ON ABSOLUTELY CONTINUOUS TRANSFORMATIONS FOR MEASURE SPACES[1]

BY

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1. Introduction

If T is a transformation from a first measure space (S, Σ, μ) onto a second measure space (S', Σ', μ'), and one wishes to transform a problem involving integration in the second space to an equivalent problem in the first space, he usually introduces a weight function W' which assigns to each point s' in S' a value reflecting its importance relative to the image under T of a subset D of S which belongs to a class $\mathfrak{D}$ having suitable properties. For the case when W' is non-negative, quite general solutions to this question are known [12], [14], [1], [3]. Brooks [2] introduced Banach-valued weight functions; as a special case, signed weight functions may now be used when the spaces are oriented. In this case it is natural to try to express such functions as the difference of two non-negative weight functions [12], [11]. One obvious approach is to consider the total variations V', together with the positive and negative variations V'_+ and V'_-, of W' relative to $\mathfrak{D}$ and to seek conditions which insure that these are non-negative weight functions for which the relations $V' = V'_+ + V'_-$, $W' = V'_+ - V'_-$ hold almost everywhere. This approach is explored in [5], where it is shown that if W' satisfies suitable relations uniformly with respect to $\mathfrak{D}$, then such Jordan-type decompositions do exist for W'. Each weight function W' determines a weight W by the relation

$$W(D) = \int_{S'} W'(\cdot, D)\, d\mu'.$$

Chaney [9] observed that when T is absolutely continuous with respect to these weights, it is possible to determine the weight function W' from the weights W. This suggests that in the absolutely continuous case, for a signed weight function W' one might consider the variations of the weights W, use the result of Chaney to associate weight functions with these and thus obtain a Jordan-type decomposition of the weight function W'. This approach is studied below. Because of the differences in the hypotheses and in the approach, the results obtained here are neither included in nor include the results obtained earlier by the authors in [5]. Our results lead to a transformation formula for signed weight functions under more general conditions than any heretofore known.

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[1] The results in this paper have been announced in the Bulletin of the American Mathematical Society under the title *On absolutely continuous transformations* [6].

2. The setting

In this section we list the standard hypotheses H1–H8 [14] under which the theory is to be developed. For brevity, E, E', D, B', O' will be generic notations for sets belonging to Σ, Σ', $\mathfrak{D}$, $\mathfrak{B}'$, $\mathfrak{O}'$ respectively.

H1, H2. (S, Σ, μ) and (S', Σ', μ') are complete σ-finite non-negative measure spaces.

H3. T is a function (transformation) from S onto S'.

H4. $\mathfrak{D}$ is a subfamily of Σ containing $\emptyset$ and S.

$T\mathfrak{D} \subseteq \Sigma'$ and the intersection of two sets from $\mathfrak{D}$ can be expressed as a countable union of disjoint sets from $\mathfrak{D}$. $S = \bigcup_{j=1}^{\infty} {}_*D_j$, where $\mu({}_*D_j) < \infty$. If $E \in \Sigma$ and $E \subseteq {}_*D_j$, for some j, then for every $\varepsilon > 0$ there exists a disjoint sequence $\{D_i\}$ such that

$$E \subseteq \bigcup D_i \quad \text{and} \quad \Sigma\mu(D_i) < \mu(E) + \varepsilon.$$

Definitions. $\Delta(D)$ $(\Delta^*(D))$ denotes the family of all finite (countable) collections of disjoint sets in $\mathfrak{D}$ contained in D. A set $E \subseteq S$ is *$\mu\mu'$-null* if $E = A_1 \cup A_2$, where $\mu(A_1) = \mu'(TA_2) = 0$. An element D is of *type* γ if it belongs to a countable partition of S, where the partition (empty set permitted) consists of sets from $\mathfrak{D}$ and a $\mu\mu'$-null set.

H5. Every member of D can be expressed as the union of a monotone increasing sequence of sets of type γ.

H6. $\mathfrak{B}'$ is a sub-σ-algebra of Σ'. $T^{-1}\mathfrak{B}' \subseteq \Sigma$. For each E' there exist sets B_1', B_2' such that $B_1' \subseteq E' \subseteq B_2'$ and $\mu'(B_2' - B_1') = 0$.

