LEVEL-PRESERVING APPROXIMATIONS AND ISOTOPIES, AND HOMOTOPY GROUPS OF SPACES OF EMBEDDINGS

BY

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1. Introduction

In 1965 Gluck [G] proved that a locally tame embedding f of a compact k-dimensional polyhedron P into a q-dimensional PL manifold Q, with $2k + 2 \leq q$, is ambient isotopic to a PL embedding by an arbitrarily short ambient isotopy fixed outside a small neighborhood of $f(P)$. His technique, a formalization of a technique of Homma [H1], emphasized two properties of certain subspaces of spaces of embeddings, denseness and solvability. Bryant and Seebeck [B-S1, B-S2] have replaced the notion of solvability by local solvability and obtained results similar to Gluck's when P is a k-manifold with $k \leq q - 3$ and f is "locally nice" (see [B-S2]). Edwards [E] has proved related results in the same dimension range when P is a polyhedron. See also Miller [M2].

Let M and Q be PL m- and q-manifolds, with M compact, and let $\mathrm{Hom}_{TOP}(M, Q)$ denote the metric space of proper embeddings of M into Q with metric

$$d(f, g) = \max_{x \in m} \{d(f(x), g(x))\}.$$

Let $\mathrm{Hom}_{LF}(M, Q)$ and $\mathrm{Hom}_{PL}(M, Q)$ denote the subspaces of locally flat and PL embeddings, respectively. Then the above result of Bryant and Seebeck says that $\pi_0(\mathrm{Hom}_{LF}(M, Q), \mathrm{Hom}_{PL}(M, Q) = 0$ in a special way, namely the path (the isotopy) in the embedding space from the locally flat embedding f to a PL embedding is short and lifts to a short path in the space of self homeomorphisms of Q which are fixed outside a small neighborhood of $f(M)$. The notions of denseness and local solvability can also be used to prove that the space of embeddings of a compact m-manifold into a q-manifold is locally arcwise connected if $m \leq q - 3$.

In this paper we consider the higher connectivity of such spaces of embeddings, considered as semi-simplicial complexes. We prove analogues of denseness and local solvability for simplices of embeddings (Theorems 2.1 and 2.2) and use them to prove (Theorem 2.3) that

$$\pi_s(\mathrm{Hom}_{LF}(M, Q), \mathrm{Hom}_{PL}(M, Q)) = 0$$

in the same special way as above (the homotopies lift to short ambient isotopies of s-simplices of $\mathrm{Hom}(Q, Q)$) when $m + s \leq q - 3$. We prove that

Received May 8, 1972.

[1] The main results of this paper form part of the author's Ph.D. thesis, written under the direction of Professor Jerome Dancis (University of Maryland, 1970). The author wishes to thank Professor Dancis for his invaluable help and encouragement during the course of this research. It was partially supported by a National Science Foundation grant.

in this dimension range the homotopy groups of certain spaces of embeddings are independent of whether we consider the spaces of embeddings as semi-simplicial complexes or topological spaces (Theorem 2.4). A corollary of this theorem is that if we let $PL(M, Q)$ denote the semi-simplicial complex of PL embeddings of M into Q, then Zeeman's conjecture in [Z] that $\pi_s PL(S^m, S^q) = 0$ if $m + s \leq q - 3$, proved by Husch [H4], is also true in the topological category. We also show that the topological space of embeddings of a PL m-manifold M into a PL q-manifold Q is locally s-connected if $m + s \leq q - 3$ (Theorem 2.5). Improvements of some of the above results have recently been announced. In particular, local connectivity of $\mathrm{Hom}_{TOP}(M, Q)$ in all dimensions, where M and Q are topological manifolds, has been announced by Cernavskii [C1], and local connectivity of $PL(X, Q)$ in all dimensions, where X is a polyhedron, has been announced by Edwards and Miller [E-M], who use it to prove the associated approximation theorem (our Theorem 2.1 with M a polyhedron and no restriction on s).

