STRONG EMBEDDINGS INTO CATEGORIES OF ALGEBRAS

BY

A. PULTR AND V. TRNKOVÁ

INTRODUCTION

A *strong embedding* of a concrete category[1] $(\mathfrak{K}, U)$ into a concrete category $(\mathfrak{K}', U')$ is a full embedding $\Phi : \mathfrak{K} \rightarrow \mathfrak{K}'$ such that there is a functor F with $U' \circ \Phi = F \circ U$. Thus, the underlying sets (mappings) of images of objects (morphisms) with equal underlying sets (mappings) coincide, the underlying mapping of an image of a morphism carried by an identity mapping is an identity mapping etc.

We see easily that if $(\mathfrak{K}, U)$ can be strongly embedded into a category of algebras (taken with the natural forgetful functor) and if, for a morphism $\varphi \in \mathfrak{K}$, $U\varphi$ is one-to-one onto, then φ is an isomorphism. Consequently, categories like the category of graphs and graph homomorphisms, category of topological spaces and continuous mappings etc. are not strongly embeddable into a category of algebras. We shall say that a concrete category is *strongly algebraic* if it is strongly embeddable into a category of algebras.

The present paper was stimulated by the questions whether some familiar categories which are not rejected by the simple criterion above (we shall give a finer one in 3.14), e.g. the category of compact Hausdorff spaces and continuous mappings, the category of topological spaces with open mappings, or, with quotient mappings, the category of complete lattices and complete homomorphisms etc. are strongly algebraic. We obtained a relatively general theorem (2.7, 2.9) from which the positive answer for the mentioned categories, and for other ones, easily follows. Another corollary of the main theorem is that one of the choices of morphisms described by Wyler in [8] (see 3.9 and 3.11) yields for certain wide family of functors strongly algebraic categories.

We work in the Gödel-Bernays set theory with one additional assumption:

(M) There exists a cardinal α such that every α-additive two-valued measure is trivial.

(Thus, (M) is a bit weaker then the assumption of non-existence of measurable cardinals—which is (M) with $\alpha = \omega_0$). The statements in §2 in the proof of which (M) is used are indicated by superscript m after their numbers —e.g. 2.7^{m}.

The notation used and some results from elsewhere are summarized in

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[1] A concrete category is, as usual, a category $\mathfrak{K}$ together with a fixed faithful functor U from $\mathfrak{K}$ into the category of sets (the forgetful functor). If x (φ resp.) is an object (morphism resp.) in $\mathfrak{K}$, Ux ($U\varphi$ resp.) is called the underlying set (mapping resp.) of x (φ resp.); we also say that x, φ are carried by Ux, $U\varphi$.

a section named simply "Notation", placed after the last, third, paragraph- It is divided into seven parts (N1–N7). Whenever a symbol or a notion un. explained before, or some of its less obvious properties, appears in the text, the corresponding Ni is quoted.

1. Categories $A(F)$

1.1 Definition. Let J be a set. Let for every $i \in J$ a set functor F_i (see N3) and a type Δ_i (see N2) be given. The category $A((F_i, \Delta_i)_{i \in J})$ is defined as follows: The objects are systems $(X, (\omega_i)_J)$ where X is a set and every ω_i is an algebraic structure of the type Δ_i on $F_i(X)$. The morphisms from $(X, (\omega_i)_J)$ into $(X', (\omega_i')_J)$ are triples $((X, (\omega_i)_J), f, (X', (\omega_i')_J))$ where $f : X \to X'$ is a mapping such that every $F_i(f)$ is a $\omega_i\,\omega_i'$-homomorphism. The category $A((F_i, \Delta_i)_{i \in J})$ will be, throughout this paper, taken always for a concrete category endowed with the forgetful functor U defined by

$$U(X, (\omega_i)_J) = X, U((X, (\omega_i)_J), f, (X', (\omega_i')_J)) = f.$$

Conventions. We shall often write simply f instead of

$$((X, (\omega_i)_J), f, (X', (\omega_i')_J)).$$

If $J = n$, we write $A((F_0, \Delta_0), (F_1, \Delta_1), \cdots, (F_{n-1}, \Delta_{n-1}))$ instead of $A((F_i, \Delta_i)_J)$. If $\Delta_i = (1)$ for every i, we write $A((F_i)_J)$ for $A((F_i, \Delta_i)_J)$. In particular, $A(F)$ is the category $A(F, (1))$.

1.2 Lemma (For notation see N3). *Let M, N be sets. For a mapping*

$$\omega : \langle N, M\rangle \to M \text{ define } \bar{\omega} : \langle N, M\rangle \to \langle N, M\rangle \text{ by } \bar{\omega}(\varphi) = \mathrm{const}_{\omega(\varphi)}.$$

If $\omega_i : M_i^N \to M_i$ $(i = 1, 2)$ and $g : M_1 \to M_2$, then

$$g \circ \omega_1 = \omega_2 \circ Q_N g \quad \text{iff} \quad Q_N g \circ \bar{\omega}_1 = \bar{\omega}_2 \circ Q_N g$$

For a mapping $\omega : M \to M$ define $\varpi : P^-(M) \to P^-(M)$ by $\varpi = P^-(\omega)$. If $\omega_i : M_i \to M_i$ $(i = 1, 2)$ and $g : M_1 \to M_2$, then

$$g \circ \omega_1 = \omega_2 \circ g \quad \text{iff} \quad P^-(g) \circ \varpi_2 = \varpi_1 \circ P^-(g)$$

Proof is trivial.

