NON-COMPACT SOLVMANIFOLDS OF DIMENSION LESS THAN 4 OR OF RANK 1 ARE VECTOR BUNDLES OVER COMPACT SOLVMANIFOLDS

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If G is an analytic group and if S is a closed subgroup with a finite number of components then G/S is a vector bundle over a compact manifold K/W, where K is a compact analytic group and W is a closed subgroup of K [16]. When G is solvable the quotient G/S is called a solvmanifold. In this case S is likely to have infinitely many components. The question is natural: Is G/S a vector bundle over a compact solvmanifold? In [12], all the 2-dimensional spaces G/S are obtained and one sees by inspection that the answer to our question is affirmative. In this paper the question is answered affirmatively for the non-compact solvmanifolds with fundamental group the integers.

Let G be a Lie group satisfying the second axiom of countability. If S is a closed subgroup of G the space of left cosets G/S is called a Klein space. Assume G/S is connected. Since G_0, the identity component of G, acts transitively on G/S by left multiplication [7, p. 114] we can suppose that G is connected. If G is solvable it is shown in [13, p. 22] that G can be assumed simply connected with S containing no proper normal analytic subgroup of G and with S_0, the identity component of S, lying in the commutator subgroup of G.

A solvmanifold is a connected Klein space G/S where G_0 is a solvable analytic group. From now on G will represent a solvable simply connected analytic group unless specifically indicated to be otherwise. We say S is full in G if S is not contained in any proper analytic subgroup of G [13, p. 13]. Now suppose that S is not full in G. Then there exists a proper closed analytic subgroup F in G which contains S. G/S is the bundle space of a fiber bundle with base G/F and fiber F/S [17, pp. 30–33]. Since G/F is Euclidean [13, p. 22] G/S is topologically $F/S \times G/F$ [13, p. 23]. So, if G/S is a non-compact 3-dimensional solvmanifold and S is not full in G then G/S is a topological product of a Euclidean space and a solvmanifold of dimension less than 3. The solvmanifolds of dimension less than 3 are a point, line, circle, plane, cylinder, Moebius band, torus, and Klein bottle [12, p. 634]. Hence if S is not full in G then G/S is indeed a vector bundle over a compact solvmanifold. Consequently, we make the standing convention that S is full in G unless we specifically indicate otherwise. We make also the following conventions: $\mathfrak{g}$ denotes the Lie algebra of G, S is a closed subgroup of G whose identity component S_0 is contained in the commutator subgroup of G, S contains no proper analytic subgroups of G which are normal in G, $\mathfrak{s}$ denotes the Lie algebra of S, N denotes the maximum nilpotent analytic subgroup of G, and $\mathfrak{N}$ its Lie algebra.

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We remark that if G is a solvable analytic group then N contains the commutator subgroup of G [2, Cor. 5, p. 67]. If B is a subset of $GL(n, \mathbf{R})$ then $[B]$ denotes the algebraic group hull of B. If D is a subset of a vector space then $\langle D \rangle$ denotes the linear hull of D.

If K is a group and c belongs to K then Γ_c denotes the map of $K \to K$ given by $\Gamma_c(k) = ckc^{-1}$. (c) denotes the cyclic group generated by c.

We say that h (in an arbitrary Lie group H) is an exp element if $h = \exp X$ for one and only one X in $\mathfrak{H}$, the Lie algebra of H. The Lie group H is said to be exponential if exp is a homeomorphism of $\mathfrak{H}$ onto H. For example, a nilpotent simply connected analytic group is exponential [9, p. 59]. We will use repeatedly the fact that if G is a simply connected analytic group and if S is a connected closed subgroup of G then the space G/S is simply connected [12, p. 617].

We define the rank of G/S to be the rank of the solvable group S/S_0 [15].

THEOREM 1. *Every non-compact solvmanifold of rank* 1 *is a vector bundle over a circle.*

In the following proof we use the Lemmas 1–7a which appear after the proof of Theorem 1.

Proof. $S = \mathbf{Z} \cdot S_0$ (semi-direct) since S/S_0 is isomorphic to $\mathbf{Z}$, where $\mathbf{Z} = (a)$ for some a in S. Since S is full in G and G/N has no torsion $SN = (a) \cdot N$ (semi-direct). SN projects onto a cyclic group $(\bar{a})$ in G/N. $(\bar{a})$ is contained in a line group W in G/N. If dim $(G/N) > 1$ then the lift of W to G is a proper analytic subgroup of G which contains S, a contradiction of the fullness of S. Hence dim $(G/N) = 1$. Since $SN = (a) \cdot N$, SN is closed in G (Lemma 1). Hence G/S is a fiber bundle with base the circle G/SN, fiber SN/S and group SN. $SN = (a) \cdot N$ and $(a) \cdot N/(a) = N$ (topologically), which is solid. Hence the group is reducible to (a) which is contained in S. The fiber $SN/S = N/S \cap N = N/S_0 = \mathbf{R}^r$ (topological equality). Since (a) is contained in S the fiber can be taken as N/S_0 with the group (a) acting by inner automorphisms. We now suppose that G is an analytic subgroup of $GL(n, \mathbf{R})$ and that N consists only of unipotent matrices [8, Thm. 3.1, p. 219]. Then Lemmas 5a and 7a give us that G/S is a vector bundle over a circle. This concludes Theorem 1 in the event that S is full in G. Now drop the assumption that S be full in G. Assume Theorem 1 is true whenever dim $(G) < n$. Suppose dim $G = n$. If S is full in G we already have Theorem 1. Suppose S is not full in G. Then there exists F, a proper analytic subgroup of G, containing S. G/S is the bundle space of a fiber bundle with base G/F and fiber F/S. Since G/F is Euclidean [13, p. 22] $G/S = G/F \times F/S$ (topologically) [13, p. 23]. Since F is a proper analytic subgroup of G the induction assumption applies to give us that F/S is a vector bundle over a circle. Therefore the product $G/S = G/F \times F/S$ is a vector bundle over a circle. Hence Theorem 1.