2. Definitions and statement of results

If X, Y, Z, W are spaces and

$$f : X \to Y \quad \text{and} \quad g : Z \to W$$

are maps, then $f \times g : X \times Z \to Y \times W$ is the map given by

$$f \times g(x, z) = (f(x), g(z)).$$

If $\prod_{i=1}^{n} X_i$ is a product space, then $p_i : \prod_{i=1}^{n} X_i \to X_i$ denotes projection onto the i-th factor. An embedding $F : X \times Y \to Z \times Y$ is *level-preserving* (*with respect to* Y) if $p_2(x, y) = p_2 F(x, y)$. If $F : X \times Y \to Z \times Y$ is a level-preserving map, then $F_y : X \to Z$ denotes the map given by

$$F(x, y) = (F_y(x), y).$$

Let $I = [0, 1]$ and $I^n = I \times \cdots \times I$. Let $0 \in \partial I^n$ denote the point $(0, 0, \cdots, 0)$. An *n-concordance* of M in Q is an embedding F of $M \times I^n$ into $Q \times I^n$ such that $F|M \times \partial I^n$ is level-preserving. An *n-isotopy* of M in Q is a level-preserving embedding of $M \times I^n$ into $Q \times I^n$. An *ambient n-isotopy* of Q is a level-preserving homeomorphism of $Q \times I^n$ onto itself. An embedding $F : M \times I^n \to Q \times I^n$ is called *proper* if

$$F^{-1}(\partial Q \times I^n) = \partial M \times I^n,$$

allowable if

$$F^{-1}(\partial Q \times I^n) = V \times I^n,$$

where V is a PL $(m - 1)$-submanifold of ∂M. An n-isotopy is *proper*, or *allowable*, if it is so when considered as a concordance. An n-isotopy

$$F : M \times I^n \to Q \times I^n$$

is *fixed* on $X \subset M$ if $F|X \times I^n = (F_0|X) \times \mathrm{id}$. A 1-isotopy $F : M \times I \to Q \times I$ is called an *isotopy*, and the embeddings F_0 and F_1 are said to be *isotopic*. If P is any space then an embedding

$$F : X \times P \times I \to Y \times P \times I$$

is a *level-preserving* isotopy of $X \times P$ in $Y \times P$ if it is level-preserving with respect to $P \times I$.

Let 1 denote the identity homeomorphism on a space as well as the element of I. If $f, g : X \to Y$ are two maps into a metric space which agree off some compact set, then $d(f, g) = \sup_{x \in X} d(f(x), g(x))$. If $A \subset X$ and $\varepsilon > 0$, then an *ε-push* K of (X, A) is an ambient isotopy of X such that $d(K, 1) < \varepsilon$ and K is fixed outside the ϵ-neighborhood of A, that is, $K \mid (X - N_\epsilon(A)) \times I = 1$.

The composition of ambient isotopies will be considered as one move followed by the other move, as in [G], rather than as a composition of homeomorphisms. The composition of isotopies is formed in the same way. Note that the composition of an ε-push and an η-push is an $(\varepsilon + \eta)$-push.

Finally, we denote by B^n the standard n-simplex, and by S^{n-1} its boundary. We assume a fixed PL homeomorphism between B^n and I^n.

The principal theorems are the following.

THEOREM 2.1 (Level-preserving approximation). *Let $H : M \times B^s \to Q \times B^s$ be an allowable level-preserving embedding and let $\varepsilon > 0$ be given. Then if $m + s \leq q - 2$ and $m \leq q - 3$ there is an allowable, level-preserving PL embedding $F : M \times B^s \to Q \times B^s$ such that $d(F, H) < \varepsilon$.*

THEOREM 2.2 (Local solvability). *Let $H : M \times B^s \to Q \times B^s$ be an allowable level-preserving embedding and let $\varepsilon > 0$ be given. Then if $m + s \leq q - 3$ there is a $\delta = \delta(\varepsilon, H) > 0$ such that if $G_0, G_1 : M \times B^s \to Q \times B^s$ are allowable level-preserving PL embeddings such that $d(G_i, H) < \delta$, $i = 0, 1$, then there is a level-preserving PL ε-push K of $(Q \times B^s, H(M \times B^s))$ such that $K_1 G_0 = G_1$.*

Theorems 2.1 and 2.2 and Gluck's techniques combine to give us the following theorem.

THEOREM 2.3 (Level-preserving taming). *Let $H : M \times B^s \to Q \times B^s$ be an s-simplex of locally flat embeddings (see Section 5 for definition) and let $\varepsilon > 0$ be given. Then if $m + s \leq q - 3$ there is a level-preserving ε-push K of $(Q \times B^s, H(M \times B^s))$ such that $K_1 H$ is PL.*

Relative versions of these theorems hold when the conclusions are satisfied to begin with on ∂B^s.

