MOORE-POSTNIKOV SYSTEMS FOR NON-SIMPLE FIBRATIONS

BY

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In [9], Moore outlined a method of factorizing a fibration into a sequence of fibrations whose fibres are Eilenberg-MacLane spaces. Several accounts of this theory have since appeared [e.g. 4, 6], but these treat only the case when the fibration is simple in that the fundamental group of the base acts trivially on the homotopy groups of the fibre. This is untrue for certain interesting fibrations, such as non-orientable sphere-bundles. This paper develops the theory without the hypothesis of simplicity.

Our main theorem (3.4) states that a fibration with fibre $K(\pi, n-1)$ is completely determined (up to fibre homotopy equivalence) by a 'characteristic class' in the n^{th} cohomology group of the base with suitably twisted coefficients π. We prove this by constructing a classifying space $\hat{K}(\pi, n)$ for fibrations whose fibres have the homotopy type of $K(\pi, n-1)$ [cf. 10]. We define "representable" cohomology with twisted coefficients in terms of the space $\hat{K}(\pi, n)$, and reconcile our definition with a classical one. The spaces $\hat{K}(\pi, n)$ seem to be useful objects in non-simple obstruction theory. The corresponding semi-simplicial complexes have been considered by Gitler [3].

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1. The complex $\hat{K}(\pi, n)$

All our spaces will have compactly-generated Hausdorff topologies, in accordance with the system described by Steenrod [11]. In a topological group G, the multiplication map $G \times G \to G$ is required to be continuous only when $G \times G$ has the compactly-generated topology. Except where the reverse is stated, all spaces have basepoints (denoted by o) and all maps and homotopies are based. We abbreviate "(X, A) has the homotopy type of a CW pair" to "(X, A) is of CW type", etc.

1.1. Let π be an abelian group, and aut π its group of (left) automorphisms. Take an Eilenberg-MacLane CW-complex $K(\pi, n)$ which is a topological abelian group on which aut π acts by cellular automorphisms. For instance, the Milnor realization of the standard semi-simplicial complex $K(\pi, n)$ [2, 7] will do. Let Q be a CW-complex of type $K(\text{aut}\ \pi, 1)$. Then the universal cover $\tilde{Q}$ is a contractible complex on which aut π acts freely and cellularly on the left. The diagonal action of aut π on $K(\pi, n) \times \tilde{Q}$ (given by

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$g \cdot (x, y) = (gx, gy)$ for $g \in \operatorname{aut} \pi$) is cellular and free, so the quotient

$$[K(\pi, n) \times \tilde{Q}]/\operatorname{aut} \pi$$

is a CW-complex which we call $\hat{K}(\pi, n)$. The projection $K(\pi, n) \times \tilde{Q} \to \tilde{Q}$ induces a map $q : \hat{K}(\pi, n) \to Q$ which is a fibre bundle with group aut π. Each fibre is a topological abelian group isomorphic to $K(\pi, n)$. The inclusion $\tilde{Q} \to K(\pi, n) \times \tilde{Q}$ given by $x \to (0, x)$ induces a standard section s of the bundle q. We have a diagram

$$K(\pi, n) \longrightarrow \hat{K}(\pi, n) \underset{s}{\overset{q}{\rightleftarrows}} Q \qquad\qquad \text{where} \quad qs = 1_Q .$$

We identify Q with the image of s in $\hat{K}(\pi, n)$. We have canonical isomorphisms $\pi_n(\hat{K}, Q) \approx \pi_n(K(\pi, n) \times \tilde{Q}, \tilde{Q}) \approx \pi$, since $K(\pi, n) \times \tilde{Q}$ is a covering of $\hat{K}(\pi, n)$ (the universal cover if $n > 1$). This covering also shows that the action of $\pi_1 Q$ on $\pi_n(\hat{K}, Q)$ is the usual action of aut π on π.

$\hat{K}(\pi, n)$ has the following universal property for $n \geqq 2$.

1.2 PROPOSITION. *Let (X, A) be a pair of CW-type where X is connected and (X, A) is $(n - 1)$-connected with $n \geqq 2$ (and $\pi_n(X, A)$ abelian if $n = 2$). Given a homomorphism $\varepsilon : \pi_1 X \to \operatorname{aut} \pi$ and an ε-equivariant homomorphism $\kappa : \pi_n(X, A) \to \pi$, there exists a unique homotopy class $\phi : (X, A) \to (\hat{K}(\pi, n), Q)$ such that*

(i) *$q\phi : X \to Q$ induces $\varepsilon : \pi_1 X \to \pi_1 Q$*
(ii) *ϕ induces $\kappa : \pi_n(X, A) \to \pi_n(\hat{K}, Q)$.*

(The statement "κ is ε-equivariant" makes sense for $\pi_1 X$ does act on $\pi_n(X, A)$ under the conditions stated).