1.3 Proposition. *Let J be a set, F_i $(i \in J)$ set functors and Δ_i $(i \in J)$ types. Then there is a set K and covariant set functors G_k $(k \in K)$ such that $A((F_i, D_i)_J)$ can be realized in $A((G_k)_K)$. If every F_i is a TB-functor (see* N6*), the functors G_k may be chosen to be TB-functors too.*

Proof. Let $\Delta_i = (\alpha_\beta^i)_{\beta < \gamma^i}$. First, we see easily that $A((F_i, \Delta_i)_{i \in J})$ is realizable (see N7) in $A((F_k, (\delta_k))_K)$, where $K = \bigcup \{\gamma^i \times \{i\} \mid i \in J\}$, $\delta_{(\beta, i)} = \alpha_\beta^i$ and $F_{(\beta, i)} = F_i$. Now, define $H_k = Q_{\delta_k} \circ F$ for $\delta_k \neq 0$, $H_k = F_k$ for $\delta_k = 0$. By the first part of Lemma 1.2 we see easily that

$$\Phi : A((F_k, (\delta_k))_K) \to A((H_k)_K)$$

defined by $\Phi(X, (\omega_k)_K) = (X, (\bar{\omega}_k)_K)$, where, for $\delta_k = 0$, $\bar{\omega}_k = \text{const}\ \omega_k$, is a realization. Finally, by the second part of Lemma 1.2 we obtain a realization of $A((H_k)_K)$ in $A((G_k)_K)$, where $G_k = H_k$ if H_k is covariant, $G_k = P^{-} \cdot H_k$ otherwise.

1.4 LEMMA. *Let F_i be covariant for every $i \in J$. Then $A((F_i)_{i \in J})$ is realizable in $A(\bigvee_J F_i)$. (For the symbol $\bigvee$, see* N4.*)*

Proof. It suffices to define $\Phi : A((F_i)_{i \in J}) \to A(\bigvee F_i)$ by

$$(X, (\omega_i)_J) = (X, \omega),$$

where $\omega(x, i) = (\omega_i(x), i)$.

1.5 LEMMA. *Let F be covariant. Then $A(F)$ is realizable in $A(F \vee C_1)$. (See* N3, N4.*)*

Proof. It suffices to replace the operations ω by $\omega \vee 1_1$.

1.6 DEFINITION. A set functor F is said to be pointed if there exists a transformation $\tau : C_1 \to F$.

As a simple corollary of 1.3–1.6 we obtain

1.7 THEOREM. *Let J be a set, Δ_i $(i \in J)$ types, F_i $(i \in J)$ set functors (TB-functors resp.). Then there exists a covariant point set functor (TB-functor resp.) F such that $A((F_i, \Delta_i)_J)$ is realizable in $A(F)$.*

1.8 PROPOSITION. *Let F be a retract (see* N3*) of G. Then $A(F)$ is realizable in $A(G)$.*

Proof. Take transformations $\mu : F \to G$ and $\varepsilon : G \to F$ such that $\varepsilon\mu = 1_F$. Define $\Phi : A(F) \to A(G)$ by $\Phi(X, \varphi) = (X, \mu^x \circ \varphi \circ \varepsilon^x)$. It is easy to see that Φ is a realization.

1.9 DEFINITION. A faithful set functor F (covariant or contravariant), is said to be algebraically selective if there exists a set M and a one-to-one full functor $\Phi : \text{Set} \to M\mathfrak{A}$ (see N2, N3) such that $F = U \circ \Phi$.

1.10 PROPOSITION. *Let F be covariant algebraically selective, Δ a type, $\Sigma\Delta \geq 2$ (see* N2*). Then there exists a strong embedding of $A(F)$ into $\mathfrak{A}(\Delta)$ (see* N2*).*

Proof. Let $\Phi : \text{Set} \to M\mathfrak{A}$ be a full embedding such that $F = U \circ \Phi$. Put $\Phi(X) = (F(x), (\overset{X}{\varphi}_m)_M)$. Thus, the following two statements are equivalent:

(1) There is an $f : X \to Y$ with $F(f) = g$.
(2) $g \circ \overset{X}{\varphi}_m = \overset{X}{\varphi}_m g$ for every $m \in M$.

Choose an element $a \notin M$ and put $N = M \cup \{a\}$. Define $\Psi : A(F) \to N\mathfrak{A}$

as follows:

$$\Psi(X, \alpha) = (F(X), (\alpha_n)_N) \text{ where } \alpha_m = \overset{x}{\varphi_m} \text{ for } m \in M \quad \text{and} \quad \alpha_a = \alpha,$$

$$U \circ \Psi((X, \alpha), f, (Y, \beta)) = F(f).$$

By the equivalence of (1) and (2) above we check easily that Ψ is a strong embedding. By [2], $N\mathfrak{A}$ is strongly embeddable into $\mathfrak{A}(\Delta)$.

2. Majorization by algebraically selective and main theorem

2.1 Lemma. *Let F, G be algebraically selective functors. Then $F \circ G$ is algebraically selective.*

Proof. Let $\Phi_i : \mathrm{Set} \to M_i \mathfrak{A}$ $(i = 1, 2)$ be full one-to-one functors such that $U \circ \Phi_1 = F$, $U \circ \Phi_2 = G$. We may assume that $M_1 \cap M_2 = \emptyset$. Let $\Phi_1(X) = (F(X), (\overset{x}{\varphi_m})_{M_1})$, $\Phi_2(X) = (G(X), (\overset{x}{\psi_m})_{M_2})$. Put $M = M_1 \cup M_2$, $\overset{x}{\chi_m} = F(\overset{x}{\psi_m})$ for $m \in M_1$, $\overset{x}{\chi_m} = \varphi_m^{G(x)}$ for $m \in M_2$ and define $\Phi : \mathrm{Set} \to M\mathfrak{A}$ by $\Phi(X) = (FG(X), (\overset{x}{\chi_m})_M)$, $U\Phi(f) = FG(f)$. It is easy to see that Φ is full and one-to-one.

