A NOTE ON THE STONG-HATTORI THEOREM

BY
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Summary. The theorem of Stong and Hattori referred to in the title asserts that the natural mapping

$$sh : \Omega_*^U(X) \to K_*(X \wedge MU)$$

is a split monomorphism whenever X is a finite complex with torsion free integral homology. In the present note we will show that the map sh remains monic (although need no longer be split) for those complexes with $\Omega_*^U(X)$ of projective dimension at most one as an Ω_*^U-module. Two proofs will be presented—one for K-theory and one for k-theory. We show by example that the result is best possible.

Let us begin by fixing our notations and conventions. We assume that we are working in a suitable category of spectra where the $\wedge$ product is defined, such as that constructed by Boardman [9]. We denote by $\Omega_*^U(\ \ )$ the complex bordism homology theory which is represented by the Thom spectrum **MU**. We write $K_*(\ \ )$ for the homology theory dual to the usual K-cohomology theory, and $k_*(\ \ )$ for the homology theory represented by the connective **bu** spectrum. The representing spectrum for $K_*(\ \ )$ is denoted by **BU**. There is the natural commutative diagram of spectra

$$\begin{tikzcd} & \mathbf{bu} \wedge \mathbf{MU} \arrow[dd, "\lambda \wedge 1"] \\ \mathbf{MU} \arrow[ur, "sh"] \arrow[dr, "SH"'] & \\ & \mathbf{BU} \wedge \mathbf{MU} \end{tikzcd}$$

which for any finite complex X yields a commutative diagram

$$\begin{tikzcd} & k_*(X \wedge MU) \arrow[dd, "\lambda"] \\ \Omega_*^U(X) \arrow[ur, "sh"] \arrow[dr, "SH"'] & \\ & K_*(X \wedge MU). \end{tikzcd}$$

Note that $sh = (1_{\mathbf{bu}} \wedge 1)_*$ and similarly for SH.

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THEOREM 1. *Let X be a finite complex with hom $dim_{\Omega_*^U}\Omega_*^U(X) \leq 1$. Then the map*

$$sh : \Omega_*^U(X) \to k_*(X \wedge MU)$$

is monic.

Proof. Since hom $\dim_{\Omega_*^U}\Omega_*^U(X) \leq 1$ the bordism spectral sequence

$$E^r \Rightarrow \Omega_*^U(X), \qquad E^2 \cong H_*(X; \Omega_*^U)$$

collapses [5; 3.11]. There is a natural transformation [5; §10]

$$\zeta : \Omega_*^U(\ \ ) \to k_*(\ \ )$$

which on coefficients is the Todd genus and a split epimorphism of abelian groups. Letting $Z[t] = k_*(pt)$ and

$$\bar{E}^r \Rightarrow k_*(X), \qquad \bar{E}^2 \cong H_*(X; Z[t])$$

be the usual k theory spectral sequence we see that ζ induces an epimorphism $\zeta : E_2 \to \bar{E}_2$. Since all the differentials d_r vanish the same must be true for the differentials $\bar{d}_r$.

Let $\mathbf{E} = \mathbf{bu} \wedge \mathbf{MU}$ and write $E_*(\ \ )$ for the corresponding homology theory. The natural map

$$k_*(—) \otimes_{Z[t]} k_*(\mathbf{MU}) \to E_*(\ \ ) \cong k_*(— \wedge \mathbf{MU})$$

is seen to be a natural equivalence. Hence the spectral sequence

$$\hat{E}_r \Rightarrow E_*(X) \cong k_*(X \wedge \mathbf{MU}), \qquad \hat{E}_2 \cong H_*(X; k_*(\mathbf{MU}))$$

is seen to be naturally isomorphic to the spectral sequence

$$\{\bar{E}_r \otimes_{Z[t]} k_*(\mathbf{MU}), \bar{d}_r \otimes 1\}$$

and since $\bar{d}_r = 0$ for all $r \geq 2$ we conclude that $\hat{d}_r = 0$ for all $r \geq 2$. Now the map

$$sh : \Omega_*^U(X) \to k_*(X \wedge \mathbf{MU})$$

induces a map of spectral sequences

$$sh : \{E^r, d^r\} \to \{\hat{E}^r, \hat{d}^r\}$$

which on the initial terms is seen to coincide with the map

$$H_*(X; \Omega_*^U) \to H_*(X; k_*(\mathbf{MU}))$$

induced by the coefficient map

$$sh : \Omega_*^U \to k_*(\mathbf{MU}).$$

The theorem of Stong [8] and Hattori [6] says that this latter map is a split monomorphism of abelian groups, and hence

$$H_*(X; \Omega_*^U) \to H_*(X; k_*(\mathbf{MU}))$$

is a split monomorphism of abelian groups. Since the spectral sequences $\{E^r, d^r\}$ and $\{\hat{E}^r, \hat{d}^r\}$ both collapse the preceding discussion shows that

$$sh : \Omega_*^U(X) \to k_*(X \wedge \mathbf{MU})$$

is a map of filtered abelian groups whose associated graded map is split monic. Simple algebra now shows that sh is monic, although perhaps not split. $\square$