Embedding spaces are studied both as topological spaces and as semi-simplicial complexes, and homotopy groups are defined somewhat differently in each case. The next theorem gives us some information about when the homotopy groups of a certain space of embeddings are independent of the method used for defining them. Let $f : M \to Q$ be a proper PL embedding, which serves as base point for both $\mathrm{Hom}_{TOP}(M, Q)$ and $\mathrm{Hom}_{PL}(M, Q)$.

Let $TOP(M, Q; f)$ and $PL(M, Q; f)$ denote the corresponding semi-simplicial complexes. (See Section 6.)

THEOREM 2.4 (Independence of definition of homotopy groups). *There is a commutative diagram of homomorphisms*

$$\begin{array}{ccc} \pi_s\, PL(M, Q; f) & \xrightarrow{\phi_{2*}} & \pi_s\, (\mathrm{Hom}_{PL}\, (M, Q), f) \\ \big\downarrow{\scriptstyle j_{1*}} & & \big\downarrow{\scriptstyle j_{2*}} \\ \pi_s\, TOP(M, Q; f) & \xrightarrow{\phi_{1*}} & \pi_s\, (\mathrm{Hom}_{TOP}\, (M, Q), f) \end{array}$$

in which ϕ_{1*} is an isomorphism for all s, and ϕ_{2*}, j_{1*} and j_{2*} are epimorphisms if $m + s \leq q - 3$ and isomorphisms onto if $m + s \leq q - 4$.

The proof of Theorem 2.4 from Theorem 2.3 does not make use of the fact that taming isotopies can be made arbitrarily short. If we use this additional information, we get the following theorem.

THEOREM 2.5 (Local s-connectedness). Hom_{TOP} (M, Q) *and* Hom_{PL} (M, Q) *are locally s-connected if $m + s \leq q - 3$.*

3. Concordance and isotopy

In this section we prove a level-preserving version of Hudson's Covering n-Isotopy Theorem [H2] and state the theorem of Morlet which is used in Section 4.

THEOREM 3.1 (Level-Preserving Covering n-Isotopy Theorem). *Let*

$$F : M \times B^s \times I^n \to Q \times B^s \times I^n$$

be an allowable level-preserving PL n-isotopy of $M \times B^s$ in $Q \times B^s$. Suppose M is compact and $m \leq q - 3$. Then there is an ambient, level-preserving PL n-isotopy K of $Q \times B^s$ such that $K \circ (F_0 \times 1) = F$. If F is fixed on $F_0^{-1}(\partial Q \times B^s)$, then we may choose K so that it is fixed on $\partial Q \times B^s$.

Proof. Choose a point $0 \in \partial B^s$. Let $F_{00} : M \to Q$ be defined to be $(F_0)_0$. There is a level-preserving PL isotopy between

$$F_0 \mid F_0^{-1}(\partial Q \times B^s) \quad \text{and} \quad F_{00} \times 1 \mid F_0^{-1}(\partial Q \times B^s).$$

By induction on the dimension of M there is a PL, level-preserving ambient isotopy of $\partial Q \times B^s$, and hence of $\partial Q \times B^s \times I^n$, which covers it. Therefore we may assume that $F_0 \mid F_0^{-1}(\partial Q \times B^s) = F_{00} \times 1 \mid F_0^{-1}(\partial Q \times B^s)$. By the Covering $(s + n)$-Isotopy Theorem, there is a PL ambient $(s + n)$-isotopy

$$H : Q \times B^s \times I^n \to Q \times B^s \times I^n$$

such that $H \circ (F_{00} \times 1 \times 1) = F$ and if F is fixed on $F_0^{-1}(\partial Q \times B^s)$, we may assume that H is fixed on $\partial Q \times B$. By the Covering s-Isotopy Theorem, there is an ambient PL s-isotopy

$$G : Q \times B^s \to Q \times B^s,$$

with $G \mid \partial Q \times B^s = 1$, such that $G \circ (F_{00} \times 1) = F_0$. Then $G \times 1$ is an ambient PL $(s + n)$-isotopy of $Q \times B^s \times I^n$ such that

$$(G \times 1) \circ (F_{00} \times 1 \times 1) = F_0 \times 1.$$

Then $H \circ (G \times 1)^{-1}$, where composition here is of homeomorphisms rather than of isotopies, is the desired isotopy. This completes the proof.