LEMMA 1. *Let c be an element of a solvable simply connected analytic group G and let L be a normal analytic subgroup of G. Then $(c)L$ is closed.*

Proof. First we show that any cyclic subgroup (c) of G is closed. If c is in N then (c) is closed since $\{c^n\} = \exp \{n \log c\}$ is a closed subset of N where $\log c$ is the unique element of $\mathfrak{N}$ such that $\exp \log c = c$. If c is not in N then (c) is sent onto a cyclic subgroup $(\bar{c})$ of G/N which is discrete in G/N since G/N is a vector group. Hence the components of $(c)N$ do not accumulate in G. Also, $a^nN = a^mN$ implies that $n = m$ since G/N has no torsion. Hence if $\{c^{n_i}\}$ is a convergent sequence from (c) the n_i's are all the same for i sufficiently large and so (c) is closed. Now G/L is a simply connected solvable Lie group since L is closed and connected [2, p. 100], [3, p. 127]. $(\bar{c})$, the image under the quotient map $\nu: G \to G/L$ of (c) is a cyclic subgroup of G/L and hence closed. Therefore $\nu^{-1}(\bar{c}) = (c)L$ is closed.

LEMMA 2. *Let G be a locally compact Hausdorff topological group satisfying the second axiom of countability and let B be a closed normal subgroup of G. Suppose A and AB are closed subgroups of G and that C is a topological group which is an abstract subgroup of A such that $i: C \to A$, the inclusion map, is continuous. Then the transformation group $(C, B/(A \cap B))$ where C acts on $B/(A \cap B)$ by inner automorphisms is topological; the transformation group $(C, AB/A)$ where C acts by left multiplication is topological; and $(C, B/(A \cap B))$ is isomorphic to $(C, AB/B)$.*

LEMMA 3. *If G is exponential then*
(a) *every analytic subgroup M of G is exponential,*
(b) *if M is normal then G/M is exponential.*

Proof. (a) Let $\mathfrak{M}$ be the Lie algebra of M. Let $\bar{M}$ denote the complement of $\mathfrak{M}$. $G = \exp \mathfrak{G} = \exp \bar{\mathfrak{M}} \cup \exp \mathfrak{M}$. Hence $\exp \mathfrak{M}$ is a closed subset of G and therefore is closed in M. Exp restricted to $\mathfrak{M}$ is a one-one continuous map of $\mathfrak{M}$ into the manifold M. Hence $\exp \mathfrak{M}$ is an open subset of M in the topology of M. Since M is connected $\exp \mathfrak{M} = M$ and so M is exponential.

(b) G/M is simply connected [12, p. 617]. Suppose G/M is not exponential. Then $\mathfrak{G}/\mathfrak{M}$ has a quotient which contains the (unique) 3-dimensional solvable Lie algebra, $\mathfrak{G}_1$, which is not exponential [6]. Suppose $(\mathfrak{G}/\mathfrak{M})/A$ is the quotient. Let $\mathfrak{A}$ be the lift of A back to $\mathfrak{G}$ under the natural mapping $\nu: \mathfrak{G} \to \mathfrak{G}/\mathfrak{M}$. Then $\mathfrak{A}$ contains $\mathfrak{M}$ and $(\mathfrak{G}/\mathfrak{M})/A$ is isomorphic to $\mathfrak{G}/\mathfrak{A}$ and hence $\mathfrak{G}$ has a quotient containing $\mathfrak{G}_1$. This contradicts G's being exponential [6]. Therefore if M is a normal analytic subgroup of G then G exponential implies that G/M is exponential.

LEMMA 4. *Suppose K is a normal analytic subgroup of the simply connected solvable Lie group H. Let $\mathfrak{L}$ be a subspace of $\mathfrak{H}$ (the Lie algebra of H) supplementary to $\mathfrak{K}$ (the Lie algebra of K). Let ϕ be the mapping on $\mathfrak{L} + \mathfrak{K} \to H$*

defined by

$$\phi: X + Y \to \exp X \cdot \exp Y,$$

X in $\mathfrak{L}$ and Y in $\mathfrak{K}$. If K and H/K are exponential then ϕ is a homeomorphism between $\mathfrak{H}$ and H.

