hypotheses of Theorem (2.2). The existence of the local fixed point index now follows from the main result of [4, Section 3] giving such an index for classes of SC's.

To see that the index assumes only integral values, note that the antiprojections in the SSC's on polyhedra and I^ω all preserve integral chains. Because each of the theorems employed preserves this property, it is also possessed by the antiprojections in the SC's on ANR's. Hence, as was noted in [17, Remark (III, 2.7)], the index is integer-valued.

The explicit construction of SC's on HLC* spaces given in [17, Chapter IV, Section 3] yielded some additional information about ANR's which can be deduced, with much less effort, by the method used in the proof of Theorem (4.1).

PROPOSITION (4.2). *The category of all SC-structures on ANR's is a fully reducible category of semicomplexes and SC-morphisms.*

Proof. As in Definition (5.7) we must show that if $f : A \to B$ is a map of ANR's and $S(A)$ and $S(B)$ are SC's, then $f : S(A) \to S(B)$ is an SC-morphism.

Let X be a prism and suppose that $r : X \to B$ is a retraction. We showed in [17, Proposition (IV, 3.1)] that any standard HLC* space Y has a preferred SC-structure, i.e., an SC such that any map from an SC into the preferred SC is an SC-morphism. (The proof of this fact would be much simpler if it was only needed for prisms, as is the present situation. In fact, it could be based on the product theorem.) Thus, we have a preferred SC-structure, which will be called $P(X)$, and an SSC-structure, which will be called $T(X)$, that was shown to exist in Theorem (4.1).

Since $P(X)$ is preferred, $P(X) \approx T(X)$ and, by Corollary (2.7), $r : P(X) \to P(B)$ is an SC-morphism. Note that any map into $P(B)$ factors into a map into $P(X)$ followed by $r : P(X) \to P(B)$. Because the composition of SC-morphisms is an SC-morphism and both of these factors are SC-morphisms, we see that P (B) is itself a preferred SC.

Both the identity map $1 : S(B) \to P(B)$ and $f : S(A) \to P(B)$ are therefore SC-morphisms. As shown in [17, Proposition (III, 3.2)], $1 : P(B) \to S(B)$ must be an SC-morphism and, since $f : S(A) \to S(B)$ is the composition of $f : S(A) \to P(B)$ and $1 : P(B) \to S(B)$, f is an SC-morphism.

COROLLARY (4.3). *The values of the local fixed point index given in Theorem*

(4.1) *on the category of ANR's are independent of the particular semicomplexes used in their computation.*

Proof. It is shown in [17, Proposition (II, 3.8)] that this is the case for the underlying spaces of any fully reducible category of SC's and SC-morphisms.

Remark (4.4). If we are only interested in finite-dimensional ANR's–including compact topological manifolds–the results given so far in this section can be simplified still further. Borsuk shows in [1, p. 122] that all finite-dimensional ANR's are homeomorphic to retracts of polyhedra. Thus, the existence of SC-structures on such spaces can be deduced without any appeal to the product theorem or any consideration of prisms.

It is an open question whether all ANR's (or even all manifolds) admit SSC- or SWSC-structures. This is of some interest, since the existence of such a structure on a space shows that the homology groups of the space are isomorphic to those of arbitrarily fine nerves. Lefschetz stated in [13, p. 322] that all ANR's admit SWSC-structures (quasi-complexes in his terminology). However, in the twenty-six years since the publication of that book, no proof of this conjecture has been found.

In light of Propositions (3.2) and (3.3), we would have the existence of an SSC-structure on an ANR, X, if the retraction of a prism, P, onto X used in the proof of Theorem (4.1) is known to be proper with respect to the covers in the SSC-structure on P. Whether such retractions exist for all ANR's is unknown, as is the answer to the corresponding but simpler question for retractions of polyhedra onto finite-dimensional ANR's or even manifolds.

5. Appendix: definitions

In this section we reproduce several of the definitions from [16] and [17] which are used in the earlier parts of this paper.

DEFINITION (5.1). A weak semicomplex (WSC), $S(X) = \{X, \Omega, C\}$, is a triple where X is a compact Hausdorff space; Ω is a function assigning to each $\lambda \in \Sigma(X)$ a cofinal subset Ω_λ of $\Sigma(X)$ which has a designated coarsest element $\alpha_0(\lambda)$ such that $\alpha_0(\lambda) > \lambda$; and C is a function assigning to each $\lambda \in \Sigma(X)$ a family, C_λ, of chain maps consisting of one or more chain maps

$$c_\alpha^\beta : C(N_\beta) \to C(N_\alpha)$$

for every pair $\alpha, \beta \in \Omega_\lambda$ such that $\alpha > \beta > \alpha_0(\lambda)$. Each $c_\alpha^\beta \in C_\lambda$ has the property that if $\sigma \in N_\beta$ then there is a set $U \in \lambda$ with $\sup(\sigma) \cup \sup(c_\alpha^\beta(\sigma)) \subseteq U$. These chain maps are called antiprojections and are assumed to satisfy the following axioms.