Definitions. $\mathfrak{O}'$ is the family of subsets O' of S' such that $T^{-1}O'$ is a countable union of disjoint sets from $\mathfrak{D}$. O' is of *type* γ' if it is a member of a countable partition (empty set permitted) of S', where the partition consists of sets from $\mathfrak{O}'$ and sets A', C' such that $\mu'(A') = \mu(T^{-1}C') = 0$. E' is of *type* ι' if it is the union of a monotone increasing sequence of sets of type γ'.

H7. $S' = \bigcup_{j=1}^{\infty} {}_*O_j'$, where $\mu'({}_*O_j') < \infty$ for every j. If $E' \subseteq {}_*O_j'$, for some j and $\mu'(E') = 0$, then for every $\varepsilon > 0$ there is a set O' such that $E' \subseteq O'$ and $\mu'(O') < \varepsilon$.

H8. For every B' there is a monotone decreasing sequence of sets E_i', each of type ι', and μ'-null sets M', N' such that $(\bigcap E_i') \cup M' = B' \cup N'$.

The above hypotheses provide a setting in which absolute continuity can be defined and transformation formulas can be derived. When S and S' are topological spaces and T is a continuous map, there is a theory [15] which includes a large class of topological spaces satisfying H1–H8. For other papers relating to transformation theory, the reader is referred to the bibliography.

Definitions. A *signed weight function* (for T) is a real-valued function W' defined on $S' \times \mathfrak{D}$ satisfying the following conditions:

(1) If $D \in \mathfrak{D}$, then $W'(\cdot, D) = 0$ a.e. μ' on $S' - TD$.

(2) If $D \in \mathfrak{D}$ and $D_j \uparrow D$, then $\lim_j W'(\cdot, D_j) = W'(\cdot, D)$ a.e. μ'.

(3) If $D \in \mathfrak{D}$ and $\{D_j\} \in \Delta^*(D)$ is such that $D - \bigcup D_j$ is $\mu\mu'$-null, then $W'(\cdot, D) = \sum W'(\cdot, D_j)$ a.e. μ'.

(4) For each $D \in \mathfrak{D}$, $W'(\cdot, D)$ is measurable.

A *non-negative weight function* (for T) is a non-negative function W' satisfying the above conditions, except (3) is replaced by the requirement that $\sum W'(\cdot, D_j) \leq W'(\cdot, D)$ a.e. μ, whenever $\{D_j\} \in \Delta^*(D)$. The term weight function will refer to either of the above types of functions. If a signed weight function is non-negative, then it is also a non-negative weight function [5, 1.7].

If W_1', W_2' are weight functions such that $W_1'(\cdot, D) \geq W_2'(\cdot, D)$ a.e. μ' for every D, write $W_1' \succeq W_2'$. If $W_1' \succeq W_2'$ and $W_2' \succeq W_1'$, write $W_1' \sim W_2'$.

Remark. One can show that if W' is a signed weight function and V' is a non-negative weight function such that $V' \geq W'$, then $V' - W'$ is a non-negative weight function for T.

3. Absolute continuity

DEFINITIONS. Let W' be a signed weight function. We extend the notion of bounded variation in [2] and say T is of *bounded variation with respect to* W' (BVW') if there exists a non-negative μ'-integrable function K' such that $\sum | W'(\cdot, D_i) | \leq K'$ a.e. μ' whenever $\{D_i\} \in \Delta^*(S)$. T is *absolutely continuous with respect to* W' (ACW') if T is BVW' and there exists a μ-integrable function f such that

$$(*) \qquad \int_D f\, d\mu = \int_{TD} W'(\cdot, D)\, d\mu' \equiv W(D) \quad \text{for every } D.$$

f is called a *generalized Jacobian for* T *relative to* W' (the term gauge was previously used). For brevity we write $f = J(W')$. It follows that $J(W')$ is unique in the a.e. μ sense. $W(D)$ is called the weight attached to D. If W' is a non-negative weight function and f is a non-negative function for which (*) holds, then f is called a glbf W (see [14]), and we say that T is ACW.