2.2$^{\mathrm{m}}$ Lemma. *There exists an algebraically selective functor F with $P^- < F$ (see* N5).

Proof.[2] Take a set M such that $\mathrm{card}\, M = \delta$, where δ is the cardinal from the condition (M). Define F by $F(X) = \langle X \times M, 2\rangle$, $F(f)(\varphi) = \varphi \circ f$. By [4], (see N5), $P^- < F$. Define unary operations $\overset{x}{\pi_m}$ $(m \in M)$, $\overset{x}{\sigma}$, $\overset{x}{p}$, $\overset{x}{c}$ on $F(X)$ as follows:

$$(\pi_m\varphi)(x, n) = \varphi(x, m), \quad (\sigma\varphi)(x, b) = 0,$$

$$(p\varphi)(x, n) = \inf\{\varphi(x, m) \mid m \in M\}, \quad (c\varphi)(x, m) = 1 - \varphi(x\, m).$$

Define, for $\varphi : X \to 2$, $\bar{\varphi} : X \times M \to 2$ by $\bar{\varphi}(x, m) = \varphi(x)$; for $\varphi : X \times M \to 2$ and $m \in M$, define $\varphi_m : X \to 2$ by $\varphi_m(x) = \varphi(x, m)$. We have

(1) $\bar{\varphi}_m = \pi_m \varphi$.

Further, put $z = \mathrm{const}_0$, $e = \mathrm{const}_1$.

If $f : X \to Y$ is a mapping, we see easily that $F(f)$ is a homomorphism with respect to π_a, σ, p, c. On the other hand, let $g : F(Y) \to F(X)$ be a homomorphism. We obtain immediately $g(z) = g(\sigma\varphi) = \sigma g(\varphi) = z$, $g(e) = g(cz) = cz = e$.

Choose an $m_0 \in M$ and define

(2) $J_x = \{\varphi : Y \to 2 \mid g(\bar{\varphi})(x, m_0) = 1\}$.

We have obviously $e \in J_x$, $z \notin J_x$.

Choose $m, n \in M$, $m \neq n$ and define, for $\varphi, \psi : Y \to 2$ a mapping

[2] The proof is, in fact, only a modification of the proof of selectivity of P^- from [1].

$\chi : Y \times M \to 2$ by $\chi(x, m) = 1 - \varphi(x)$, $\chi(x, n) = 1 - \psi(x)$, $\chi(x, k) = e$ otherwise. We have $\overline{\sup(\varphi,\psi)} = cp\chi$, $\pi_m\chi = c\bar{\varphi}$, $\pi_n\chi = c\bar{\psi}$ and $\pi_k\chi = e$ for $k \neq m, n$. Thus,

$$g(\overline{\sup(\varphi,\psi)})(x, m_0) = 1 - pg(\chi)(x, m_0).$$

Since

$$g(\chi)(x, k) = \pi_k g(\chi)(x, m_0) = g(\pi_k \chi)(x, m_0)$$

we see that $\sup(\varphi,\psi) \in J_x$, i.e., $pg(\chi)(x, m_0) = 0$, iff either $\varphi \in J_x$ or $\psi \in J_x$.

Let $\varphi^m : Y \to 2$ $(m \in M)$ be mappings. Define $\varphi : Y \times M \to 2$ by $\varphi(x, m) = \varphi^m(x)$. We obtain $\varphi_m = \varphi^m$, $\inf\{\varphi^m \mid m \in M\} = p\varphi$, so that

$$g(\inf \varphi^m)(x, m_0) = g(p\varphi)(x, m_0) = pg\varphi(x, m_0) = \inf\{g(\varphi)(x, m) \mid m \in M\}$$
$$= \inf\{\pi_m g(\varphi)(x, m_0) \mid m \in M\} = \inf\{\bar{\varphi}^m(x, m_0) \mid m \in M\}$$

where the last equation is obtained by (1). Thus, if $\varphi^m \in J_x$ for every $m \in M$, then $\inf \varphi^m \in J_x$.

Thus, J_x is an ultrafilter closed with respect to the infima of systems of cardinality δ. Consequently, it is principal (by (M)), so that there is exactly one $y \in Y$ such that

(3) $J_x = \{\varphi \mid \varphi(y) = 1\}$

Let us denote this y by $f(x)$. We obtained a mapping $f : X \to Y$. By (2) and (3), $g(\bar{\varphi})(x, m_0) = \varphi f(x)$ for every $\varphi : Y \to 2$. Let $\psi : Y \times M \to 2$ be a mapping. We have

$$g(\psi)(x, m) = \pi_m g(\psi)(x, m_0) = g(\pi_m\psi)(x, m_0) = g(\bar{\psi}_m)(x, m_0) = \psi_m(f(x))$$
$$= \psi(f(x), m) = (F(f)(\psi))(x, m).$$

2.3 LEMMA. *Let F_i $(i \in J)$ be covariant algebraically selective functors. Then there is an algebraically selective functor G such that $\bigvee_J F_i < G$.*

Proof. Let $\Phi_i : \mathrm{Set} \to M_i\mathfrak{A}$ be full embeddings such that $U \circ \Phi_i = F_i$. We may assume M_i disjoint, $J \cap \bigcup M_i = \emptyset$ and $0, 1 \notin J \cup \bigcup M_i$. Put $\Phi_i(X) = (F_i(X), (\varphi_m^X)_{m \in M_i})$, $M = 2 \cup J \cup \bigcup M_i$.