Proof. We may assume (X, A) is a CW-pair where $X - A$ has no cells in dimensions less than n. Let $\tilde{X}$ be the universal cover of X, and $\tilde{A}$ the subcomplex covering A. The homotopy classes $(X, A) \to (\hat{K}, Q)$ which induce

$$\varepsilon : \pi_1 X \to \pi_1 \hat{K} \approx \operatorname{aut} \pi$$

correspond exactly to the (based) equivariant homotopy classes of ε-equivariant maps

$$(\tilde{X}, \tilde{A}) \to (K(\pi, n) \times \tilde{Q}, \tilde{Q}).$$

There is a unique ε-equivariant homotopy class $\tilde{X} \to \tilde{Q}$, because there is a unique homotopy class $X \to Q$ inducing $\varepsilon : \pi_1 X \to \pi_1 Q$. It only remains to show there is a unique ε-equivariant homotopy class $(\tilde{X}, \tilde{A}) \to (K(\pi, n), o)$ inducing $\kappa : \pi_n(X, A) \to \pi_n K(\pi, n)$.

$\tilde{A}$ is connected, and $\pi_1 \tilde{A}$ acts trivially on $\pi_n(\tilde{X}, \tilde{A})$ (indeed $\pi_1 \tilde{A} = 0$ unless $n = 2$). By the relative Hurewicz and universal coefficient theorems,

$$H^n(\tilde{X}, \tilde{A}; \pi) \approx \operatorname{Hom}(\pi_n(\tilde{X}, \tilde{A}), \pi).$$

Let $k_0 : (\tilde{X}, \tilde{A}) \to (K(\pi, n), o)$ represent the cohomology class corresponding to $\kappa \in \operatorname{Hom}(\pi_n(\tilde{X}, \tilde{A}), \pi)$. Then k_0 is homotopic to an equivariant map. For the fact that $\pi_1 X$ acts freely on the set of cells of $\tilde{X} - \tilde{A}$ makes it possible to construct, skeleton by skeleton, a homotopy from k_0 to an equivariant map k; and the fact that κ is equivariant ensures that the only obstruction to this procedure is zero. If k' is any other equivariant map inducing κ, then $k' \simeq k$ since they represent the same cohomology class; and there is no obstruction to deforming a homotopy $k' \simeq k$ into an equivariant homotopy. Thus $k : (\tilde{X}, \tilde{A}) \to (K(\pi, n), o)$ is unique up to equivariant homotopy.

1.3 *Modification to* 1.2 *for small n.* If $n = 2$ and $\pi_2(X, A)$ is non-abelian, replace $\pi_2(X, A)$ in the statement of 1.2 by the abelianization $\bar{\pi}_2(X, A)$. Then

$$H^2(\tilde{X}, \tilde{A}; \pi) \approx \operatorname{Hom}(\bar{\pi}_2(X, A), \pi)$$

and the argument goes through.

If $n = 1$, then $\hat{K}(\pi, 1)$ is a space of type $K(\pi \tilde{\times} \operatorname{aut} \pi, 1)$ where $\pi \tilde{\times} \operatorname{aut} \pi$ is the semi-direct product. If X and A are connected, then the homotopy classes

$$\phi : (X, A) \to (\hat{K}(\pi, 1), Q)$$

for which $q\phi$ induces $\varepsilon : \pi_1 X \to \operatorname{aut} \pi$ correspond to functions ('crossed homomorphisms') $\psi : \pi_1 X \to \pi$ satisfying

$$\psi(ab) = \psi a + (\varepsilon a)\cdot\psi b \quad \text{and} \quad \psi(\operatorname{im} \pi_1 A) = 0.$$

1.4. We shall need a reformulation of 1.2 in terms of fibrewise homotopy classes. For any pair (X, A) and any map $e : X \to Q$, we consider the maps

$$f : (X, A) \to (\hat{K}(\pi, n), Q)$$

which make the diagram

$$\begin{tikzcd} (X, A) \arrow[r] \arrow[dr, "e"'] & (\hat{K}(\pi, n), Q) \arrow[d, "q"] \\ & Q \end{tikzcd}$$

commute. We call such maps *fibrewise over e*, and we denote by

$$[(X, A), (\hat{K}, Q)]_e$$

the set of fibrewise homotopy classes of such maps.