The following theorem, due to Morlet, is a generalization to n-concordances of the "Concordance implies Isotopy" theorem of Hudson [H3]. It is the source of the dimension restrictions in the main theorems of this paper.

THEOREM 3.2 (Morlet [M4, M5]). *Let $F : M \times I^n \to Q \times I^n$ be a proper PL n-concordance of M in Q such that $F \mid \partial(M \times I^n)$ is level preserving. Suppose M is compact and $m + n \leq q - 2$. Then there is an ambient PL isotopy K of $Q \times I^n$, fixed on $\partial(Q \times I^n)$, such that $K_1 F$ is level preserving.*

4. Level-preserving approximation and local solvability

In this section we prove level-preserving analogues of the denseness and local solvability properties used by Homma [H1], Gluck [G], and Bryant and Seebeck [B-S2] for moving certain embeddings to PL embeddings.

Let C be a compact submanifold of M and let $\mathrm{Hom}_s\,(M, C; Q)$ denote the set of level-preserving allowable embeddings $F : M \times B^s \to Q \times B^s$ which agree with some fixed PL embedding on $(M - C) \times B^s$ and have the property that

$$F^{-1}(\partial Q \times B^s) = V \times B^s$$

for some fixed PL submanifold V of ∂M. $\mathrm{Hom}_s\,(M, C; Q)$ is a metric space under the metric

$$d(F, G) = \sup_{x \in m,\, t \in B^s} d(F(x, t), G(x, t)).$$

DEFINITION. A subset A of $\mathrm{Hom}_s\,(M, C; Q)$ is *dense* if for each $H \in \mathrm{Hom}_s\,(M, C; Q)$ and each $\varepsilon > 0$ there is an $F \in A$ such that $d(F, H) < \varepsilon$. The set A is *locally solvable* if for each $H \in \mathrm{Hom}_s\,(M, C; Q)$ and each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, H) > 0$ such that if G_0 and G_1 belong to A and $d(G_i, H) < \delta$ for $i = 0, 1$, then there is a level-preserving ε-push K of $(Q \times B^s, H(C \times B^s))$ such that $K_1 G_0 = G_1$.

The core of this paper is the proof that under certain conditions the PL embeddings form a dense and locally solvable subset of $\mathrm{Hom}_s\,(M, C; Q)$. We assume always that $m \leq q - 3$.

The proof is by induction, so we begin by stating the main theorems indexed by the dimensions of M and B^s. They are Theorems 2.1 and 2.2 in the rather complex relative form needed to complete the induction step.

THEOREM 4.1 (m, s) (Denseness). *Let $H \in \mathrm{Hom}_s\,(M, C; Q)$ and let N be an m-submanifold of M such that $N \cap \partial M$ is either empty or a PL $(m - 1)$-submanifold of both ∂N and ∂M. Suppose $m + s \leq q - 2$ and $\mathrm{Cl}(M - N)$ is* *compact. Then for each* $\varepsilon > 0$ *there is a* $\delta = \delta(\varepsilon, H) > 0$ *such that if*

$$G : (N \cup \partial M) \times B^s \cup M \times \partial B^s \to Q \times B^s$$

is an allowable level-preserving PL embedding with

$$d(G, H \mid (N \cup \partial M) \times B^s \cup M \times \partial B^s) < \delta,$$

then there is a PL element F *of* $\mathrm{Hom}_s\,(M, C; Q)$ *such that* F *extends* G *and* $d(F, H) < \varepsilon$.

THEOREM 4.2 (m, s) (Local solvability). *Let* $H \in \mathrm{Hom}_s\,(M, C; Q)$ *and suppose* $m + s \leq q - 3$. *Then for each* $\varepsilon > 0$ *there is a* $\delta = \delta(\varepsilon, H) > 0$ *such that if*

$$G_0, G_1 : M \times B^s \to Q \times B^s$$

are two PL elements of $\mathrm{Hom}_s\,(M, C; Q)$ *with* $d(G_i, H) < \delta$ *for* $i = 0, 1$, *and* G_0 *and* G_1 *agree on* $H^{-1}(\partial Q \times B^s)$, *then there is a level-preserving PL* ε*-push* K *of* $(Q \times B^s, H(C \times B^s))$, *fixed on* $\partial Q \times B^s$, *such that* $K_1 G_0 = G_1$. *If in addition* G_0 *and* G_1 *agree on* $M \times \partial B^s$, *then* K *can be chosen fixed on* $Q \times \partial B^s$.