By [5, Lemma 1], (F is there assumed one-to-one; it suffices however to assume F to be faithful) there exist monotransformations $\mu_i : I \to F_i$. Define a functor G by

$$G(X) = \bigvee F_i(X) \cup X \times \{0\} \cup 2 \times \{1\},$$

$$G(f)(x, i) = (F_i(f)(x), i), \quad G(f)(x, 0) = (f(x), 0), \; G(f)(j, 1) = (j, 1).$$

Define unary operations ψ_m^X $(m \in M)$ on $G(X)$ as follows:

$$\psi_0(x, 0) = (x, 0), \quad \psi_0(x, k) = (0, 1) \text{ for } k \neq 0, 1, \quad \psi_0(j, 1) = (1 - j, 1),$$

$$\psi_1 = \mathrm{const}_{(0,1)},$$

$$\text{for } x \neq 0, 1, \psi_i(x, 0) = (\mu_i(x), i), \ \psi_i(x, i) = (x, i),$$

$$\psi_i(x, k) = (0, 1) \text{ for } k \neq i, 0, 1, \ \psi_i(j, 1) = (1 - j, 1),$$

$$\psi_m(x, i) = (\varphi_m(x), i) \ \text{ for } m \in M_i, \ \psi_m(x, k) = (0, 1) \text{ otherwise.}$$

If $f : X \to Y$ is a mapping, $G(f)$ is obviously a homomorphism with respect to the operations described above. On the other hand, let $g : G(X) \to G(Y)$ be a homomorphism. Considering ψ_1 we see that $g(0, 1) = (0, 1)$. Thus, using ψ_0, we obtain $g(1, 1) = (1, 1)$. Now, considering the fixed points of ψ_0 and ψ_i $(i \in J)$ we obtain

$$g(X \times \{0\}) \subset Y \times \{0\}, \quad g(F_i(X) \times \{i\}) \subset F_i(Y) \times \{i\}$$

Thus, we may define $f : X \to Y$ by $g(x, 0) = (f(x), 0)$ and $g_i : F_i(X) \to F_i(Y)$ by $g(x, i) = (g_i(x), i)$. We have, for any $m \in M_i$,

$$(g_i \varphi_m(x), i) = g\psi_m(x, i) = \psi_m g(x, i) = (\varphi_m g_i(x), i)$$

and hence $g_i = F_i(f_i)$ for some $f_i : X \to Y$. Finally, for $i \in J$,

$$(\mu_i^Y f(x), i) = \psi_i(f(x), 0) = \psi_i g(x, 0) = g\psi_i(x, 0) = g(\mu_i^X(x), i)$$
$$= (F_i(f_i)\mu_i^X(x), i) = (\mu_i^Y f_i(x), i).$$

Thus, $f_i = f$ and hence $g = G(f)$.

2.4 LEMMA. *If* card $M \geq 2$, *then* V_M (*see* N3) *is algebraically selective.*

Proof. Choose $m, n \in M$, $m \neq n$, and define unary operations φ_m^x $(m \in M)$ on $X \times \{0\} \cup M \times \{1\}$ by $\varphi_m^x(x, 0) = (x, 0)$, $\varphi_m^x(m, 1) = (n, 1)$, $\varphi_m^x(k, 1) = (m, 1)$; otherwise, for $k \neq m$ put $\varphi_k^x = \text{const}_{(k,1)}$. If $f : X \to Y$, $g = V_M(f)$ is obviously a homomorphism. On the other hand, let $g : V_M(X) \to V_M(Y)$ be a homomorphism. We have

$$g(x, 0) = g(\varphi_m(x, 0)) = \varphi_m(g(x, 0)),$$

so that $g(X \times \{0\}) \subset Y \times \{0\}$, $g(k, 1) = g\varphi_k(\xi) = \varphi_k g(\xi) = (k, 1)$ for $k \neq m$, and finally

$$g(m, 1) = g\varphi_m(n, 1) = \varphi_m(n, 1) = (m, 1).$$

2.5 PROPOSITION. *For every covariant TB-functor F there is an algebraically selective G with* $F < G$.

Proof. By [3, Theorem 3.12] and N3 there is an ordinal α and a set M (we may assume card $M \geq 2$) with $F < P^\alpha \circ V_M$. According to 2.1 and 2.4 (see also N5) it suffices to prove that for every α there is an algebraically selective G with $P^\alpha < G$. For $\alpha = 1$, this follows by 2.1 and 2.2 (since $P = P^- \circ P^-$). If the statement is true for α, it is true for $\alpha + 1$ by 2.1 and 2.2. Now, let α be a limit ordinal and let there be, for every $\beta < \alpha$, an algebraically selective G_β with $P^\beta < G_\beta$. Then, by [3] (see N6) and 2.3 we have $P^\alpha < \bigvee P^\beta < \bigvee G_\beta < G$ for some algebraically selective G.