1.5 LEMMA. *Let (X, A) be a pair of CW-type with the homotopy extension property, where X is connected. Choose one map $e_i : X \to Q$ in each homotopy class. Then the set $[(X, A), (\hat{K}, Q)]$ of ordinary (based) homotopy classes is isomorphic to $\bigcup_i [(X, A), (\hat{K}, Q)]_{e_i}$.*

Proof. Every map $(X, A) \to (\hat{K}, Q)$ is homotopic to a fibrewise map over some e_i, by the relative homotopy lifting lemma of [12, Theorem 4] applied to the fibration q. If H is a homotopy from f to $f' : (X, A) \to (\hat{K}, Q)$, where f, f' are fibrewise over e_i, e_j respectively, then $qH : e_i \simeq e_j$. Hence $i = j$ and qH is a (based) homotopy from e_i to itself. Since X is connected and Q has no homotopy groups above π_1, qH can be deformed rel $X \times \partial I$ to the constant homotopy at e_i. Application of [12, Theorem 4] to this deformation turns H into a fibrewise homotopy $f \simeq f'$.

2. The universal fibration

Suppose $n \geqq 1$. Let $P = P(\pi, n)$ be the space of unbased paths in $\hat{K}(\pi, n)$ which have initial point in Q and which lie entirely in some fibre of $q : \hat{K}(\pi, n) \to Q$. That is,

$$P = \{w \in (\hat{K}, Q)^{(I,0)} \mid qw(I) \text{ is a point}\}.$$

The map $p : P \to \hat{K}(\pi, n)$ defined by $p(w) = w(1)$ is a fibration with fibre $\Omega K(\pi, n) \simeq K(\pi, n - 1)$. There is a canonical section σ over $Q \subset \hat{K}(\pi, n)$ given by $\sigma(x) =$ (constant path at x).

Let $f : E \to B$ be a fibration with fibre $F = f^{-1}o$. Let $(\phi, \psi) : f \to p$ be a map of fibrations, i.e. there is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\psi} & P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\phi} & \hat{K}(\pi, n). \end{array}$$

Let M_f be the reduced mapping cylinder of f. Define

$$\chi : (M_f, E) \to (\hat{K}, Q) \quad (\text{the } \textit{adjoint} \text{ of } (\phi, \psi))$$

by $\chi b = \phi b$ $(b \in B)$, $\chi(e, t) = (\psi e)(t)$ for $(e, t) \in E \times I$. χ is continuous since E is compactly generated. The homotopy sequence of (M_f, E) is identified with that of the fibration f, and the following diagram commutes:

$$(2.1) \qquad \begin{array}{ccc} \pi_{n-1} F & \xrightarrow{\psi \mid F} & \pi_{n-1}(\Omega K(\pi, n)) \\ \downarrow \wr & & \wr \\ & & \pi \\ & & \wr \\ \pi_n(M_f, E) & \xrightarrow{\chi} & \pi_n(\hat{K}, Q). \end{array}$$

2.2 THEOREM. *Let $n \geqq 2$. Let $F \subset E \xrightarrow{f} B$ be a fibration with F and B of CW-type, B connected, F $(n - 2)$-connected.*

Then for any isomorphism $\kappa : \pi_{n-1} F \to \pi$ there is a map $(\phi, \psi) : f \to p$ of fibrations, unique up to homotopy, such that $\psi \mid F$ induces κ.

If f admits a section, then ϕ can be deformed into $Q \subset \hat{K}(\pi, n)$.

Proof. Since F and B are of CW-type, so is E[10]; hence the pair (M_f, E) is of CW-type. $\pi_1 M_f$ acts on $\pi_n(M_f, E) \approx \pi_{n-1} F$ and hence on π (via the isomorphism κ). Therefore there is a unique homomorphism $\varepsilon : \pi_1 M_f \to \operatorname{aut} \pi$ such that κ is ε-equivariant. By 1.2, there is a map

$$k : (M_f, E) \to (\hat{K}(\pi, n), Q),$$

unique up to homotopy, which induces $\kappa : \pi_n(M_f, E) \to \pi_n(\hat{K}, Q)$. Using the homotopy lifting property of q, deform k to a map k' which takes each line $e \times I$ in $M_f (e \in E)$ into a fibre of q. Define $\phi : B \to \hat{K}$ to be $k' \mid B$ and $\psi : E \to P$ to be the adjoint of

$$E \times I \longrightarrow M_f \xrightarrow{k'} \hat{K}(\pi, n).$$

Then $p\psi = \phi f$ and by diagram 2.1, $\psi \mid F$ induces κ.

