ON FOURS GROUPS

BY

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In a recent paper [2], Ernest Shult has discovered remarkable necessary and sufficient conditions for a conjugacy class of involutions of a group to be the set of non-identity elements of a subgroup of order greater than two. Even more striking is the fact that this result is a consequence of a theorem characterizing the symplectic groups over two-element fields. In trying to give a direct proof of this corollary we have, in fact, come up with a stronger result, namely:

THEOREM. *If V is a fours subgroup of a group G and V intersects $O_2(G)$ trivially, then there is an involution of G, conjugate to an element of V, which commutes with no involution of V.*

It turns out that this strange theorem can even be used in studying doubly transitive groups, in places where Shult's result is not strong enough [3]. We shall now proceed by first proving the theorem and then stating and deriving Shult's result from it. All our notation is standard [1].

Since $V \cap O_2(G) = 1$, Baer's theorem [1, p. 105] yields that each involution of V has a conjugate together with which it generates a subgroup of order not a power of two. However, this subgroup is dihedral as it is generated by two involutions. Thus, it follows that each element of V inverts a non-identity element of odd order of G.

Let $a \in V^{\#}$ and choose such an element x. If a^x centralizes no element of $V^{\#}$ we are done; thus we may assume a^x does centralize an element b of $V^{\#}$. But $a^x = x^{-1}ax = ax^2$ and V is abelian so x^2 centralizes b. In particular, $b \neq a$. Moreover, x centralizes b since x is a power of x^2 as x has odd order.

Similarly, b inverts a non-identity element y of odd order and we may assume that y centralizes an element of $V^{\#}$ other than b. If y centralizes a then we have the following symmetrical relations:

$$x^a = x^{-1}, \quad x^b = x, \quad y^a = y, \quad y^b = y^{-1}.$$

On the other hand, suppose that y centralizes ab, the other element of $V^{\#}$. We set $a' = ab$ so a' inverts x, as x does and y centralizes x. As a' is assumed to centralize y, we see that if we replace a by a' then we again have the above symmetrical relations. Hence, we shall assume this is done.

There are now two possibilities to consider: x and y commute or they do not. First, we claim that if x and y do commute then $(ab)^{xy}$ is the desired

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involution. Indeed, it follows that

$$(ab)^{xy} = (ax^2b)^y = ax^2by^2 = ab(xy)^2.$$

Hence, if $(ab)^{xy}$ commutes with an element of $V^{\#}$ then so does xy. But, if $(xy)^a = xy$ then $x^{-1}y = xy$ and $x^2 = 1$, a contradiction. Similarly, xy and b do not commute. Finally, if $(xy)^{ab} = xy$ then $x^{-1}y^{-1} = xy$ so $x^2 = y^2$. But a inverts x^2 and centralizes y^2 so this is a contradiction and our assertion is demonstrated.

Finally, we may assume that x and y do not commute. In this case, the required involution is b^{yx}. Indeed,

$$b^{yx} = (by^2)^x = b(y^x)^2$$

so that if b^{yx} centralizes an element of $V^{\#}$ then so does y^x. However, if $y^xa = ay^x$, then

$$x^{-1}yxa = ax^{-1}yx = xayx = xyax = xyx^{-1}a$$

so $y^x = y^{x^{-1}}$ and x^2 and y commute. Hence, x and y commute, which is a contradiction. On the other hand, if y^x and b commute then

$$x^{-1}yxb = bx^{-1}yx = x^{-1}y^{-1}xb$$

and $y = y^{-1}$, again a contradiction. Finally, if x^y and ab commute then

$$x^{-1}yxab = abx^{-1}yx = ax^{-1}y^{-1}xb = xy^{-1}x^{-1}ab$$

so $y^x = (y^{-1})^{x^{-1}}$ and $y^{x^2} = y^{-1}$. Hence, x has even order. This final contradiction establishes the theorem.

We now turn to Shult's result and its derivation.

THEOREM (Shult [2]). *If K is a conjugacy class of involutions of a group then K consists of the non-identity elements of a subgroup of order larger than two if, and only if the following conditions are satisfied:*

(1) *There are distinct commuting elements of K;*

(2) *If t and u are any such elements then $tu \in K$ and every element of K commutes with one of t, u or tu.*

The implication one way is clear. As for the other, we need only show that conditions (1) and (2) imply that any two elements of K commute, as then the rest is obvious.

Proof. First, we claim that $K \subseteq O_2(G)$. Indeed, if $t \in K$ then by (1) and the fact that K is a conjugacy class there is $u \in K$, $u \neq t$ such that t and u commute. Let $V = \langle t, u\rangle$. Applying our theorem to V yields immediately that $V \cap O_2(G) \neq 1$ and so some element of K lies in $O_2(G)$. Hence, all elements of K do.

We set $L = \langle K\rangle$ and choose some $x \in K$ It suffices to prove that $x \in Z(L)$ as then x commutes with each element of K. We assume $x \notin Z(L)$ and shall derive a contradiction. Since $L \supset C_L(x)$ and L is a 2-group there is a subgroup H of L containing $C_L(x)$ as a proper normal subgroup. Hence, if $h \in H$, $h \notin C_L(x)$ then $x^h \neq x$ and $x^h \in C_L(x)$. Thus, x and x^h are commuting elements of K so $xx^h = x^2[x, h] = [x, h]$ is in K, by (2). However, $[x, h] \subseteq L'$ so $K \subseteq L'$ and $L = \langle K \rangle \subseteq L'$. But L is a 2-group so $L \supset L'$, a contradiction, and the theorem is proved.

References

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