By an argument similar to the one in [2], it can be shown that $f = J(W')$ satisfies the following condition: if $\mu'(E') = 0$, then $f = 0$ a.e. μ on $T^{-1}E'$. In the sequel, any function which satisfies this condition is said to satisfy *condition IN*.

Suppose that W' is a non-negative weight function. One can show that if T is ACW and f is a glbf W, then W' is a signed weight function, T is ACW' and $f = J(W')$.

4. Preliminaries

LEMMA 1. *Let W_1', W_2' be signed weight functions for which T is both ACW_1', ACW_2'. Then there exists a function f such that $f = J(W_1') = J(W_2')$ if and only if $W_1' \sim W_2'$.*

Proof. Sufficiency is obvious. To show necessity, since the difference $W' = W_1' - W_2'$ is a signed weight function for which T is ACW', it suffices to prove that if for each D, $\int_{S'} W'(\cdot, D)\, d\mu' = 0$, then for each D and every E', $\int_{E'} W'(\cdot, D)\, d\mu' = 0$. Use the methods employed in the proofs in [14; 4.6, 4.7, 4.8] to verify these equalities in three stages—first for sets O' of type γ', next for sets E' of type ι', lastly for arbitrary sets E'.

LEMMA 2. *Let f be an extended real-valued μ-integrable function defined on S. For each D define a function $I(\cdot, D)$ on $\mathfrak{B}'$ by*

$$I(B', D) = \int_{D \cap T^{-1}B'} f\, d\mu.$$

Then $I(\cdot, D)$ is countably additive. It is absolutely continuous with respect to the measure $\mu'/\mathfrak{B}'$ for each D if and only if f satisfies condition IN. Assume that f satisfies IN. Then there exists a μ'-integrable function $W'(\cdot, D)$ defined on S' such that

$$I(B', D) = \int_{B'} W'(\cdot, D)\, d\mu', \quad B' \in \mathfrak{B}', \quad D \in \mathfrak{D}.$$

The function W' is a signed weight function for T, T is ACW', $f = J(W')$. Finally, W' is unique in the sense that if W_*' is any signed weight function for which T is ACW_*' and $f = J(W_*')$, then $W_*' \sim W'$.

Proof. Use the properties of the integral, the Radon-Nikodym theorem, and the above lemma to construct a proof. Alternately, observe that if f satisfies IN then so do $|f|$ and $|f| + f$, and use the corresponding result for non-negative functions established in [9, 4, 5] to devise a proof in the signed case.

The preceding result has the following

COROLLARY 3. *Let W' be a signed weight function for which T is ACW' and let $f = J(W')$. Then*

$$\int_{D \cap T^{-1}B'} f\, d\mu = \int_{B'} W'(\cdot, D)\, d\mu', \quad B' \in \mathfrak{B}', D \in \mathfrak{D}.$$

Given D, any collection $\{D_i\}$ in $\Delta^*(D)$ for which $D - \bigcup D_i$ is $\mu\mu'$-null is termed an *almost T partition of D*. Use an inductive argument to prove the following:

LEMMA 4. *If $\{D(h, i_h) : 1 \leq i_h\}$, $1 \leq h \leq p$, are p almost T partitions of D, then there is an almost T partition $\{D(i_1, \cdots, i_p, k)\}$ of D such that for each (h, i_h),*

$$\{D(i_1, \cdots i_h, \cdots, i_p, k) : 1 \leq i_1, \cdots, i_{h-1}, i_{h+1}, \cdots, i_p, k\}$$

is an almost T partition of $D(h, i_h)$ and for each set $\{i_h : 1 \leq h \leq p\}$,

$$\bigcap\{D(h, i_h) : 1 \leq h \leq p\} = \bigcup\{D(i_1, \cdots, i_h, k) : 1 \leq k\}.$$

Let $\Delta_\gamma(D)$ be the set of those collections $\{D_i\}$ in $\Delta(D)$ each of whose elements D_i is of type γ. Use induction to show the following:

LEMMA 5. *Every* $\{D_i : 1 \leq i \leq m\}$ *in* $\Delta_\gamma(D)$ *is a subset of an almost T partition* $\{D_k\}$ *of D.*