The induction scheme is given by the following diagram:

$$\left.\begin{array}{l}\text{Theorem 4.1}\,(m, s)\\ \text{Theorem 4.2}\,(m - 1, s)\\ \text{Theorem 4.2}\,(m, s - 1)\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{Theorem 4.2}\,(m, s)\\ \text{Theorem 4.1}\,(m, s + 1)\end{array}\right.$$

We begin the induction by noting that Theorem 4.1 $(m, 0)$, in a slightly weaker relative form, has been proved for all m by Richard T. Miller [M3]. Theorem 4.2 $(m, 0)$ has been proved by Bryant and Seebeck [B-S2], and can be used together with Miller's theorem to establish the above form of Theorem 4.1 $(m, 0)$. To simplify the proofs of Theorems 4.1 (m, s) and 4.2 (m, s) we consider proper embeddings only and prove only the absolute versions; i.e., we assume $C = M$ in both theorems and $N = \emptyset$ in Theorem 4.1 (m, s). We also leave to the reader the inductive calculation of δ from ε in both theorems. Complete details can be found in [L].

Proof of Theorem 4.2 (m, s). By Theorems 4.2 $(m - 1, s)$ and 4.2 $(m, s - 1)$ we may assume that G_0 and G_1 agree on $V \times B^s$ and on $M \times \partial B^s$. The rest of the proof is in two steps; first we prove the theorem for the special case in which H is of the form $h \times 1$ for some allowable embedding h of M into Q. Then we apply this first step locally in B^s to obtain the general result. There is a handle decomposition of M such that the image under h of each handle has diameter less than $\varepsilon/3(m + 1)$ [R-S2, p. 82]. Let K_i denote the union of all the handles of handle dimension i, let $J_i = \bigcup_{j=0}^{i-1} K_j$, and let $L_i = \mathrm{Cl}(M - K_i')$, where K_i' is a neighborhood of K_i such that for each component D of K_i', $\mathrm{diam}\ h(D) < \varepsilon/3(m + 1)$. The idea of the proof is to construct an ambient isotopy which moves G_0 to G_1 in $m + 1$ stages, at each stage adjusting the embedding constructed at the last stage to agree with

G_1 on K_i while keeping L_i fixed. The fact that the components of K_i are small implies that each step can be carried out by a short ambient isotopy. The argument is similar to that in [B-S1].

By induction on the dimension of the handles of M, we may assume that G_0 agrees with G_1 on J_i. We show how to adjust G_0 to agree with G_1 on J_{i+1} by an ε-push. If G_0 and G_1 are sufficiently close to $h \times 1$, there is by Theorem 4.1 (m, s) a level-preserving approximation $G' : M \times B^s \to Q \times B^s$ to $h \times 1$ which agrees with G_1 on $J_{i+1} \times B^s$ and with G_0 on $L_i \times B^s$. Provided G' is close enough to $h \times 1$, there is by Theorem 4.1 $(m + s + 1, 0)$ an approximation

$$F : M \times B^s \times I \to Q \times B^s \times I$$

to $h \times 1 \times 1$ with the following properties:

(a) $F \mid M \times B^s \times 0 = G_0$; $F \mid M \times B^s \times 1 = G'$;

(b) $F \mid (L_i \cup J_i) \times B^s \times I = (G_0 \mid (L_i \cup J_i) \times B^s) \times 1$;

(c) $\operatorname{diam} p_1 F(D \times B^s \times I) < \varepsilon$ for each component D of K_i, and if D_1 and D_2 are distinct components of K_i, then

$$p_1 F(D_1 \times B^s \times I) \cap p_1 F(D_2 \times B^s \times I) = \emptyset.$$

There are regular neighborhoods M' of K_i and Q' of $p_1 F(K_i \times B^s \times I)$ such that each component of Q' has diameter less than ε and $F \mid M' \times B^s \times I$ is a proper $(s + 1)$-concordance of M' in Q' which is fixed on $\partial M' \times B^s$ and level-preserving on $\partial(M \times B^s \times I)$. Therefore by Theorem 3.2 we can assume F is an $(s + 1)$-isotopy of M' in Q'. By Theorem 3.1, F, which we now think of as a level-preserving isotopy of $M' \times B^s$ in $Q' \times B^s$, can be covered by a level-preserving ambient isotopy K of $Q \times B^s$, fixed outside $Q' \times B^s$. Because each component of Q' has diameter less than ε, K is the desired ε-push of $(Q \times B^s, h(M) \times B^s)$ such that $K_1 G_0 = G'$. The induction ends with $G' = G_1$.