If $(\phi', \psi') : f \to p$ also induces κ, its adjoint

$$k'' : (M_f, E) \to (\hat{K}, Q)$$

is homotopic to k' by 1.2. Take a homotopy $k' \simeq k''$ and deform it $(\operatorname{rel} M_f \times \partial I)$ until each line $e \times I \times t \subset M_f \times I$ lies in a fibre of q. The adjoint of the new homotopy is a homotopy $(\phi, \psi) \simeq (\phi', \psi')$.

Suppose now that f has a section $\tau : B \to E$. Then a deformation of ϕ into Q is given by $\Phi(b, t) = k'(\tau b, 1 - t)$ for $(b, t) \in B \times I$.

By a $K(\pi, n - 1)$*-fibration* we shall mean a fibration over a CW-type base in which each fibre has the homotopy type of $K(\pi, n - 1)$. Theorem 2.2 enables us to classify these up to fibre homotopy equivalence (c.f. Stasheff [10]).

2.3 Corollary. (i) *$\hat{K}(\pi, n)$ is a classifying space for $K(\pi, n - 1)$-fibrations (in the sense of Stasheff) and $p : P \to \hat{K}(\pi, n)$ is a universal fibration.*

(ii) *A $K(\pi, n - 1)$-fibration has a section if and only if its classifying map can be deformed into $Q \subset \hat{K}(\pi, n)$.*

Proof. For (i), we remark that the fibre of p is $\Omega K(\pi, n)$, which has the homotopy type of $K(\pi, n - 1)$ by a result of Milnor [8, Theorem 3]. To prove (ii), recall that p has a standard section σ over Q. The rest follows easily from 2.2.

2.4. We note that 2.3 (i) can be interpreted in two ways. Let us define a $K(\pi, n - 1)$*-fibration with fixed fibre* to be a $K(\pi, n - 1)$-fibration together with a specific homotopy equivalence between the fibre over the basepoint and the space $K(\pi, n - 1)$. There is an evident notion of fibre homotopy equivalence (preserving the extra structure) between such entities. Then

(a) the based homotopy classes $X \to \hat{K}(\pi, n)$ correspond to equivalence classes of $K(\pi, n - 1)$-fibrations (with fixed fibre) over X and

(b) the unbased homotopy classes $X \to \hat{K}(\pi, n)$ correspond to ordinary fibre homotopy equivalence classes of fibrations (cf. [1, §16.7]).

We show next that the inclusion $Q \subset \hat{K}(\pi, n)$ is a classifying map for the $K(\pi, n-1)$-fibration $q_{n-1} : \hat{K}(\pi, n-1) \to Q$ of §1. For let $PK(\pi, n)$ be the usual path-space $(K(\pi, n), o)^{(I,0)}$. Then P is homeomorphic to $(PK)(\pi, n) \times \tilde{Q})/\text{aut}\ \pi$. Choose a homotopy equivalence

$$g : K(\pi, n-1) \to \Omega K(\pi, n).$$

Then

$$\hat{K}(\pi, n-1) = \frac{K(\pi, n-1) \times \tilde{Q}}{\text{aut}\ \pi} \xrightarrow{g \times 1} \frac{PK(\pi, n) \times \tilde{Q}}{\text{aut}\ \pi} \approx P$$

gives a fibration-map $q_{n-1} \to p$ over the inclusion $Q \subset \hat{K}(\pi, n)$ which carries the standard section of q_{n-1} to the standard section σ of p over Q.

The inverse image of $Q \subset \hat{K}(\pi, n)$ by the fibration p is the space of 'fibrewise loops' on $\hat{K}(\pi, n)$. Denote this by Ω. We have shown the following.

2.5. The restriction $p \mid \Omega : \Omega \to Q$ of the universal fibration is fibre homotopy equivalent to $q : \hat{K}(\pi, n-1) \to Q$ in a standard, section-preserving way.

3. Cohomology with twisted coefficients

As before, we denote $K(\text{aut}\ \pi, 1)$ by Q and the projection $\hat{K}(\pi, n) \to Q$ by q.