2.6. LEMMA. *Let F be covariant pointed, let G be algebraically selective and et $F < G$. Then there exists an algebraically selective H such that F is a retract f H.*

Proof. Since $F < G$, we have a monotransformation $\mu : F \to G'$ and an epitransformation $\varepsilon : G \to G'$ (see N5). Let $\Phi : \mathrm{Set} \to N\mathfrak{A}$ be a full embedding such that $U \circ \Phi = G$, let $\Phi(X) = (G(X), (\varphi_n^x)_{n \in N})$. We may assume $N \cap 2 = \emptyset$. Put

$$H = F \vee G' \vee G \vee C_2 \qquad \text{(see N4)}$$

and define unary operations ψ_m^x $(m \in M = B \cup 2)$ on

$$H(X) = F(X) \times \{0\} \cup G'(X) \times \{1\} \cup G(X) \times \{2\} \cup 2 \times \{3\}$$

as follows: $\psi_0(x, 0) = (\mu^x(x), 1), \psi_0(x, 1) = (0, 3), \psi_0(x, 2) = (1, 3), \psi_0(i, 3) = (0, 3), \psi_1(x, 0) = (0, 3), \psi_1(x, 1) = (1, 3), \psi_1(x, 2) = (\varepsilon^x(x), 1), \psi_1(i, 3) = (1, 3)$; for $n \in N, \psi_n(x, 0) = \psi_n(x, 1) = \psi_n(i, 3) = (0, 3), \psi_n(x, 2) = (\varphi_n(x), 2)$. It is easy to see that for every $f : X \to Y, H(f)$ is a homomorphism with respect to the operations described. Now, let $g : H(X) \to H(Y)$ be a homomorphism. Since $(0, 3)$ is the only fixed point of ψ_0, we have $g(0, 3) = (0, 3)$, similarly, using ψ_1, we obtain $g(1, 3) = (1, 3)$. If $x \in G(X)$, we have $\psi_0 g(x, 2) = g\psi_0(x, 2) = (1, 3)$, so that $g(x, 2) \in G(Y) \times \{2\}$ and we can define $h : G(X) \to G(Y)$ by $g(x, 2) = (h(x), 2)$. For $n \in N$ we have $(h\varphi_n(x), 2) = g\psi_n(x, 2) = \psi_n g(x, 2) = (\varphi_n h(x), 2)$, so that, by definition of $(\varphi_n)_N$, $h = G(f)$ for some $f : X \to Y$. Let $x \in G'(X)$. Since ε^x is onto, we have

$$\begin{aligned} g(x, 1) = g(\varepsilon^x(y), 1) &= g\psi_1(y, 2) = \psi_1(G(f)(x), 2) = (\varepsilon^Y G(f)(x), 1) \\ &= (G'(f)(x), 1). \end{aligned}$$

Let $x \in F(X)$. We have

$$\psi_0 g(x, 0) = g(\mu^x(x), 1) = (G'(f)\mu^x(x), 1) = (\mu^x F(f)(x), 1) = \psi_0(F(f)(x), 0).$$

Thus, since $\psi_0 g(x, 0) = (y, 1), g(x, 0)$ is in $F(Y) \times \{0\}$. Since ψ_0 is one-to-one on $F(Y) \times \{0\}$, we have $g(x, 0) = (F(f)(x), 0)$.

Let $\tau : C_1 \to F$ be a transformation. Define transformations $\alpha : F \to H$ and $\beta : H \to F$ by $\alpha^x(x) = (x, 0), \beta^x(x, 0) = x, \beta^x(x, i) = \tau^x(0)$ for $i \neq 0$. Obviously $\beta\alpha = 1_F$.

2.7[m] THEOREM. *Let J be a set, Δ_i $(i \in J)$ types, F_i $(i \in J)$ TB-functors, Δ a type with $\Sigma\Delta \geq 2$. Let $(\mathfrak{K}, U)$ be a concrete subcategory (see* N1) *of* $A((F_i, \Delta_i)_J)$. Then there exists a strong embedding of $(\mathfrak{K}, U)$ *into* $\mathfrak{A}(\Delta)$.

Proof. By 1.7, $(\mathfrak{K}, U)$ is realizable in an $A(F)$ with a covariant pointed TB-functor F. By 2.5 and 2.6, F is a retract of an algebraically selective G, so that, by 1.8, $(\mathfrak{K}, U)$ is realizable in $A(G)$ and, by 1.10, there is a strong embedding of $(\mathfrak{K}, U)$ into $\mathfrak{A}(\Delta)$.

2.8 DEFINITION. A concrete category $(\mathfrak{K}, U)$ is said to be strongly algebraic if there is a strong embedding of $(\mathfrak{K}, U)$ into some $\mathfrak{A}(\Delta)$.

Remark. A strongly algebraic category can be strongly embedded into any $\mathfrak{A}(\Delta)$ with $\Sigma\Delta \geq 0$. By [6], [7], it can be strongly embedded into the category of semigroups.

In view of the previous definition, we may reformulate 2.7 as follows:

2.9^m THEOREM. *Every concrete subcategory of $A((F_i, \Delta_i)_J)$ with TB-functors F_i is strongly algebraic.*

3. Consequences and remarks

3.1 PROPOSITION. *The category of complete lattices and all their complete homomorphisms and the category of complete Boolean algebras and all their complete homomorphisms are strongly algebraic.*

Proof. The first category is obviously a concrete subcategory of

$$A((I, (2, 2)), (P^+, (1, 1))),$$

the second one of

$$A((I, (0, 0, 1, 2)), (P^+, (1, 1))).$$

3.2 PROPOSITION. *The category of all topological spaces and their continuous closed mappings, the category of compact Hausdorff spaces and their continuous mappings and the category of all topological spaces and their continuous open mappings are strongly algebraic.*

Proof. The first one is realizable in $A(P^+)$, since f is continuous closed iff $P^+(f) \circ \mathrm{Cl} = \mathrm{Cl} \circ P^+(f)$ (where Cl is the closure operator). The second one is a concrete subcategory of the first one. The third one is realizable in $A(P^-)$, since f is continuous open iff $P^-(f) \circ \mathrm{Int} = \mathrm{Int} \circ P^-(f)$ (where Int is the interior operator).

3.3 PROPOSITION. *The category of pointed compact Hausdorff spaces is strongly algebraic.*

Proof. It is obviously realizable in $A((P^+, 1), (I, 0))$.