Proof. Suppose $\exp X \exp Y = \exp X_1 \exp Y_1$. Then

$$\exp X = \exp X_1 \exp Y_1 \exp(-Y).$$

Now $\exp Y_1 \exp(-Y)$ is in K. Using the commutative diagram

$$\begin{array}{ccc} \mathfrak{H} & \xrightarrow{\ -\ } & \mathfrak{H}/\mathfrak{K} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp'} \\ H & \xrightarrow{\ \sim\ } & H/K \end{array}$$

we get

$$\exp' \bar{X} = (\exp X)^{\sim} = (\exp X_1)^{\sim} = \exp' \bar{X}_1.$$

Therefore $\exp' \bar{X} = \exp \bar{X}_1$. Hence $\bar{X} = \bar{X}_1$ (Lemma 3). $X - X_1$ is in $\mathfrak{L} \cap \mathfrak{K}$ and so $X = X_1$. Since $\exp X = \exp X_1$, $\exp Y = \exp Y_1$ and therefore $Y = Y_1$. Hence ϕ is one-one. To show that ϕ is surjective let h be an arbitrary element of H. Then $\tilde{h} = \exp' \bar{W}$ (Lemma 4) $= (\exp W)^{\sim}$, W in $\mathfrak{H}$. So $h = \exp W \pmod K$. Now, $W = X' + Y'$, X' in $\mathfrak{L}$, Y' in $\mathfrak{K}$. Since $\mathfrak{K}$ is an ideal in $\mathfrak{H}$, $\exp W = \exp X' \cdot k$ [12, p. 620]. Therefore $h = \exp X' \pmod K$. Since K is exponential (Lemma 4), $h = \exp X' \cdot \exp Y''$, X' in $\mathfrak{L}$, Y'' in $\mathfrak{K}$. Hence ϕ is surjective. Since ϕ is continuous we get that ϕ is a homeomorphism between $\mathfrak{L} + \mathfrak{K}$ and H.

Corollary 1. *Let M be a simply connected nilpotent analytic group. Let H be a closed subgroup of M and let W be an analytic subgroup of M which is normal in H such that H/W is isomorphic to $\mathbf{Z}$. Then there is a Y in the Lie algebra $\mathfrak{M}$ of M such that* $\exp\{\mathbf{Z}Y \oplus \mathfrak{W}\} = H$ *where $\mathfrak{W}$ is the Lie algebra of W.*

Proof. Let $\exp Y$ be a mod W generator of H [9, p. 59]. If w is in H then $\bar{W}$ in H/W has the form

$$\overline{(\exp Y)}^n = \overline{\exp nY}.$$

Hence $w = \exp nY \cdot m$ for some m in W. Since $\exp Y$ normalizes W so does $\exp tY$ [11, p. 284]. Hence W is in the analytic group $\exp\{\mathbf{R}Y \oplus \mathfrak{W}\}$ [9]. Hence $\exp nY \cdot m = \exp(tY + W) = \exp tY \cdot m'$, W in $\mathfrak{W}$ [12, p. 620]. Therefore $\exp(n - t)Y$ is in W. Since exp is one-one on $\mathbf{R}Y \oplus \mathfrak{W}$, $n = t$. Hence Corollary 1.

Corollary 2. *$\eta : (X, k) \to \exp X \cdot k$ where X is in $\mathfrak{L}$ and k is in K gives a homeomorphism between $\mathfrak{L} \times K$ and H because* $\log : K \to \mathfrak{K}$ *is a homeomorphism onto.*

COROLLARY 3. *Suppose that G is an exponential group and that*

$$G = M_1 \supset M_2 \supset M_3 \supset \cdots \supset M_p$$

is a sub-invariant sequence of analytic subgroups. Let $\mathfrak{M}_i$ be the Lie algebra of M_i and let $\mathfrak{U}_i$ be a linear supplement to $\mathfrak{M}_i$ in $\mathfrak{M}_{i-1}$ for $i = 2, 3, \cdots, p$. Then the map

$$\phi : \sum_{i=2}^{p} Y_i + X \to \exp Y_2 \exp Y_3 \cdots \exp Y_p \, exp \, X$$

is a homeomorphism of $\mathfrak{G}$ onto G where Y_i is in $\mathfrak{U}_i$ and X is in $\mathfrak{M}_p$.

LEMMA 5. *Let M be a unipotent analytic subgroup of $GL(n, \mathbf{R})$, H a closed subgroup of M, W an analytic subgroup of M normal in H such that H/W is isomorphic to $\mathbf{Z}$. Let B be a fully reducible subgroup of $GL(n, \mathbf{R})$ which normalizes M, H and W. Let T denote the real numbers mod $\mathbf{Z}$ and let V denote a finite-dimensional real vector space. Let $(B, M/H)$ denote the transformation group with action given by $\nu : B \times M/H \to M/H$ defined by $\nu(b, mH) = bmb^{-1} \cdot H$. Then there exists a V and an action of B on V which is linear and an action of B on T which satisfies the condition $b\bar{t} = \pm \bar{t}$ for all b in B and all $\bar{t}$ in T and a surjective homeomorphism $\phi : V \times T \to M/H$ which is equivariant with respect to the action of B on M/H and the action of B on $V \times T$ defined by $b(v, \bar{t}) = (bv, b\bar{t})$.*