Clearly one can make exactly the same argument as above with $K_*(\ )$ in place of $k_*(\ )$. However a more interesting proof of the $K_*(\ )$ result may be obtained by using the formalisms of [1] and [2]. Recall from [1; §3] that for any finite complex X, $K_*(X)$ is a comodule over the Hopf algebra $K_*(\mathbf{BU})$. An element $x \in K_*(X)$ is called primitive iff $\psi(x) = 1 \otimes x$ where

$$\psi : K_*(X) \to K_*(\mathbf{BU}) \otimes_{K_*(S^0)} K_*(X)$$

is the coaction map. Write $PK_*(X)$ for the set of primitive elements in $K_*(X)$.

THEOREM 2. *Let X be a finite complex with hom $\dim_{\Omega_*^U} U\Omega_*^U(X) \leq 1$. Then the map*

$$SH : \Omega_*^U(X) \to K_*(X \wedge MU)$$

is monic. (*Note this implies Theorem* 1)

Proof. Choose a partial U-bordism resolution [5; §2]

$$\Sigma^s X \to A \to Y.$$

Thus

$$0 \leftarrow \tilde{\Omega}_*^U(\Sigma^s X) \leftarrow \tilde{\Omega}_*^U(A) \leftarrow \tilde{\Omega}_*^U(Y) \leftarrow 0$$

is exact and $H_*(A; Z)$ is a free Z-module. From [4; VI.2.3] and the fact that

$$\text{hom} \dim_{\Omega_*^U} \Omega_*^U(\Sigma^s X) \leq 1$$

we conclude $\tilde{\Omega}_*^U(Y)$ is a projective Ω_*^U-module and hence by [5; 3.31], $H_*(Y; Z)$ is a free Z-module. From the commutative diagram

$$\begin{array}{ccccccc}
0 \leftarrow & \Omega_*^U(\Sigma^s X) & \leftarrow & \Omega_*^U(A) & \leftarrow & \Omega_*^U(Y) & \leftarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& K_*(\Sigma^s X) & \leftarrow & K_*(A) & \leftarrow & K_*(Y) &
\end{array}$$

we find as in [5; §9] that

$$0 \leftarrow K_*(\Sigma^S X) \leftarrow K_*(A) \leftarrow K_*(Y) \leftarrow 0$$

is exact, and since $K_*(\mathbf{MU})$ is a free $K_*(S^\circ)$-module

$$K_*(—) \otimes_{K_*(S^\circ)} K_*(\mathbf{MU}) \to K_*(—\mathbf{MU})$$

is an equivalence of functors. Thus the sequence

$$0 \leftarrow K_*(\Sigma^s X \wedge \mathbf{MU}) \leftarrow K_*(A \wedge \mathbf{MU}) \leftarrow K_*(Y \wedge \mathbf{MU}) \leftarrow 0$$

is also exact. Routine verification shows that the functor $P(—)$ is left exact and thus we obtain the diagram

$$\begin{tikzcd}
0 & \tilde{\Omega}^U_*(\Sigma^s X) \arrow[l] \arrow[d, "SH"] & \tilde{\Omega}^U_*(A) \arrow[l] \arrow[d, "SH_A"] & \tilde{\Omega}^U_*(Y) \arrow[l] \arrow[d, "SH_Y"] & 0 \arrow[l] \\
 & PK_*(\Sigma^s X \wedge \mathbf{MU}) & PK_*(A \wedge \mathbf{MU}) \arrow[l] & PK_*(Y \wedge \mathbf{MU}) \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

The formulation of [2; 14.1] for the Stong-Hattori theorem says that SH_A and SH_Y are isomorphisms since $H_*(A; Z)$ and $H_*(Y; Z)$ are free Z-modules. Routine diagram chasing therefore shows that

$$SH : \Omega^U_*(\Sigma^s X) \to PK_*(\Sigma^s X \wedge \mathbf{MU})$$

is monic, and the result follows from stability. □

We close by discussing an example to show that the preceding results are best possible in a certain sense.