This completes the proof of Theorem 4.2 (m, s) in the case $H = h \times 1$; we write $\delta(\varepsilon, h)$ for $\delta(\varepsilon, H)$ in this case. Now suppose ε and H are given with H not of the form $h \times 1$. We may assume that B^s is triangulated so finely that if σ_i is a simplex with barycenter t_i in the triangulation and $t \in \sigma_i$, then

$$d(H_t, H_{t_i}) < \min\{\varepsilon/4, (1/3)\delta(\varepsilon/4, H_{t_i})\}.$$

For the rest of this proof we write H_i for H_{t_i}. Choose

$$\eta < \min\{\varepsilon/2, (1/3) \min_i \delta(\varepsilon/4, H_i)\},$$

and let $(B^s)^{s-1}$ denote the $(s - 1)$-skeleton of B^s. Using Theorem 4.2 $(m, s - 1)$, a $\delta < \eta$ can be found so that if $d(G_0, G_1) < \delta$ then there is a level-preserving η-push K' of $(Q \times B^s, H(M \times B^s))$ such that

$$K'_1 G_0 \mid M \times (B^s)^{s-1} = G_1 \mid M \times (B^s)^{s-1}.$$

Let $\delta(\varepsilon, H)$ be this δ. One may now easily compute that for $j = 0, 1$, if

$d(G_j, H) < \delta$, then for each i,

$$d(K_1' G_0 \mid M \times \sigma_i, (H_i \times 1) \mid M \times \sigma_i) < \delta(\varepsilon/4, H_i).$$

Therefore for each i there is a level-preserving $(\varepsilon/4)$-push K^i of

$$(Q \times \sigma_i, (H_i \times 1)(M \times \sigma_i)),$$

fixed on $\partial(Q \times \sigma_i)$, such that $K_1^i G_0 \mid M \times \sigma_i = G_1 \mid M \times \sigma_i$. Since

$$d(H_i \times 1 \mid M \times \sigma_i, H \mid M \times \sigma_i) < \varepsilon/4,$$

each K^i is an $(\varepsilon/2$-push of $(Q \times \sigma_i, H(M \times \sigma_i))$. If we let K'' be the union of all the K^i's and let K be the composition (as pushes) of K'' and K', then K is the desired level-preserving ϵ-push of $(Q \times B^s, H(M \times B^s))$ such that $K_1 G_0 = G_1$.

Proof of Theorem 4.1 $(m, s + 1)$. Again we first consider the case in which $H = h \times 1$. We think of B^{s+1} as $B \cup (\partial B \times I)$, where B is a small $(s + 1)$-ball in the interior of B^{s+1}. Using Theorem 4.1$(m, 0)$ we may approximate $h : M \to Q$ by a PL embedding $f : M \to Q$, so that $f \times 1$ approximates $h \times 1$ on $M \times B$. If f is close enough to h, then by Theorem 4.2(m, s) the level-preserving PL embeddings $F \mid M \times \partial B$ and $H \mid M \times \partial B^{s+1}$ are isotopic by a PL isotopy which stays close to H. This isotopy extends F to an approximation of H. The case in which H is not of the form $h \times 1$ is done by subdividing B^{s+1} as in the proof of Theorem 4.2(m, s).

5. Taming locally flat simplices of embeddings

In this section we give the definitions relevant to the proof of Theorem 2.3.

DEFINITION. A level-preserving allowable embedding $F : M \times B^s \to Q \times B^s$ is called a *locally tame s-simplex of embeddings* if for each (x, t) in $M \times B^s$ there are

(1) an open neighborhood U of x in M and a closed neighborhood N of t in B^s which is homeomorphic to B^s,

(2) a neighborhood W of $F(x, t)$ in $Q \times B^s$,

(3) a PL q-manifold V and a homeomorphism $K : V \times N \to W$ of $V \times N$ onto W which is level-preserving with respect to N,

such that $F(U \times N) = W \cap F(M \times B^s)$ and $K^{-1}F \mid U \times N : U \times N \to V \times N$ is PL with respect to the product PL structures of $U \times N$ and $V \times N$, where U has the PL structure inherited from M.