Proof. $H = \exp\{\mathbf{Z}Y \oplus \mathfrak{W}\}$ where $\mathfrak{W}$ is the Lie algebra of W (Cor. 1) and since $\exp Y$ normalizes W so does $\exp tY$ for all real t [11, p. 284]. Let $M_1 = [H]$, the algebraic group hull of H. $M_1 = (\exp \mathbf{R}Y) \cdot W$ (semi-direct) and is invariant under B [14, p. 205]. Define inductively, $M_{i+1} =$ normalizer in M of M_i. We get a B-invariant sequence of analytic groups,

$$W \subset [H] = M_1 \subset M_2 \subset \cdots \subset M_p = M.$$

$\mathfrak{H} = \mathbf{R}Y + \mathfrak{M}$ is the Lie algebra of $[H]$. If b is in B then

$$b(\exp Y)b^{-1} = \Gamma_b(\exp Y) = \exp(\pm Y + W'),$$

W' in $\mathfrak{W}$ (Cor. 1). Hence $Adb(Y) = \pm Y + W'$.

Letting $\mathfrak{M}_i$ denote the Lie algebra of M_i we get the AdB-invariant sequence $\mathfrak{W} \subset \mathfrak{H} \subset \mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{M}_p = \mathfrak{M}$. Since Ad is Zariski continuous and B is fully reducible so is AdB fully reducible. Hence there exists AdB-invariant subspaces $\mathfrak{U}_i$ of $\mathfrak{M}_{i+1}$ and an element Q in $\mathfrak{W}$ such that $\mathfrak{M}_i \oplus \mathfrak{U}_i = \mathfrak{M}_{i+1}$, $\mathfrak{H} = \mathbf{R}Q \oplus \mathfrak{W}$ and $AdB(Q) \subset \mathbf{R}Q$. We can therefore suppose that Y in $\exp\{\mathbf{Z}Y \oplus \mathfrak{W}\} = H$ is chosen so that $AdB(Y) = \{\pm Y\}$.

Define

$$\sigma(b) = +1 \quad \text{if } Adb(Y) = Y$$
$$= -1 \quad \text{if } Adb(Y) = -Y.$$

Let ρ be the isomorphism between T and $\mathbf{R}Y/\mathbf{Z}Y$ given by $\rho : \bar{t} \to tY \oplus \mathbf{Z}Y$.

Define the action of B on T by

$$\begin{aligned}(b, \bar{t}) &\rightarrow (b, {}_{\rho}(\bar{t})) = (b, tY \oplus \mathbf{Z}Y) \\ &\rightarrow tAdb(Y) \oplus \mathbf{Z}Y = t\sigma(b)Y \oplus \mathbf{Z}Y \\ &\rightarrow \overline{t\sigma(b)} = \sigma(b)\bar{t}.\end{aligned}$$

Let $V = \sum_{i=1}^{p-1} \mathfrak{U}_i$. Define the action of B on V by $(b, \sum u_i) \rightarrow \sum_i Adb(u_i)$. Define the action of B on $V \times T$ to be the direct product of the two actions defined above. Define the map $\phi : V \times T \rightarrow M/H$ by

$$\phi[\textstyle\sum_{i=1}^{p-1} u_i, \bar{t}] = \exp u_1 \cdots \exp u_{p-1} \exp tY \cdot H.$$

LEMMA 5a. *Let M be a unipotent analytic subgroup of $GL(n, \mathbf{R})$, W an analytic subgroup of M. Let B be a fully reducible subgroup of $GL(n, \mathbf{R})$ which normalizes M and W. Let V denote a finite-dimensional real vector space. Let $(B, M/W)$ denote the transformation group with action given by $\nu(b, \bar{m}) = \overline{bmb^{-1}}$. Then there exists a V and an action of B on V which is linear and a surjective homeomorphism $\phi : V \rightarrow M/W$ which is equivariant with respect to the action of B on M/W and the action of B on V.*

Proof. Similar to the proof for Lemma 5.

LEMMA 6. *If B is a solvable subgroup of $GL(n, \mathbf{R})$ and $[B] = A \cdot U$ is a semi-direct product decomposition of $[B]$ into a maximal fully reducible subgroup A and the group of unipotent matrices of $[B]$ then $(BU) \cap A$ is fully reducible.*

Proof. Let $C = [(BU) \cap A]$. Then $(BU) \cap A \subset C \subset A$ since A is algebraic. The Lie algebra $\mathfrak{C}$ of C is contained in the Lie algebra $\mathfrak{A}$ of A. Since each element of $\mathfrak{A}$ is semi-simple [14, p. 208] so is each element of $\mathfrak{C}$. Hence $\mathfrak{C}$ is fully reducible and so C is fully reducible [14, p. 206]. Since a linear group is fully reducible if and only if its algebraic group hull is fully reducible, $(BU) \cap A$ is fully reducible.

Let M and H be as in Lemma 5.