(i) If $\alpha > \beta > \gamma > \alpha_0(\lambda)$, $\alpha, \beta, \gamma \in \Omega_\lambda$ and c_α^β, $c_\alpha^\gamma \in C_\lambda$ then c_α^β is chain homotopic ($\sim$) to $c_\alpha^\gamma \pi_\gamma^\beta$.

(ii) If $\alpha > \beta > \gamma > \alpha_0(\lambda)$, $\alpha, \beta, \gamma \in \Omega_\lambda$ and c_β^γ, $c_\alpha^\gamma \in C_\lambda$ then $c_\beta^\gamma \sim \pi_\beta^\alpha c_\alpha^\gamma$.

(iii) If $\alpha > \alpha_0(\lambda)$, $\alpha \in \Omega_\lambda$ and $c_\alpha^\alpha \in C_\lambda$ then

$$c_{\alpha *}^\alpha : H(N_\alpha; Q) \to H(N_\alpha; Q)$$

is an idempotent endomorphism whose image is exactly the image of the projection homomorphism $p_\alpha : H(X; Q) \to H(N_\alpha; Q)$.

DEFINITION (5.2). A weak semicomplex, $\{X, \Omega, C\}$, is called simple (SWSC) if for each λ, $\lambda \in \Sigma(X)$, $\alpha \in \Omega_\lambda$ and $c_\alpha^\alpha \in C_\lambda$, $c_\alpha^\alpha \sim 1 : C(N_\alpha) \to C(N_\alpha)$.

DEFINITION (5.3). A semicomplex (SC), $S(X) = \{X, \mathcal{I}, \Omega, \alpha_0, C\}$, is a quintuple where X is a compact Hausdorff space; $\mathcal{I}$ is a collection of finite covers of X by connected open sets which is cofinal in $\Sigma(X)$; Ω is a cofinal subset of $\Sigma(X)$; α_0 is a function from $\mathcal{I}$ into Ω such that for each $\lambda \in \mathcal{I}$, $\alpha_0(\lambda) > \lambda$; and C is a function assigning to $\lambda \in \mathcal{I}$ a family, C_λ, of chain maps consisting of one or more chain maps $c_\alpha^\beta : C(N_\beta) \to C(N_\alpha)$ for every pair $\alpha, \beta \in \Omega$ such that $\alpha > \beta > \alpha_0(\lambda)$. These chain maps c_α^β are called antiprojections and are assumed to preserve the Kronecker index as well as satisfy the following axioms.

(i) If $\alpha > \beta > \gamma > \alpha_0(\lambda)$, $\alpha, \beta, \gamma \in \Omega$ and c_α^β, $c_\alpha^\gamma \in C_\lambda$ then there exists a chain homotopy Δ_α^β connecting c_α^β and $c_\alpha^\gamma \pi_\gamma^\beta$ such that for each $\sigma \in N_\beta$ there is a set $U \in \lambda$ with

$$\sup(\sigma) \cup \sup(c_\alpha^\beta(\sigma)) \cup \sup(\Delta_\alpha^\beta(\sigma)) \subseteq U.$$

(ii) If $\alpha > \beta > \gamma > \alpha_0(\lambda)$, $\alpha, \beta, \gamma \in \Omega$ and c_β^γ, $c_\alpha^\gamma \in C_\lambda$ then there exists a chain homotopy Γ_β^γ connecting c_β^γ and $\pi_\beta^\alpha c_\alpha^\gamma$ such that for each $\sigma \in N_\gamma$ there is a set $V \in \lambda$ with

$$\sup(\sigma) \cup \sup(c_\beta^\gamma(\sigma)) \cup \sup(\Gamma_\beta^\gamma(\sigma)) \subseteq V.$$

(iii) If $\alpha > \alpha_0(\lambda)$, $\alpha \in \Omega$ and $c_\alpha^\alpha \in C_\lambda$, then

$$c_{\alpha *}^\alpha : H(N_\alpha; Q) \to H(N_\alpha; Q)$$

is an idempotent endomorphism whose image is exactly the image of the projection homomorphism $p_\alpha : H(X; Q) \to H(N_\alpha; Q)$.