3.4 PROPOSITION. *The category of all sets and all one-to-one mappings* and the category of all sets and all mappings onto are strongly algebraic.

Proof. Define a unary operation δ^x on $V_2 Q_2(X) = X \times X \times \{0\} \cup 2 \times \{1\}$ by $\delta(x, x, 0) = (0, 1)$, $\delta(x, y, 0) = (1, 1)$ for $x \neq y$, $\delta(i, 1) = (i, 1)$. Obviously f is one-to-one iff $V_2 Q_2(f) \circ \delta = \delta \circ V_2 Q_2(f)$. Thus, the first category is realizable in $A(V_2 Q_2)$. The second one is evidently realizable in $A((P^+, 0))$. X being the nullary operation on $P^+(X)$.

3.5 *Remark.* Combining the previous considerations we see easily that the category of all topological spaces and their closed (open) homeomorphisms into is strongly algebraic etc.

3.6 PROPOSITION. *The category of topological spaces and their quotient mappings onto is strongly algebraic.*

Proof. If (X, τ) is a space, define a unary operation $\sigma(X, \tau)$ on $V_2 P^-(X)$ by $\sigma(M, 0) = (0, 1)$ if M is open, $\sigma(M, 0) = (1, 1)$ otherwise, $\sigma(i, 1) = (i, 1)$. We see easily that a mapping onto is a quotient mapping iff $V_2 P^-(f) \circ \sigma = \sigma \circ V_2 P^-(f)$. Thus, the category is realizable in $A((P^+, 0), (V_2 P^-, 1))$.

3.7 Definition. Let F be a set functor. Define a concrete category $\bar{S}(F)$ ($\bar{\bar{S}}(F)$ respectively) as follows: The objects are couples (X, r) where X is a set and $r \subset F(X)$, the morphisms from (X, r) into (Y, s) are triples $((X, r), f, (Y, s))$ where $f : X \to Y$ is a mapping such that

$$F(f)(r) \subset s \quad \text{and} \quad F(f)(F(X) - r) \subset F(Y) - s(F(f)(r) = s \text{ resp.};$$

in the case of contravariant F interchange r and s, X and Y). The forgetful functors in $\bar{S}(F)$ and $\bar{\bar{S}}(F)$ are defined by

$$U(X, r) = X, \qquad U((X, r), f, (Y, s)) = f.$$

Remark. Compare the definition with the definitions of $S(F)$, $S((F_i, \Delta_i)_J)$ in [1], [4]. In a way analogous to the one above we can, of course, define more generally concrete categories $\bar{S}((F_i, \Delta_i)_J)$, $\bar{\bar{S}}((F_i, \Delta_i)_J)$.

3.8 Proposition. *If F is a TB-functor, $\bar{S}(F)$ and $\bar{\bar{S}}(F)$ are strongly algebraic.*

Proof. If (X, r) is an object of $\bar{S}(F)$, define a unary operation $\sigma^{(x,r)}$ on $V_2 F(x)$ by $\sigma(x, 0) = (0, 1)$ if $x \in r$, $\sigma(x, 0) = (1, 1)$ if $x \notin r$, $\sigma(i, 1) = (i, 1)$. We see easily that $F(f)(r) \subset s$ and $F(f)(F(X) - r) \subset F(Y) - s$ iff $V_2 F(f) \circ \sigma = \sigma \circ V_2 F(f)$. Thus, $\bar{S}(F)$ is realizable in $A(V_2 F)$. $\bar{\bar{S}}(F)$ is obviously realizable in $A((P^+F, 0))$.

Remark. Analogous statements about $\bar{S}((F_i, \Delta_i)_J)$ and $\bar{\bar{S}}((F_i, \Delta_i)_J)$ are valid.

The choices of morphisms used in the following definition are particular cases of the choices introduced by Wyler in [8].

3.9 Definition. Let F, G be covariant set functors. Define concrete category $W_a(F, G)$ ($W^*(F, G)$ resp.) as follows. The objects are couples (X, r) where X is a set and $r : F(X) \to G(X)$ a mapping ($r \subset G(X) \times F(X)$ a subset, respectively). The morphisms from (X, r) into (Y, s) are triples $((X, r), f, (Y, s))$ where $f : X \to Y$ is a mapping such that $G(f) \circ r = s \circ F(f)$ (in the case of $W^*(F, G)$, $\circ$ is the usual composition of relations). The forgetful functors are defined by $U(X, r) = X$, $U((X, r), f, (Y, s)) = f$. $W_a(F, G)$ and $W^*(F, G)$ for contravariant F, G are defined quite analogously.

Remark. Of course, we can define $W_a((F_i, G_i)_J)$, $W^*((F_i, G_i)_J)$ is an obvious way.

3.10 Lemma. *$W^*(F, G)$ is realizable in $W_a(F, G)$.*

Proof.[3] For $r \subset G(X) \times F(X)$ define $\tilde{r} : F(X) \to P^+G(X)$ by

$$\tilde{r}(x) = \{y \mid (y, x) \in r\}.$$

It suffices to prove that, for $f : X \to Y$ and $s \subset G(Y) \times F(Y)$, the following two statements are equivalent:

(I) $G(f) \circ r = s \circ F(f)$;
(II) $P^+G(f) \circ \tilde{r} = \tilde{s} \circ F(f)$.