LEMMA 7. *Let M be a unipotent analytic subgroup of $GL(n, \mathbf{R})$. Let B be a solvable topological group which is an abstract subgroup of $GL(n, \mathbf{R})$ such that $i : B \rightarrow GL(n, \mathbf{R})$ is continuous (i the inclusion map). Suppose B normalizes M and H.*

(0) *Then $(B, M/H)$ is a topological transformation group and*

(1) *if $\mathfrak{B}$ is a fiber bundle having structure group and fiber $(B, M/H)$ then the structure group of $\mathfrak{B}$ may be replaced by a fully reducible subgroup of $GL(n, \mathbf{R})$.*

Proof. (0) Since M is a unipotent analytic group it is algebraic [5, Prop. 17, p. 127] and so is a topological subgroup of $GL(n, \mathbf{R})$. The mapping $\nu : B \times M \rightarrow M$ given by $\nu(b, m) = bmb^{-1}$ is the composition of the maps in the sequence:

$$B \times M \xrightarrow{i \times i_1} GL(n, \mathbf{R}) \times GL(n, \mathbf{R}) \xrightarrow{\tau} GL(n, \mathbf{R})$$

where i, i_1 are inclusion maps and $\tau(g, h) = ghg^{-1}$. Hence ν is continuous. Since B normalizes H the map $\nu : B \times M/H \to M/H$ given by $\nu(b, \bar{m}) = \nu(b, m)\cdot H$ is continuous and so $(B, M/H)$ is a topological transformation group.

(1) As in the proof of Lemma 5, AdB leaves $\mathfrak{M}$ and $\mathbf{Z}Y \oplus \mathfrak{W}$ invariant and $Adb(Y) = \pm Y + W_b$, W_b in $\mathfrak{W}$, for all b in B. Let $W_1, \cdots, W_r$ be a basis for $\mathfrak{W}$. Let $X_1, \cdots, X_l$ be a basis for a subspace of $\mathfrak{M}$ supplementary to $\mathfrak{K}$. Then if b is in B, Adb on $\mathfrak{M}$ has a matrix with respect to

$$\{X_1, \cdots, X_l, Y, W_1, \cdots, W_r\}$$

as follows:

$$\begin{array}{c|c}
 & X_1, \cdots, X_l, \quad Y, W_1, \cdots, W_r \\
\hline
\begin{matrix} X_1 \\ \vdots \\ X_l \\ Y \\ W_1 \\ \vdots \\ W_r \end{matrix} &
\left(\begin{array}{c|c|c}
 & \begin{matrix} 0 \\ \vdots \\ 0 \\ \delta \end{matrix} & \begin{matrix} 0 & \cdots & 0 \\ \vdots & \cdots 0 \cdots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} \\
 & & \\
\end{array}\right)
\end{array}$$

where $\delta = \pm 1$. Hence Adb on $\mathfrak{M}$ satisfies the algebraic conditions

$$u_{l+1,l+1}^2 - 1 = 0;$$

$$u_{j,l+1} = 0, \quad 1 \le j \le l;$$

$$u_{k,s} = 0, \quad 1 \le k \le l+1, \quad l+2 \le s \le l+r+1;$$

for all b in B. Hence $[AdB]$ leaves $\mathfrak{M}$ and $\mathbf{Z}Y \oplus \mathfrak{W}$ invariant. Since Ad is continuous in the Zariski topology and B is contained in $Ad^{-1}[AdB]$ so is $[B]$. If c is in $[B]$ then $Adc(Y) = \pm Y + W'$, W' in $\mathfrak{W}$. Therefore $\Gamma_c(\exp Y) = \exp Adc(Y) = \exp(\pm Y + W')$ is in H and so $[B]$ normalizes M and H. Let $[B] = A \cdot U$ (semi-direct) where A is a maximal fully reducible subgroup of $[B]$ and U is the group of unipotent matrices [14, p. 217]. If we put the relative $GL(n, \mathbf{R})$-topology on BU then $(BU, M/H)$ is a topological transformation group with the elements of BU acting on M/H by conjugation. If $\{g_{kl}\}$ is a set of coordinate transformations for the bundle $\mathfrak{B}$ then, since $i : B \to BU$ is continuous, the set $\{i \circ g_{kl}\}$ is a set of coordinate transformations for $\mathfrak{B}$ with the larger group BU as structure group. Because A is an algebraic subgroup of $GL(n, \mathbf{R})$ $A \cap BU$ is closed in BU.

$$\frac{BU}{A \cap BU} = \frac{(A \cap BU) \cdot U}{A \cap BU} = U \qquad \text{(topologically)}.$$

But U is solid [14, p. 205] and so the group BU for $\mathfrak{B}$ is reducible to the group $A \cap BU$ which is a fully reducible group by Lemma 6.

LEMMA 7a. *Let M be a unipotent analytic subgroup of $GL(n, \mathbf{R})$. Let B be a solvable topological group which is an abstract subgroup of $GL(n, \mathbf{R})$ such that $i : B \to GL(n, \mathbf{R})$ is continuous (i the inclusion map). Suppose B normalizes M and W.*

(0) *Then $(B, M/W)$ is a topological transformation group and*

(1) *if $\mathfrak{B}$ is a fiber bundle having structure group and fiber $(B, M/W)$ then the structure group of $\mathfrak{B}$ may be replaced by a fully reducible subgroup of $GL(n, \mathbf{R})$.*

Proof. Same as for Lemma 7 but simpler.