DEFINITIONS. Let L be a real valued function on $\mathfrak{D}$ for which $L(\emptyset) = 0$. Designate by $V(\cdot, L)$, $V_+(\cdot, L)$, $V_-(\cdot, L)$ respectively the total, positive and negative variations of L on $\mathfrak{D}$, having the extended real values

$$V(D, L) = \sup \sum | L(D_i) |, \qquad V_\pm(D, L) = \sup \sum \pm L(D_i),$$

where the sup is taken over sets $\{D_i\}$ in $\Delta(D)$. From the definitions it follows that $V_\pm \leq V \leq V_+ + V_-$. A real-valued function L defined on $\mathfrak{D}$ is termed *inner continuous* if $D_j \uparrow D$ implies $\lim_j L(D_j) = L(D)$. When L is inner continuous, the classes $\Delta_\gamma(D)$ determine $V(D, L)$, $V_\pm(D, L)$. A real-valued function L on $\mathfrak{D}$ is termed *almost T additive* if for each D and every almost T partition $\{D_i\}$ of D, it is true that $\sum L(D_i) = L(D)$. When L is almost T additive, $L(\emptyset) = 0$.

LEMMA 6. *If L is a real-valued inner continuous almost T additive function on $\mathfrak{D}$, then $V(D, L) = V_+(D, L) + V_-(D, L)$. Moreover, if either $V_+(D, L)$ or $V_-(D, L)$ is finite, then both are finite and $L(D) = V_+(D, L) - V_-(D, L)$.*

Proof. If either of $V_\pm(D, L)$ is 0, the first equality follows. Otherwise, let $r_\pm$ be positive numbers such that $r_\pm < V_\pm(D, L)$, and choose $\{D_{\pm i}\}$ in $\Delta_\gamma(D)$ such that $r_\pm < \sum \pm L(D_{\pm i})$. Extend $\{D_{\pm i}\}$ to almost T partitions $\{D_{\pm k}\}$ of D, then choose an almost T partition $\{D_{klm}\}$ of D such that for each k, $\{D_{klm} : 1 \leq l, m\}$ is an almost T partition of D_{+k} and for each l, $\{D_{klm} : 1 \leq k, m\}$ is an almost T partition of D_{-l}. Use the almost T additivity of L to obtain

$$\sum L(D_{+i}) = \sum\nolimits_{ilm} L(D_{ilm}) \leq \sum \{L(D_{klm}) : L(D_{klm}) > 0\},$$

$$-\sum L(D_{-j}) = -\sum\nolimits_{kjm} L(D_{kjm}) \leq -\sum \{L(D_{klm}) : L(D_{klm}) < 0\},$$

$$r_+ + r_- < \sum | L(D_{klm}) | \leq V(D, L).$$

Thus $V_+(D, L) + V_-(D, L) \leq V(D, L)$, and the first equality follows.

Next suppose that $V(D, L)$ is finite. Consider any $\{D_i : 1 \leq i \leq m\}$ in $\Delta_\gamma(D)$ and extend it to an almost T partition $\{D_k\}$ of D. Then

$$\sum \{L(D_i) : 1 \leq i \leq m\} = L(D) - \sum\nolimits_{m<k} L(D_k) \leq L(D) + V_-(D, L).$$

Hence $V_+(D, L)$ is finite, and $V_+(D, L) \leq L(D) + V_-(D, L)$. Similarly if $V_+(D, L)$ is finite, then so is $V_-(D, L)$, and $V_-(D, L) \leq -L(D) + V_+(D, L)$. This concludes the proof.

THEOREM 7. *Let f be a μ-integrable function defined on S, and consider the function L defined on $\mathfrak{D}$ by $L(D) = \int_D f\,d\mu$. Then L is inner continuous, and $V(D, L) \le \int_D |f|\,d\mu$. If f satisfies condition IN, then L is almost T additive and*

$$V(D, L) = \int_D |f|\,d\mu, \quad V_\pm(D, L) = \int_D f^\pm\,d\mu.$$

Finally the variations $V(\cdot, L)$, $V_\pm(\cdot, L)$ are also inner continuous and almost T additive.