Let (I) hold and let $x \in F(X)$. If $y \in P^+G(f)(\tilde{r}(x))$, there is a z with $y = G(f)(z)$ and $(z, x) \in r$. Thus, $(y, x) \in G(f) \circ r = s \circ F(f)$ and hence $(y, F(f)(x)) \in s$, i.e. $y \in \tilde{s}(F(f)(x))$. If $y \in \tilde{s}(F(f)(x))$, we have $(y, F(f)(x)) \in s$, hence $(y, x) \in s \circ F(f) = G(f) \circ r$ so that there is a z such that $y = G(f)(z)$ and $(z, x) \in r$. Thus, $y \in P^+G(f)(\tilde{r}(x))$. Let (II) hold. If $(y, x) \in G(f) \circ r$, we have $y = G(f)(z)$ and $(z, x) \in r$ for some z. Thus,

$$z \in \tilde{r}(x) \quad \text{and} \quad y \in P^+G(f)(\tilde{r}(x)) = \tilde{s}(F(f)(x))$$

and hence $(y, x) \in s \circ F(f)$. If $(y, x) \in s \circ F(f)$, we have $y \in \tilde{s}(F(f)(x)) = P^+G(f)(\tilde{r}(x))$. Thus, there is a $z \in \tilde{r}(x)$ with $y = G(f)(z)$ and we have $(y, x) \in G(f) \circ r$.

3.11 PROPOSITION. *If F, G are TB-functors, $W_a(F, G)$ is strongly algebraic. It is easy to see that $W_a(F, G)$ is realizable in $A(F \times G)$—define, for $\varphi : F(X) \to G(X)$, a unary operation $\bar{\varphi}$ on $F(X) \times G(X)$ by $\bar{\varphi}(x, y) = (x, \varphi(x))$.*

3.12 PROPOSITION. *The category $\mathfrak{R}^*$ of graphs and their strong homomorphisms (if R, S are binary relations on X, Y respectively, we say that a mapping $f : X \to Y$ is a strong homomorphism from (X, R) into (Y, S) iff $f \circ R = S \circ f$) is strongly algebraic.*

Proof. In fact, $\mathfrak{R}^* = W^*(I, I)$.

3.13 *Remark.* By [6], [7], the categories mentioned in 3.1–3.12 are strongly embeddable into the category of semigroups.

3.14 PROPOSITION. *Let $(\mathfrak{K}, U)$ be strongly algebraic.*

(1) *If $\alpha : a \to c$ and $\beta : b \to c$ are morphisms in $\mathfrak{K}$, $U\beta$ one-to-one and $U\alpha = U\beta \circ f$, there is a $\gamma : a \to b$ in $\mathfrak{K}$ such that $U\gamma = f$.*

(2) *If $\alpha : c \to a$ and $\beta : c \to b$ are morphisms in $\mathfrak{K}$, $U\beta$ onto and $U\alpha = f \circ U\beta$, there is a $\gamma : b \to a$ in $\mathfrak{K}$ such that $U\gamma = f$.*

Proof. $\mathfrak{A}(2)$ has obviously the properties (1) and (2). Now, let $(\mathfrak{K}, U)$ be strongly algebraic, let $\Phi : \mathfrak{K} \to \mathfrak{A}(2)$ be a strong embedding of $(\mathfrak{K}, U)$ into $\mathfrak{A}(2)$, let $U \circ \Phi = F \circ U$. We see easily that F is faithful. We shall prove statement (1); the proof of (2) is analogous. We have $FU(\alpha) = FU(\beta) \circ F(f)$ and hence $U(\Phi\alpha) = U(\Phi\beta) \circ F(f)$. Thus, there is a homo-

[3] This proof is a modification of a proof proposed by Hedrlin.

morphism $h : \Phi(a) \to \Phi(b)$ with $F(f) = U(h)$. Since Φ is full, $h = \Phi(\gamma)$ for some $\gamma : h \to b$. We have $F(f) = U\Phi(\gamma) = F(U\gamma)$ and hence $f = U\gamma$.

3.15 PROPOSITION. *The category of compact spaces and their open mappings is not strongly algebraic.*

Proof. Let C be the subspace of the real line consisting of 0 and the numbers of the form $1/n$ where n is a natural number. Let A, B be two non-homeomorphic countable compact spaces, let α (β resp.) be a one-to-one mapping of A (B resp.) onto $C - \{0\}$. Obviously, α and β are open and determine a unique $f : A \to B$ such that $\beta \circ f = \alpha$. Thus f is a one-to-one mapping of A onto B. Thus, if it were open, it would be a homeomorphism.

3.16 *Problem.* Is the category of Banach spaces and their linear (continuous) mappings strongly algebraic?

Notation

N1. $(\mathfrak{K}, U)$ is said to be a concrete subcategory of a concrete category $(\mathfrak{K}', U')$ if $\mathfrak{K}$ is a subcategory of $\mathfrak{K}'$ and $U = U' \mid \mathfrak{K}$. Two concrete categories $(\mathfrak{K}, U)$ and $(\mathfrak{K}', U')$ are said to be concretely isomorphic if there is an isofunctor $\Phi : \mathfrak{K} \to \mathfrak{K}'$ with $U' \circ \Phi = U$.

N2. An ordinal α is taken, as usual, as the set of all ordinals less than α. A type $\Delta = (\alpha_\beta)_{\beta<\gamma}$ is a sequence of ordinals indexed by ordinals. The sum of a type Δ, written $\Sigma\Delta$, is the usual ordinal sum of the sequence. The notions of an algebra of the type Δ and of a homomorphism is used in the standard way. We denote by $\mathfrak{A}(\Delta)$ the concrete category of all algebras of the type Δ and their homomorphisms, endowed with the natural forgetful functor.