THEOREM 2. *Every non-compact 3-dimensional solvmanifold is a vector bundle over a compact solvmanifold. This results from the following Lemmas* 8, 9, 10, 12, 13 *and Theorem* 1.

LEMMA 8.[1] *If G is 3-dimensional then G contains no discrete $S = \mathbf{Z}^2$ such that $S \cap N = (e)$, the identity subgroup of G.*

Proof. We first remark that G is not nilpotent. For suppose that G is nilpotent. Then S full implies that S is uniform which implies that rank $(S) = 3$. This contradicts $S = \mathbf{Z}^2$ [11, p. 291]. Hence dim $(G/N) = 1$ [10, p. 12]. Since $S \cap N = (e)$, SN/N is algebraically isomorphic to $\mathbf{Z}^2$. Hence if SN/N is discrete in G/N then dim $(G/N) > 1$, a contradiction. Therefore SN/N is dense in G/N and so $\overline{SN} = G$. Hence there exists a regular element b in S which lies on a 1-parameter subgroup $\gamma(t)$ of G [13, p. 12]. Let B be the centralizer of b. Since b is regular $\gamma(t)$ is not contained in N and since $\gamma(t) \subset B$, $BN = G$. Since B is a closed subgroup of G, $B/B \cap N$ is homeomorphic to BN/N. Because $B \cap N$ is the centralizer in N of $b = \exp X$, and because N is exponential and G/N is a vector group it follows that $B \cap N$ is connected. Hence B is connected. But S is abelian and therefore lies in B. Since S is full $B = G$ and so S lies in the center of G. Since

$$AdG = Ad\overline{SN} \subset \overline{AdSN} \subset \overline{AdN},$$

AdG is nilpotent and therefore G is nilpotent. But this contradicts dim $(G/N) = 1$. Hence Lemma 8.

Recall the hypotheses: dim $G/S = 3$, S_0 is contained in N, and S is full in G.

LEMMA 9. *If S/S_0 is isomorphic to $\mathbf{Z}^2$, $S \cap N = S_0$, and* dim $S_0 \geq 1$ *then G/S is a line bundle over a torus.*

[1] The proof of Lemma 8 which appears here is due to G. D. Mostow.

Proof. (a) Let L be the normalizer of S_0. Suppose L_0, the identity component of L, has codimension 2 in G. Since L is closed in G and S is contained in L the components of L do not accumulate in G and so SL_0, a union of components of L, is closed in G. Hence G/S is the bundle space of a fiber bundle with base G/SL_0 and fiber SL_0/L_0. Let M denote the normalizer in N of S_0. Since M is connected [11, p. 284], M is contained in L_0. Since S_0 is contained properly in N the normalizer in N contains S_0 properly [2, p. 56]. Therefore the codimension of M in G must be less than or equal to 2. Since the codimension of L_0 in G is 2, $M = L_0$. Therefore, since $S \cap N = S_0$, $S \cap L_0 = S_0$. Hence $SL_0/S = L_0/S \cap L_0 = L_0/S_0 = \mathbf{R}$ (topological equality). Since $\mathbf{R}$ is solid the group of the bundle is reducible to S [17, p. 56]. Since G/SL_0 is a 2-dimensional solvmanifold whose fundamental group SL_0/L_0 is $\mathbf{Z}^2$ then G/SL_0 is a torus, T^2 [12, p. 624]. The fiber can be taken to be $L_0/S \cap L_0 = L_0/S_0$ and the group of the bundle to be $\Gamma = S/S_2$ where S_2 is the intersection of all the isotopy subgroups of S, S acting on L_0/S_0 by inner automorphisms. Let $\mathfrak{L}$ be the Lie algebra of L_0. Since $\mathfrak{S}$ is an ideal of codimension 1 in $\mathfrak{L}$ there is a mapping, $\overline{\exp}$, the exponential of the 1-dimensional Lie algebra $\mathfrak{L}/\mathfrak{S}$ onto the vector group L_0/S_0, which is an isomorphism between $\mathfrak{L}/\mathfrak{S}$ and L_0/S_0. Now,

$$\overline{\exp}\ \overline{Ads}(\bar{X}) = \bar{\Gamma}_s\ \overline{\exp}\ (\bar{X})$$

where $\overline{Ads}$ denotes the $\mathfrak{L}/\mathfrak{S}$-part of Ads. Hence $\overline{\exp}$ provides an isomorphism between the topological transformation groups $(\overline{Ad}S,\ \mathfrak{L}/\mathfrak{S})$ and $(\Gamma,\ L_0/S_0)$. Hence the fiber and group of the bundle are a vector space and a linear group, respectively. Therefore in case (a) G/S is a vector bundle over a torus.