Proof. Use the above lemma, properties of the integrals and the variations to see that all the conclusions in the theorem follow readily if one shows that when f satisfies condition IN, then

$$V(D, L) \ge \int_D |f|\,d\mu. \tag{0}$$

Consider first the case when there is a j such that

$$D \subseteq {}_*D_j, \tag{1}$$

where ${}_*D_j$ is one of the sets described in H4. Given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\int_M |f|\,d\mu < \varepsilon/4 \quad \text{if} \quad \mu(M) < \delta. \tag{2}$$

Consider the measurable partition M_+, M_- of D defined by

$$M_+ = \{s \in D : f(s) \ge 0\}, \qquad M_- = \{s \in D : f(s) < 0\}. \tag{3}$$

Use H4, H5 to conclude that there are $\{D_{\pm i} : 1 \le i \le I_\pm\}$ in $\Delta_\gamma(D)$ such that

$$\mu(M_\pm \,\Delta \bigcup D_{\pm i}) < \delta/2. \tag{4}$$

Use (2), (3), (4) to conclude that

$$\sum L(D_{+i}) + \sum |L(D_{-j})| > \int_D |f|\,d\mu - \varepsilon/2. \tag{5}$$

Extend $\{D_{\pm i}\}$ to almost T partitions $\{D_{\pm k}\}$ of D; then choose an almost T partition $\{D_{klm}\}$ of D such that for each k $\{D_{klm} : 1 \le l, m\}$ is an almost T partition of D_{+k}, for each l $\{D_{klm} : 1 \le k, m\}$ is an almost T partition of D_{-l}, and for each k, l,

$$D_{+k} \cap D_{-l} = \bigcup\{D_{klm} : 1 \le m\}. \tag{6}$$

From (5) one now obtains

$$\begin{aligned} &\sum \{\,|L(D_{klm})| : 1 \le k \le I_+,\ 1 \le l, m\} \\ &\quad + \sum \{|L(D_{klm})| : 1 \le l \le I_-,\ 1 \le k, m\} > \int_D |f|\,d\mu - \varepsilon/2. \end{aligned} \tag{7}$$

On the other hand,

$$(8)\qquad \begin{aligned} &V(D, L) \geq \\ &\qquad \sum \{ \mid L(D_{klm}) \mid : \text{either } 1 \leq k \leq I_+ \quad \text{or} \quad 1 \leq l \leq I_-, 1 \leq m \}. \end{aligned}$$

Now the left side of (7) minus the right side of (8) equals

$$(9)\qquad \sum \{ \mid L(D_{klm}) \mid : 1 \leq k \leq I_+, 1 \leq l \leq I_-, 1 \leq m \},$$

and in view of (6), the sum in (9) does not exceed

$$(10)\qquad \int_M |f|\, d\mu, \quad \text{where } M = \left(\bigcup_{i=1}^{I_+} D_{+i} \right) \cap \left(\bigcup_{j=1}^{I_-} D_{-j} \right).$$

Relations (3), (4), (10) imply that $\mu(M) < \delta$, whence it follows from (7), (8), (9), (10) that

$$V(D, L) > \int_D |f|\, d\mu - \varepsilon.$$

Thus (0) is established when (1) holds.

Now consider the case when (1) does not hold. Use H4 to write $D = \bigcup D_h$ so that for each h there is a $j(h)$ such that $D_h \subseteq {}_*D_{j(h)}$. Express D as the union of disjoint sets M_h so that

$$(11)\qquad D = \bigcup M_h, \qquad M_h \subseteq D_h \subseteq {}_*D_{j(h)}.$$

Given $\varepsilon > 0$, there is an integer p such that

$$(12)\qquad \sum_{h=1}^{p} \int_{M_h} |f|\, d\mu > \int_D |f|\, d\mu - \varepsilon/2.$$

Use (11), (12), H4, H5 to conclude that for each h there are

$$\{D(h, i_h) : 1 \leq i_h \leq I_h\} \in \Delta_\gamma(D_h)$$

such that

$$(13)\qquad \mu(M_h \Delta \bigcup D(h, i_h)) < \delta/2,$$

where

$$(14)\qquad \int_M |f|\, d\mu < \varepsilon/p^2 \quad \text{if} \quad \mu(M) < \delta$$

and

$$(15)\qquad \sum \left\{ \int_{D(h, i_h)} |f|\, d\mu : 1 \leq i_h \leq I_h, 1 \leq h \leq p \right\} > \int_D |f|\, d\mu - \varepsilon/2.$$