3.1 Definition. Let (X, A) be an unbased pair of CW-type with the homotopy extension property, and $e : X \to Q$ an unbased map. We call e a *coefficient map* for (X, A), and we define the cohomology group $H^n(X, A; e)$ to be the set of fibrewise homotopy classes of maps

$$\phi : (X, A) \to (\hat{K}(\pi, n), Q) \quad \text{satisfying} \quad q\phi = e : X \to Q.$$

Since $q : \hat{K}(\pi, n) \to Q$ is a bundle of topological abelian groups, $H^n(X, A; e)$ has the structure of an abelian group.

$H^n(\ ,\ ;\ )$ is evidently a functor of pairs of spaces over Q. We define a coboundary

$$\delta : H^{n-1}(A, \phi; e \mid A) \to H^n(X, A; e)$$

as follows. Let

$$f : (A, \phi) \to (\hat{K}(\pi, n-1), Q)$$

represent a class $\alpha \in H^{n-1}(A, \phi; e \mid A)$. Compose f with the standard equivalence (2.5) $\hat{K}(\pi, n-1) \to \Omega$, where Ω is the space of 'fibrewise loops' on $\hat{K}(\pi, n)$. The adjoint of the composition is a self-homotopy of

$$e \mid A : A \to Q \subset \hat{K}(\pi, n),$$

fibrewise over $e : A \to Q$. Extend this homotopy to $F_t : X \to \hat{K}(\pi, n)$, fibrewise over e, starting at

$$X \xrightarrow{e} Q \subset \hat{K}(\pi, n)$$

(this requires Theorem 4 of Strøm [12]) and define $\delta\alpha$ to be the class of $F_1 : (X, A) \to (\hat{K}, Q)$.

3.2. One easily proves:

(i) the resulting cohomology sequence

$$\cdots \to H^{n-1}(X, A; e) \to H^{n-1}(X, \phi; e) \to H^{n-1}(A, \phi; e \mid A) \xrightarrow{\delta} H^n(X, A; e) \to \cdots$$

is exact;

(ii) if (X, A) is a CW pair with $\dim (X - A) < n$, then $H^n(X, A; e) = 0$;
(iii) connectivity $(X, A) \geqq n$ implies $H^n(X, A; e) = 0$.

A homotopy $e \simeq e' : X \to Q$ yields an isomorphism

$$H^n(X, A; e) \approx H^n(X, A; e')$$

but the isomorphism depends, in general, on the particular homotopy chosen.
If e is the trivial map to the basepoint of Q, then

$$H^n(X, A; e) = [(X, A), (K(\pi, n), o)]$$

which is the usual cohomology group $H^n(X, A; \pi)$.

3.3. Consider the special case where X and A are connected and non-empty. Fix a base-point $o \in A$. Since Q is connected, it suffices to consider based coefficient maps. If $e, e' : X \to Q$ are based-homotopic, there is a unique homotopy class of based homotopies $e \simeq e'$ (since X is connected) and hence a *canonical* isomorphism $H^n(X, A; e) \approx H^n(X, A; e')$. Thus we can speak of cohomology with coefficients in a homotopy class $X \to Q$. Such homotopy classes correspond to $\pi_1 X$-module structures on π. From this, 2.4 and 1.5 we have:

3.4 THEOREM. *The fibre-homotopy classes of $K(\pi, n-1)$-fibrations (with fixed fibre) over X are in one-one correspondence with the set $\bigcup_\lambda H^n(X, o; \lambda)$ where λ runs over $\pi_1 X$-module structures on π. The cohomology element ("characteristic class") corresponding to a fibration is the cohomology class represented by the classifying map $X \to \hat{K}(\pi, n)$.*

A $K(\pi, n-1)$-fibration has a section if and only if its characteristic class is the zero of some $H^n(X, o; \lambda)$.