(b) Note now that $S \not\subset L_0$ because S is full in G (standing hypothesis). Suppose L_0 has codimension 1 in G. We first show that $(S(N \cap L_0)) \cap L_0$ has rank 1 in L_0. G/S is a fiber bundle with G/SL_0 (a circle) for base and

$$SL_0/S = (L_0/S_0)/(S \cap L_0/S_0)$$

for fiber. If $S \cap L_0/S_0$ has rank 2 then $L_0/S \cap L_0 = T^2$ which contradicts G/S noncompact. Therefore $S \cap L_0 = \mathbf{Z} \cdot S_0$ (semi-direct) and so

$$(S \cap L_0)(N \cap L_0) = \mathbf{Z} \cdot (N \cap L_0)$$

is closed in L_0 (with $\mathbf{Z}$ contained in $S \cap L_0$). This holds because L_0 is simply connected [4] and so Lemma 1 applies. Now

$$(S \cap L_0) \cdot (N \cap L_0) = (S(N \cap L_0)) \cap L_0$$

since $N \cap L_0 \subset L_0$. Therefore, $(S(N \cap L_0)) \cap L_0$ is closed in L_0. Since the components of a closed Lie subgroup are separated $S(N \cap L_0)$ is closed in SL_0. Therefore, since SL_0 is closed in G, $S(N \cap L_0)$ is closed in G. Hence G/S is a bundle with base $G/S(N \cap L_0)$ and fiber $S(N \cap L_0)/S$. Since $G/S(N \cap L_0)$ is a 2-dimensional solvmanifold with fundamental group $\mathbf{Z}^2$ it is a torus [12

p. 624]. $S(N \cap L_0)/S$ is homeomorphic to

$$N \cap L_0/S \cap (N \cap L_0) = N \cap L_0/S_0.$$

Since the normalizer in N of S_0 is connected [11, p. 284], $N \cap L_0$ is connected and hence $N \cap L_0/S_0$ is a line. Hence the group of the bundle is reducible to S, acting by inner automorphisms on $N \cap L_0/S_0$. Just as in case (a) $\overline{\exp}$ provides an isomorphism between $(\overline{AdS}, \mathfrak{N} \cap \mathfrak{L}/\mathfrak{S})$ and $(\Gamma, N \cap L_0/S_0)$ where $\overline{AdS}$ denotes the $\mathfrak{N} \cap \mathfrak{L}/\mathfrak{S}$-part of AdS and Γ is the $N \cap L_0/S_0$-part of the group of inner automorphisms on G determined by S. Hence G/S is a line bundle over a torus.

LEMMA 10. *If G/S is a non-compact three-dimensional solvmanifold such that S/S_0 is isomorphic to $\mathbf{Z}\cdot\mathbf{Z}$ (semi-direct, non-abelian), then G/S is a line bundle over the Klein bottle.*

Proof. $S/S_0 = (\bar{a})\cdot(\bar{b})$ (semi-direct) with $\overline{aba^{-1}} = \bar{b}^{-1}$. Let a and b be representatives of $\bar{a}$ and $\bar{b}$, respectively. $aba^{-1} = b^{-1} \bmod S_0$. Since S_0 is normal in S, $aba^{-1}b^{-1} = b^{-2} \bmod S_0$. Hence b^n is contained in N. Since the vector space G/N has no torsion b is in N. Therefore (b) is contained in N, and so $SN = (a)(b)S_0N = (a)N$. If a is in N then S is not full in G, a contradiction. Since G/N has no torsion, a^n is not in N for $n \neq 0$. Hence $SN = (a)\cdot N$, semi-direct. Since SN projects onto a cyclic subgroup of the vector group G/N, SN is closed. Therefore G/S is a fiber bundle with base G/SN, fiber SN/S, and group SN acting on SN/S by left translations.

$(a)\cdot N$ is contained in a 1-parameter subgroup $\mathbf{R}$ of the vector group G/N. The pre-image of $\mathbf{R}$ is a connected closed subgroup $\mathbf{R}^{\sim}$ of G containing S. Since S is full, this shows $\mathbf{R}^{\sim} = G$. But then $G/N = \mathbf{R}$ and so $\dim(G/N) = 1$. Hence $G/SN = G/N/(SN/N)$ is the circle T. Since G is simply connected and solvable we can assume that G is given as an analytic subgroup of $GL(n, \mathbf{R})$ with N unipotent [8, p. 219]. By Corollary 1 and Lemmas 5 and 7 we have that G/S is a fiber bundle over a circle with fiber $V \times T$, V a vector space, $T = \mathbf{R}/\mathbf{Z}$, and group B acting on $V \times T$ as follows: $b(v, \bar{t}) = (bv, b\bar{t})$ with B acting linearly on V and acting as $\mathbf{Z}_2$ on T. Since the action on $V \times T$ by B is the direct product of the actions of B on V and T it follows that G/S is a fiber bundle with fiber V, structure group linear on V, and a base K which is a circle bundle over a circle [18, p. 712]. Hence K is a torus or a Klein bottle. Hence G/S is a line bundle over a torus or a Klein bottle. Use of the exact homotopy sequence [17, §15] gives us that $\Pi_1(G/S)$ is injected into Π_1 of the base. If the base were a torus then $\Pi_1(G/S)$ would be abelian. This contradiction gives the assertion of Lemma 10.