For each h extend

$$\{D(h, i_h) : 1 \leq i_h \leq I_h\}$$

to an almost T partition $\{D(h, i_h) : 1 \leq i_h\}$ of D, and choose an almost T

partition $\{D(i_1, \cdots, i_p, k)\}$ of D with the properties described in Lemma 4. For each set (h, i_h) designate by $I(h, i_h)$ the set of indices

$$\{1 \le i_1, \cdots, 1 \le i_{h-1}, 1 \le i_{h+1}, \cdots, 1 \le i_p, 1 \le k\}.$$

Then

$$\sum\left\{\int_{D(i_1,\cdots,i_p,k)} |f|\, d\mu : I(h, i_h)\right\} = \int_{D(h,i_h)} |f|\, d\mu;$$

hence by (15)

$$\begin{aligned}(16) \quad \sum\left\{\sum\left\{\int_{D(i_1,\cdots,i_p,k)} |f|\, d\mu : I(h, i_h)\right\} : 1 \le i_h \le I_h, 1 \le h \le p\right\} \\ > \int_D |f|\, d\mu - \varepsilon/2.\end{aligned}$$

Let I designate the subset of $\{i_1, \cdots, i_p, k\}$ for each of which there is at least one value of h such that $1 \le i_h \le I_h$. Then

$$V(D, L) \ge \sum\{V(D(i_1, \cdots, i_p, k) : I\},$$

hence by the special case

$$(17) \qquad V(D, L) \ge \sum\left\{\int_{D(i_1,\cdots,i_p,k)} |f|\, d\mu : I\right\}.$$

Clearly every term occurring in the right side of (17) occurs in the left side of (16), and the left side of (16) minus the right side of (17) can be expressed as the sum of not more than $p(p-1)/2$ terms, each having the form

$$\sum\left\{\int_{M(h_1,h_2)} |f|\, d\mu : 1 \le h_1 < h_2 \le p\right\},$$

where

$$M(h_1, h_2) \subseteq [\cup\{D(h_1, i_{h_1}) : 1 \le i_{h_1} \le I_{h_1}\}] \cap [\cup\{D(h_2, i_{h_2}) : 1 \le i_{h_2} \le I_{h_2}\}]$$

so that $\mu(M(h_1, h_2)) < \delta$. In view of (14), the value of the above difference does not exceed $\varepsilon/2$. Thus one concludes that

$$V(D, L) > \int_D |f|\, d\mu - \varepsilon.$$

Hence inequality (0) always holds.

By applying Lemma 6 and Theorem 7 to the weights, we derive:

COROLLARY 8. *Let W' be a signed weight function for which T is BVW'. Then the weights $W(D) = \int_{S'} W'(\cdot, D)\, d\mu'$ satisfy*

$$V(D, W) = V_+(D, W) + V_-(D, W), W(D) = V_+(D, W) - V_-(D, W).$$

Moreover, if T is ACW' and $f = J(W')$, then

$$V(D, W) = \int_D |f|\, d\mu, \quad V_\pm(D, W) = \int_D f^\pm\, d\mu, \quad D \in \mathfrak{D}.$$

THEOREM 9 (Jordan decomposition of signed weight functions). *Assume* H1–H8. *Let W' be a signed weight function for T. Suppose that T is ACW'*

and $f = J(W')$. *Then there exist non-negative weight functions* V', $V'_{\pm}$ *which satisfy the following conditions*:

(1) $V' = V'_+ + V'_-$, $W' = V'_+ - V'_-$.

(2) T *is* ACV *and* $|f|$ *is a* glbf V, T *is* $ACV_{\pm}$ *and* $f^{\pm}$ *is a* glbf $V_{\pm}$.

(3) $V(D) = \int_D |f|\, d\mu = V(D, W)$, $V_{\pm}(D) = \int_D f^{\pm}\, d\mu = V_{\pm}(D, W)$.