If M is a set, $M\mathfrak{A}$ is the category whose objects are systems $(X, (\varphi_m)_{m \in M})$ where X is a set and $\varphi_m : X \to X$ mappings, the morphisms from $A = (X, (\varphi_m)_M)$ into $A' = (X', (\varphi'_m)_M)$ are triples (A, f, A') where $f : X \to X'$ is a mapping such that $f \circ \varphi_m = \varphi'_m \circ f$ for every $m \in M$. $M\mathfrak{A}$ is taken as a concrete category endowed by the obvious forgetful functor. Of course, $M\mathfrak{A}$ is a concretely isomorphic with some $\mathfrak{A}(1, 1, 1, \cdots)$.

N3. The category of sets is denoted by Set, the functors $F : \text{Set} \to \text{Set}$ are called set functors. We use the following notation for some particular set functors (M is a set, $\langle X, Y\rangle$ is the set of all mappings from X into Y). I is the identity functor,

$$P^+(X) = P^-(X) = \{Y \mid Y \subset X\}, \quad P^+(f)(Y) = f(Y), \quad P^-(f)(Y) = f^{-1}(Y),$$
$$P = P^- \circ P^-,$$
$$Q_M(x) = \langle M, X\rangle, \quad Q_M(f)(\varphi) = f \circ \varphi,$$
$$P_M(X) = \langle X, M\rangle, \quad P_M(f)(\varphi) = \varphi f,$$
$$V_M(X) = X \times \{0\} \cup M \times \{1\}, \quad V_M(f)(x, 0) = (f(x), 0),$$
$$V_M(f)(m, 1) = (m, 1),$$

C_M is the constant functor, $C_M(f) = 1_M$.

A functor F is said to be a retract of G if there are transformations $\alpha : F \to G$ and $\beta : G \to F$ with $\beta\alpha = 1_F$.

N4. For the product (coproduct) of functors F_m $(m \in M)$ the symbol $\times_m F_m (\vee_M F_m)$ is used. Thus, $(\times_M F_m)(X) = \times_M F_m(X)$ (where $\times$ on the right hand side designates the cartesian product), $(\vee_M F_m)(X) = \bigcup_M F_m(X) \times \{m\}$. The product (coproduct) of two functors is denoted by $F \times G (F \vee G)$. Similarly, we use the symbol $F \vee G \vee H$ etc. If F is covariant and G contravariant, $\langle F, G\rangle$ is the contravariant functor defined by

$$\langle F, G\rangle(X) = \langle F(X), G(X)\rangle, \quad \langle F, G\rangle(f)(\varphi) = G(f) \circ \varphi \circ F(f).$$

Similarly, the covariant $\langle F, G\rangle$ for contravariant F and covariant G is defined.

N5. A monotransformation (epitransformation) $\tau : F \to G$ is a transformation such that every τ^x is one-to-one (onto). We write $F < G$ if there is a functor H, a monotransformation $\mu : F \to H$ and an epitransformation $\varepsilon : G \to H$ (thus, $F < G$ means that F is a subfunctor of a factorfunctor of G). $<$ is transitive (see [3, 3.2]. In the text, we use the following properties of $<$:

$$F < F', G < G' \Rightarrow F \circ G < F' \circ G' \qquad \text{(see [4, 3.7], [3, 2.7])},$$

$$F_m < G_m \ (m \in M) \Rightarrow \vee F_m < \vee G_m \qquad \text{(see [3, 3.5])},$$

$$P^+ < P^- \circ P^- = P \qquad \text{(see [4, 4.5])},$$

$$P^- < F$$

where F is defined by

$$F(X) = \langle X \times M, 2\rangle, F(f)(\varphi) = \varphi \circ (f \times 1_M) \quad \text{for card } M \geq 1$$

(see [4, 4.7, 4.4]).

N6. Constructive functors are defined recursively as follows.

(1) I, P_M, Q_M, C_M are constructive.
(2) If F, G are constructive then $F \circ G$ is constructive.
(3) If M is a set, F_m $(m \in M)$ constructive, all F_m covariant or all F_m contravariant, then $\vee_M F_m$ and $\times_M F_m$ are constructive.

F is said to be a TB-functor if there is a constructive G with $F < G$. The definition of constructive functors differs from that given in [3]. By the theorems from [3], however, the resulting TB-functors coincide.

The family of TB-functors is closed with respect to composition, the operation $\langle F, G\rangle$ (see N4) and all limits and colimits over small diagrams. P^+ is a TB-functor (see N5). In text, we use the following theorem from [3]: For every covariant TB-functor F there is an ordinal α and a set M such that $F < P^\alpha \circ V_M$ (for P, V_M see N3). The powers P^α are defined by induction, $P^{\alpha+1} = P \circ P^\alpha$, P^α for a limit ordinal is a colimit of P^β with $\beta < \alpha$. (The precise description is given in [3]; here we need only the fact that $P^\alpha < \vee_{\beta<\alpha} P^\beta$ for a limit ordinal α.)

N7. If $(\mathfrak{K}, U)$, $(\mathfrak{K}', U')$ are concrete categories, a full embedding $\Phi : \mathfrak{K} \to \mathfrak{K}'$ is called a realization of $(\mathfrak{K}, U)$ in $(\mathfrak{K}', U')$ if $U' \circ \Phi = U$. If there is a realization of $(\mathfrak{K}, U)$ in $(\mathfrak{K}', U')$ we say that $(\mathfrak{K}, U)$ is realizable in $(\mathfrak{K}', U')$. Thus (1) a realization is a particular case of a strong embedding ($F = I$); (2) $(\mathfrak{K}, U)$ is realizable in $(\mathfrak{K}', U')$ iff it is concretely isomorphic with some full concrete subcategory of $(\mathfrak{K}', U')$.

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Universita Karlova
Prague, Czechoslovakia