We now relate our definition of cohomology to the classical idea of cohomology with twisted coefficients. Assume (X, A) is a based CW pair with X and A connected, and $X^0 \subset A$. Write X_A^n for $X^n \cup A$. From 3.2 (i), (ii), (iii), and easy manipulation with exact sequences, one deduces that $H^*(X, A; e)$ is the cohomology of the cochain complex

$$\cdots \xleftarrow{\delta} H^n(X_A^n, X_A^{n-1}; e) \xleftarrow{\delta} H^{n-1}(X_A^{n-1}, X_A^{n-2}; e) \xleftarrow{\delta} \cdots \xleftarrow{\delta} H^0(X_A^0, A; e) \leftarrow 0 \tag{3.5}$$

For $n \geq 3$, 1.2 and 1.3 imply

$$H^n(X_A^n, X_A^{n-1}; e) \approx \operatorname{Hom}(\pi_n(X_A^n, X_A^{n-1}), \pi)$$

where π has the $\pi_1 X$-module structure determined by e and Hom denotes homomorphisms of $\pi_1 X$-modules; and one knows that $\pi_n(X_A^n, X_A^{n-1})$ is a free $\pi_1 X$-module on the oriented n-cells of $X - A$. $H^2(X_A^2, X_A^1; e)$ and $H^1(X_A^1, X_A^0; e)$ have similar interpretations. $H^0(X_A^0, A; e) = 0$ since $X_A^0 = A$. The coboundary

$$\delta : H^n(X_A^n, X_A^{n-1}; e) \to H^{n+1}(X_A^{n+1}, X_A^n; e)$$

is, up to sign at least, equal to

$$\mathrm{Hom}\,(\partial, 1) : \mathrm{Hom}\,(\pi_n(X_A^n, X_A^{n-1}), \pi) \to \mathrm{Hom}\,(\pi_{n+1}(X_A^{n+1}, X_A^n), \pi)$$

where ∂ is the homotopy boundary. Hence, using (3.5):

3.6 Proposition. *$H^*(X, A; e)$ is the cohomology of the complex of $\pi_1 X$-modules*

$$\cdots \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \to \cdots \xrightarrow{\partial_1} F_1 \to 0$$

with coefficients in the $\pi_1 X$-module π, where F_n is a free $\pi_1 X$-module on the set of oriented n-cells of $X - A$ and ∂_n is the homomorphism induced by the attaching maps of the n-cells.

4. Moore-Postnikov systems

Using the foregoing theory, we perform a construction which generalises that of [4] to the non-simple case.

Let

$$F \subset E \xrightarrow{f} B$$

be a fibration, where F, E, B are connected spaces of CW-type and $\pi_1 F$ is abelian. We apply Theorem 2.2 with $\pi = \pi_1 F$ and κ the identity homomorphism. There results a map of fibrations

$$\begin{array}{ccc} E & \xrightarrow{\psi} & P(\pi_1 F, 2) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\phi} & \hat{K}(\pi_1 F, 2). \end{array}$$

Let $E_2 \to B$ be the fibration induced from p by ϕ: this has fibre $K(\pi_1 F, 1)$, and $f : E \to B$ factorises into $E \to E_2 \to B$. ϕ represents the characteristic class of $E_2 \to B$, which we denote by $k^2 \in H^2(B, o; \tilde{\pi}_1 F)$: here $\tilde{\pi}_1 F$ is the group $\pi_1 F$ with the $\pi_1 B$-module structure determined by the action in the fibration f.

We make $E \to E_2$ into a fibration: its fibre F^1 is 1-connected, and $\pi_i F^1 \approx \pi_i F$ for $i \geq 2$. We apply 2.2 to $E \to E_2$ with $\pi = \pi_2 F$. This yields as above a factorisation $E \to E_3 \to E_2$ where $E_3 \to E_2$ has fibre $K(\pi_2 F, 2)$. Let k^3 be the characteristic class of $E_3 \to E_2$.

Proceeding inductively, we obtain a factorisation

$$E \to \cdots \to E_n \to E_{n-1} \to \cdots \to E_2 \to B \tag{4.1}$$

of f where $E_{i+1} \to E_i$ has fibre $K(\pi_i F, i)$ and characteristic class

$$k^{i+1} \in H^{i+1}(E_i, o; \tilde{\pi}_i F);$$

here $\tilde{\pi}_i F$ is $\pi_i F$ with the $\pi_1 E_i$-module structure determined by the action of $\pi_1 E (\approx \pi_1 E_i)$ on $\pi_i F$ in the fibration f.

We call (4.1) a *Moore-Postnikov system* for f, and the k^{i+1} the *k-invariants* of f. The uniqueness part of 2.2 implies that the spaces and maps in (4.1), and the classes k^i, are essentially unique.

By elaborating the constructions along the lines of those in [5], one can construct modified Postnikov towers (in the sense of [5]) for non-simple fibrations. These are not constructed by a canonical process, but are useful in computations.

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