(4) *If* U', $U'_{\pm}$ *are non-negative functions on* $S' \times \mathfrak{D}$ *such that* $U' = U'_+ + U'_-$, $W' = U'_+ - U'_-$ *and* U' *is a non-negative weight function for which* T *is* ACU, *then* $U'_{\pm}$ *are non-negative weight functions for which* T *is* $ACU_{\pm}$ *and* $U'_{\pm} \succ V'_{\pm}$.

Proof. Since f is a μ-integrable function on S satisfying condition IN, $|f|$ satisfies condition IN. By the results in [9; 4, 5] (cf. the proof of Lemma 2), there exists a non-negative weight function V' such that for each D and every B'

$$\int_{B'} V'(\cdot, D)\, d\mu' = \int_{D \cap T^{-1}B'} |f|\, d\mu.$$

Since by Corollary 3,

$$\int_{B'} W'(\cdot, D)\, d\mu' = \int_{D \cap T^{-1}B'} f\, d\mu,$$

it follows that one may choose $V' \geq |W'|$. Then let the equality in (1) define $V'_{\pm}$ and use Corollary 8 to show that (1), (2), (3) are satisfied.

Now let U', $U'_{\pm}$ be as described in (4), and let g be a glbf U. Thus U' is a signed weight function for which T is ACU' and $g = J(U')$. Hence $U'_{\pm}$ are signed weight functions for which T is $ACU'_{\pm}$, and $(g \pm f)/2 = J(U'_{\pm})$. Since the $U'_{\pm}$ are non-negative, it follows that $U'_{\pm}$ are non-negative weight functions for which T is $ACU_{\pm}$ and $g \pm f \geq 0$ a.e. μ, or $g \geq |f|$ a.e. μ. Thus $U'_{\pm} \succ V'_{\pm}$, and the proof is complete.

THEOREM 10 (transformation formula). *Assume* H1–H8. *Let* W' *be a signed weight function for* T. *Suppose that* T *is* ACW' *and* $f = J(W')$. *Assume that* H' *is a real-valued measurable function defined on* S'. *Then for each* D *such that* $H' \circ Tf$ *is* μ*-integrable on* D *it is true that* $H'W'(\cdot, D)$ *is* μ'*-integrable and the following transformation formula holds*:

$$(\#) \qquad \int_D H' \circ Tf\, d\mu = \int_{S'} H'W'(\cdot, D)\, d\mu'.$$

Proof. Since f satisfies condition IN, the measurability of the functions involved follows. Let $V'_{\pm}$ be the non-negative weight functions described in the above theorem; then T is $ACV_{\pm}$ and $f^{\pm}$ are glbf $V_{\pm}$. Since $H' \circ Tf$ is μ-integrable on D so are $H' \circ Tf^{\pm}$. By the results for non-negative weight functions [14; 4], we conclude that $H'V'_{\pm}(\cdot, D)$ are μ'-integrable and

$$\int_D H' \circ Tf^{\pm}\, d\mu = \int_{S'} H'V'_{\pm}(\cdot, D)\, d\mu'.$$

Now the conclusions follow.

Remark 1. Contrary to the situation for non-negative weight functions, the μ'-integrability of $H'W'(\cdot, D)$ does not imply the μ-integrability on D of $H' \circ Tf$, as the example in Remark 1 in [4] shows.

Remark 2. For real-valued weight functions, this theorem generalizes Theorem 3.2 of Brooks [2].

Remark 3. A proof of the above theorem can be given which avoids Theorem 9. We very briefly indicate the proof. Assuming the hypotheses in Theorem 10 and using the technique in Section 4 of [14] (the dominated convergence theorem is used instead of the monotone convergence theorem), one can obtain (#) for bounded and measurable H'. Since $|f|$ satisfies condition IN, the method in [9] can be used to obtain a non-negative weight function V' such that T is ACV and $|f|$ is a glbf V. Observe that one may choose $V'(\cdot, D) \geq |W'(\cdot, D)|$. The integrability of $|H' \circ Tf|$ implies the integrability of $H'V'(\cdot, D)$, which in turn implies the integrability of $H'W'$. By using the fact that (#) holds for simple functions and the dominated convergence theorem, one establishes the conclusion of the theorem.

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