(iv) If $\mu, \lambda \in \mathcal{I}$ with $\mu > \lambda$ then $\alpha_0(\mu) > \alpha_0(\lambda)$ and if $\alpha > \beta > \alpha_0(\mu)$, $\alpha, \beta \in \Omega$, $c_\alpha^\beta(\mu) \in C\mu$ and $c_\alpha^\beta(\lambda) \in C_\lambda$, then there exists a chain homotopy Θ_α^β connecting these two antiprojections such that for each $\sigma \in N_\beta$ there is a set $W \in \lambda$ with

$$\sup(\sigma) \cup \sup(c_\alpha^\beta(\lambda)(\sigma)) \cup \sup(\Theta_\alpha^\beta(\sigma)) \subseteq W.$$

DEFINITION (5.4). A semicomplex $\{X, \mathcal{I}, \Omega, \alpha_0, C\}$ is called simple (SSC) if it satisfies the following axiom.

(v) If $\lambda \in \mathcal{I}$ and $c_\alpha^\alpha \in C_\lambda$ then there exists a chain homotopy Λ_α^α connecting c_α^α and $1 : C(N_\alpha) \to C(N_\alpha)$ such that for each $\sigma \in N_\alpha$ there is a set $U \in \lambda$ with

$$\sup(\sigma) \cup \sup(c_\alpha^\alpha(\sigma)) \cup \sup(\Lambda_\alpha^\alpha(\sigma)) \subseteq U.$$

DEFINITION (5.5). Suppose that

$$S(X) = \{X, \mathcal{I}, \Omega, \alpha_0, C\} \quad \text{and} \quad S(Y) = \{Y, \mathcal{J}, \Psi, \beta_0, D\}$$

are SC's and that $h : X \to Y$ is a continuous map of spaces. h is called an SC-morphism from $S(X)$ into $S(Y)$ and written as $h : S(X) \to S(Y)$ if for each $\lambda \in \mathcal{I}$ and $\mu \in \mathcal{J}$ there are covers $\lambda_1 = \lambda_1(\lambda, \mu) \in \mathcal{I}$ and $\mu_1 = \mu_1(\lambda, \mu) \in \mathcal{J}$ with $\lambda_1 > \lambda$ and $\mu_1 > \mu$ which have the following property. For any four covers $\varphi, \psi \in \Sigma(X)$ and $\chi, \omega \in \Sigma(Y)$, four covers $\alpha, \beta \in \Omega$ and $\gamma, \delta \in \Psi$ can be picked successively so that $\delta = \delta(\mu, \omega)$ is a common refinement of $\beta_0(\mu_1)$ and ω; $\beta = \beta(\delta, \psi)$ is a common refinement of $\alpha_0(\lambda_1)$, $h^{-1}(\delta)$ and ψ; $\gamma = \gamma(\delta, \chi)$ is a common refinement of δ and χ; and $\alpha = \alpha(\beta, \gamma, \varphi)$ is a common refinement of β, $h^{-1}(\gamma)$ and φ. Further, these are assumed to have the property that if

$$h_\gamma^\alpha : C(N_\alpha) \to C(N_\gamma) \quad \text{and} \quad h_\delta^\beta : C(N_\beta) \to C(N_\beta)$$

are chain maps induced by h, $c_\alpha^\beta \in C_{\lambda_1}$ and $d_\gamma^\delta \in D_{\mu_1}$ then there exists a chain homotopy Δ connecting $h_\gamma^\alpha c_\alpha^\beta$ and $d_\gamma^\delta h_\delta^\beta$ such that for each $\sigma \in N_\beta$ there is a set $U \in \mu$ with

$$h(\sup(\sigma)) \cup \sup(\Delta(\sigma)) \subseteq U.$$

DEFINITION (5.6). Two SC's, $S(X)$ and $T(X)$, are equivalent ($\approx$) if the identity map $1_X : X \to X$ is an SC-morphism from $S(X)$ into $T(X)$.

As shown in [17, Proposition (III, 3.7)], the definition of fully reducible category given below is equivalent to the definition of that concept given in [17, Definition (III, 3.6)]. We use the alternative definition here since it is more easily verified than the original.

If $\mathcal{S}$ is a category of SC's and SC-morphisms, then let $\mathcal{S}'$ be the category of all spaces which have an SC-structure in $\mathcal{S}$ and all maps of such spaces.

DEFINITION (5.7). A category $\mathcal{S}$ of SC's and SC-morphisms is fully reducible if every map $f : X \to Y$ in $\mathcal{S}'$ is also an SC-morphism between any SC-structures on X and Y which are in $\mathcal{S}$.

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