DEFINITION. A level-preserving embedding $F : M \times B^s \to Q \times B^s$ is called a *locally flat s-simplex of embeddings* if for each (s, t) in $M \times B^s$ there are

(1) a neighborhood N of t in B^s,

(2) a level-preserving allowable embedding H of either $E^m \times N$ or

$E_+^m \times N$ into $M \times N$ (depending on whether x is in Int M or ∂M) with $H(x, 0) = (x, t)$,

(3) a level-preserving allowable embedding G of either $E^q \times N$ or $E_+^q \times N$ into $Q \times N$ (depending on whether $F_t(x)$ is in Int Q or ∂Q) with $G(0, t) = (x, t)$,

such that $G^{-1}FH$ is of the form $i \times 1$, where i is the natural inclusion of E^m into E^q, E_+^m into E^q, or E_+^m into E_+^q, as the case may be.

Remark 5.1. In the definition of a locally flat s-simplex of embeddings, we may assume without loss of generality that the embedding H is of the form $h \times 1$, where $h : E^m(E_+^m) \to M$ is a coordinate chart on M.

THEOREM 5.2. *A locally flat s-simplex of embeddings is a locally tame s-simplex of embeddings.*

Proof. Let F be a simplex of locally flat embeddings and let N, $H = h \times 1$, and G be as given in the definition and lemma. We may assume without loss of generality that N is PL homeomorphic to B^s. We consider only the case in which $x \in \operatorname{Int} M$ and $F_t(x) \in \operatorname{Int} Q$. Let $U = h(E^m)$ and $W = G(E^q \times N)$. Since h is a coordinate chart, it is PL. Therefore $H^{-1} = h^{-1} \times 1$ gives $E^m \times N$ a PL structure with respect to which H is PL. The embedding $G^{-1}FH$ carries this PL structure onto $E^m \times N \subset E^q \times N$, giving $E^m \times N$ a PL structure which extends to a PL structure of $E^q \times N$ with respect to which $G^{-1}FH$ is PL. Let $V = E^q$ and $K = G$. Then $K^{-1}F \mid U \times N = (G^{-1}FH)H^{-1}$, which is the composition of PL maps and hence PL.

Theorem 2.3 now follows from Theorems 2.1 and 2.2 by techniques of Gluck [G]. Some modifications of Gluck's proof are necessary since Gluck's proof involves solvability (The embeddings G_i are required to be δ-close to each other instead of to a reference embedding H.) rather than the local solvability we use here. These modifications are indicated in [B–S2].

6. Homotopy groups of spaces of PL embeddings

Let $TOP(M, Q)$ denote the semi-simplicial complex or Δ-set whose k-simplices are level-preserving proper embeddings of $M \times B^k$ into $Q \times B^k$. Let $PL(M, Q)$ be the subcomplex whose simplices are PL embeddings. The homotopy theory of semi-simplicial complexes and Δ-sets can be found in [M1] and [R-S1]. For our purposes the following description will be sufficient. Let p be a fixed point in ∂B^{s+1}. An element α of $\pi_s TOP(M, Q; f)$ ($\pi_s PL(M, Q; f)$) is represented by a (PL) level-preserving proper embedding

$$H : M \times \partial B^{s+1} \to Q \times \partial B^{s+1}$$

such that $H_p = f$. If H^0 and H^1 represent α and β respectively, then $\alpha = \beta$ if there is a (PL) proper level-preserving isotopy

$$K : M \times \partial B^{s+1} \times I \to Q \times \partial B^{s+1} \times I$$

such that $K_i = H^i$, $i = 0, 1$ and $K \mid M \times p \times I = f \times 1 \times 1$. An element α of

$$\pi_s TOP(M, Q; f)(\pi_s PL(M, Q; f))$$

is trivial if any (and therefore each) of its representatives extends to a level-preserving (PL) proper embedding of $M \times B^{s+1}$ into $Q \times B^{s+1}$. If f is PL the inclusion map j_1 on representatives induces a homomorphism

$$j_{1*} : \pi_s\, PL(M, Q; f) \to \pi_s\, TOP(M, Q; f).$$

We can also study the homotopy of $\mathrm{Hom}_{TOP}(M, Q)$ and $\mathrm{Hom}_{PL}(M, Q)$ by defining a map which relates them to $TOP(M, Q)$ and $PL(M, Q)$. Let