LEMMA 11. *If $S \cap N$ is not connected and S/S_0 is isomorphic to $\mathbf{Z}^2$ then $SN = (c)\cdot N$ (semi-direct) where (c) is a cyclic subgroup of S.*

Proof.

$$2 = \operatorname{rank}(S/S_0) = \operatorname{rank}(S \cap N)/S_0 + \operatorname{rank} S/(S \cap N) \quad ([15]).$$

Since G/N has no torsion, rank $S/S \cap N = 0$ implies that $S/S \cap N = (e)$ which contradicts the fullness of S in G. Therefore rank $S/S \cap N \geq 1$. If rank $S \cap N/S_0 = 0$ then $S \cap N/S_0$ has only torsion elements. $S \cap N$ is contained in the normalizer M in N of S_0 which is connected [11, p. 284]. The quotient M/S_0 is a simply connected nilpotent Lie group [12, p. 617] and hence has no torsion. Therefore rank $S \cap N/S_0 = 0$ implies that $S \cap N$ is connected which contradicts a hypothesis of this lemma. Hence rank $S/S \cap N \geq 1$. Since rank $S/S_0 = 2$, rank $S/S \cap N = 1 =$ rank $S \cap N/S_0$. Since $S/S \cap N$ has no torsion $S/S \cap N$ is isomorphic to $\mathbf{Z}$. By selecting any c in S which is a mod $S \cap N$ generator of $S/S \cap N$ we have $SN = (c) \cdot N$ semi-direct.

LEMMA 12. *If G/S is a non-compact three-dimensional solvmanifold such that S/S_0 is isomorphic to $\mathbf{Z}^2$ and $S \cap N$ is not connected then G/S is a line bundle over a torus.*

Proof. By Lemma 11, $SN = (c) \cdot N$ (semi-direct) with (c) contained in S. Since the base of a vector bundle is a deformation retract of the bundle the fundamental group of G/S must be the same as the fundamental group of the base. Arguing as in the proof of Lemma 10 we have that the base is either a Klein bottle (with fundamental group $\mathbf{Z} \cdot \mathbf{Z} \neq \mathbf{Z}^2$) or a torus. Hence Lemma 12.

LEMMA 13. *If G/S is non-compact then S/S_0 is isomorphic to (e) (the group of only one element) $\mathbf{Z}$, $\mathbf{Z}^2$, or $\mathbf{Z} \cdot \mathbf{Z}$ (the fundamental group of the Klein bottle).*

Proof. Let $X = K \times V$ be a regular finite abelian covering space of G/S where K is a compact solvmanifold and V is a Euclidean space [13, p. 25]. Let $\tilde{\Pi}$ and Π be the fundamental groups $\Pi_1(K \times V)$ and $\Pi_1(G/S)$, respectively. Since $K \times V$ is a covering of G/S there exists an injection of $\tilde{\Pi}$ into Π. We identify $\tilde{\Pi}$ with its image Π. $\Pi/\tilde{\Pi}$ is isomorphic to A, a finite abelian group. Since the fundamental group of a solvmanifold is a finitely generated solvable group [15] we have rank $(\Pi/\tilde{\Pi}) =$ rank $A = 0 =$ rank $\Pi -$ rank $\tilde{\Pi}$ [15]. Since $\tilde{\Pi} = \Pi_1(K \times V) = \Pi_1(K)$ and dimension of $K <$ dim (G/S) the rank of Π equals the rank of the fundamental group of a compact solvmanifold of dimension less than 3. All the fundamental groups for the solvmanifolds of dimension less than 3 are (e), $\mathbf{Z}$, $\mathbf{Z}^2$, $\mathbf{Z} \cdot \mathbf{Z}$ [12, p. 624]. Hence the rank of $\Pi \leq 2$. For Π there is the following exact sequence

$$(s): \; e \to \Delta \to \Pi \to \mathbf{Z}^k \to e$$

where Δ is the fundamental group of a nilmanifold [1, p. 6]. Therefore

$$2 \geq \text{rank } \Pi = \text{rank } \Delta + \text{rank } \mathbf{Z}^k = \text{rank } \Delta + k \qquad [15].$$

For the case where $k = 1$ and rank of $\Pi = 2$ we have rank $\Delta = 1$ and therefore $\Delta = \mathbf{Z}$ [1, p. 5]. Hence (s) is $e \to \mathbf{Z} \to \Pi \to \mathbf{Z} \to e$. Therefore $\Pi/\mathbf{Z} = \mathbf{Z}$ and (s) is split. Since $\mathbf{Z}$ has only two automorphisms $\Pi = \mathbf{Z}^2$ or $\mathbf{Z} \cdot \mathbf{Z}$. Hence $\Pi = (e)$, $\mathbf{Z}$, $\mathbf{Z}^2$, or $\mathbf{Z} \cdot \mathbf{Z}$.

Added in proof. L. Auslander and R. Tolimieri have recently proved that all solvmanifolds are vector bundles over compact solvmanifolds. See their paper *Splitting theorems and the structure of solvmanifolds* in the July 1970 Annals of Mathematics.

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