Let p denote an odd prime and $V(1)$ the space constructed in [7]. Then

$$\Omega^U_*(V(1)) \cong \Omega^U_*/(p, [\mathbf{C}P(p-1)])$$

and by [5; 11.2],

$$k_*(V(1)) \cong Z[t]/(p, t^{p-1})$$

and by the theorem of Conner and Floyd [5; 9.1],

$$\tilde{K}_*(V(1)) = 0.$$

From the isomorphism

$$\tilde{K}_*(V(1) \wedge \mathbf{MU}) \cong \tilde{K}_*(V(1)) \otimes_{K_*(pt)} \tilde{K}_*(\mathbf{MU})$$

we therefore conclude

$$\tilde{K}_*(V(1) \wedge \mathbf{MU}) = 0.$$

Hence the map

$$SH : \tilde{\Omega}^U_*(V(1)) \to \tilde{K}_*(V(1) \wedge \mathbf{MU})$$

is certainly not monic. It is however still conceivable that the map

$$sh : \Omega^U_*(V(1)) \to k_*(V(1) \wedge \mathbf{MU})$$

is monic. To see that this is not the case we shall require the following useful formula (implicit in the work of Stong [8] and explicit in the proof of [3]):

$$sh[V^{2p^r-2}] = pb_{p^r-1} + t^{p-1}b^p_{p^{r-1}-1} + \text{other terms divisible by } p$$

where we have suitably chosen $b_i \in k_{2i}(\mathbf{MU})$ so that

$$k_*(MU) \cong Z[t][b_1, b_2, \cdots]$$

and $[V^{2p^r-2}] \in \Omega^U_{2p^r-2}$ is a suitably chosen Milnor manifold for the prime p.

From the commutative diagram

$$\begin{tikzcd} \Omega_*^U(S^\circ) \arrow[r, "sh"'] \arrow[d] & k_*(S^\circ \wedge \mathbf{MU}) \arrow[d] \\ \Omega_*^U(V(1)) \arrow[r, "sh"] & k_*(V(1) \wedge \mathbf{MU}) \end{tikzcd}$$

we find that

$$sh([V^{2p^r-2}]\gamma) = pa + pb_{p^r-1} + t^{p-1}b^p_{p^{r-1}-1} \in k_*(V(1) \wedge \mathbf{MU})$$

where $\gamma \in \Omega_0(V(1))$ is the canonical class. However

$$k_*(V(1) \wedge \mathbf{MU}) \cong \frac{Z[t]}{(p, t^{p-1})}[b_1, b_2, \cdots]$$

as $Z[t]$-modules and hence

$$pa + p^b_{p^r-1} + t^{p-1}b^p_{p^{r-1}-1} = 0 \in k_*(V(1) \wedge \mathbf{MU})$$

and hence

$$[V^{2p^r-2}]\gamma \neq 0 \in \ker\{sh : \Omega_*^U(V(1)) \to k_*(V(1) \wedge \mathbf{MU})\}$$

for all $r > 1$. As hom $\dim_{\Omega_*^U}\Omega_*^U(V(1)) = 2$ this example shows that in a certain sense Theorems 1, 2 are best possible. However, note that

$$sh : \Omega_*^U(X) \to k_*(X \wedge \mathbf{MU}), \qquad SH : \Omega_*^U(X) \to K_*(X \wedge \mathbf{MU})$$

are always monic mod torsion and from [5; §5] that there exist complexes with hom $\dim_{\Omega_*^U}\Omega_*^U(X)$ as large as we please and $\Omega_*^U(X)$ torsion free.

References

1. J. F. Adams, *Lectures on generalized homology theories*, Lecture Notes in Mathematics, vol. 99, Springer, New York, 1969.
2. ———, *Quillens work on formal group laws and complex cobordism*, Univ. of Chicago Lecture Notes, 1970.
3. J. F. Adams and A. Liulevicius, *On the Hattori-Stong theorem*, Preprint, 1970.
4. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, New Jersey, 1952.
5. P. E. Conner and L. Smith, *On the complex bordism of finite complexes*, Inst. Hautes Études, Sci. Publ. Math., vol. 37 (1970), pp. 117–221.
6. A. Hattori, *Integral characteristic numbers for weakly almost complex manifolds*, Topology, vol. 5 (1966), pp. 259–280.
7. L. Smith, *On realizing complex bordism modules*, Amer. J. Math., vol. 92 (1970), pp. 793–856.
8. R. E. Stong, *Relations among characteristic numbers I*, Topology, vol. 4 (1965), pp. 267–281.
9. R. Vogt, *Lectures on Boardman's category of spectra*, Aarhus University Lecture Notes, Aarhus, Denmark, 1969.

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