$$H : M \times \partial B^{s+1} \to Q \times \partial B^{s+1}$$

represent an element of $\pi_s\, TOP(M, Q; f)$. Then we may define a map

$$\phi_1(H) : \partial B^{s+1} \to \mathrm{Hom}_{TOP}(M, Q)$$

by $\phi_1(H)(t) = H_t$. Then $\phi_1(H)$ represents an element of $\pi_s(\mathrm{Hom}_{TOP}(M, Q), f)$, and ϕ_1 induces a homomorphism $\phi_{1*} : \pi_s\, TOP(M, Q; f) \to \pi_s(\mathrm{Hom}_{TOP}(M, Q), f)$ which is clearly well defined and an isomorphism for all s. If $f : M \to Q$ is PL the restriction ϕ_2 of ϕ_1 to $PL(M, Q)$ induces a homomorphism

$$\phi_{2*} : \pi_s\, PL(M, Q; f) \to \pi_s(\mathrm{Hom}_{PL}(M, Q), f).$$

Note that ϕ_2 is not a surjection as ϕ_1 is, since the piecewise linearity of a representative H with respect to ∂B^{s-1} as well as M is lost in $\mathrm{Hom}_{PL}(M, Q)$. Some information about the homomorphisms induced by the above maps is given in Theorem 2.4. We isolate the application of the previous section's results to the proof of Theorem 2.4 in the following lemma.

LEMMA 6.1. *Let P be a finite complex of dimension s and let*

$$F : M \times P \to Q \times P$$

be a level-preserving allowable embedding such that $F \mid M \times P_0$ is PL for some sub-complex P_0 of P. Suppose $m + s \leq q - 3$ and $\varepsilon > 0$. Then there is a (non-ambient) level-preserving ε-isotopy K of $M \times P$ in $Q \times P$, fixed on $M \times P_0$, such that $K_1 F$ is PL and for each t in $(0, 1]$ and y in P, $K_t F_y : M \to Q$ is PL.

Proof. By induction we may assume that $F \mid M \times (P)^{s-1}$ is PL. By Theorem 4.1 we can choose a sequence $F^i : M \times P \to Q \times P$, $i = 0, 1, 2, \cdots$ of level-preserving allowable PL embeddings such that $d(F^i, F) < \delta(\varepsilon/2^i, F)$, where each F^i agrees with F on $M \times P_0$ and $\delta(\varepsilon/2^i, F)$ is chosen by Theorem 4.2. Then for each i there is a PL $(\varepsilon/2^i)$-push H^i of $(Q \times P, F(M \times P))$, fixed on $Q \times P_0$, such that $H_1^i F^i = F^{i+1}$. Define $H : M \times P \times I \to Q \times P \times I$ by

$$H(x, t) = H^0(x, t) \quad \text{for} \quad x \in M \times P,\ t \in [0, 1/2]$$

$$H(x, t) = H^i(H_1^{i-1} \cdots H_1^0(x), i(i+1)(t - 1 + (1/i))$$
$$\text{for} \quad t \in [1 - (1/i), 1 - (1/i+1)]$$
$$H(x, 1) = (x, 1)$$

Then if we let $K_t = H_{1-t}$, K is the desired ε-isotopy.

Proof of Theorem 2.4. To prove the claim about j_{1*}, we must show that any representative of an element of $\pi_s(TOP(M, Q; f)$ is level-preservingly isotopic to a representative of an element of $\pi_s(PL(M, Q, f)$ if $m + s \leq q - 3$ and that if a representative

$$H : M \times \partial B^{s+1} \to Q \times \partial B^{s+1}$$

of an element of $\pi_s(M, Q; f)$ extends to a topological embedding

$$H' : M \times B^{s+1} \to Q \times B^{s+1}$$

then it extends to a PL embedding

$$H'' : M \times B^{s+1} \to Q \times B^{s+1}$$

if $m + s \leq q - 4$. Both of these statements follow from Lemma 6.1.

To show ϕ_{2*} is onto we must show that every element α of $\pi_s(\mathrm{Hom}_{PL}(M, Q), f)$ has a representative in the image of ϕ_2. Suppose α is represented by

$$h : \partial B^{s+1} \to \mathrm{Hom}_{PL}(M, Q),$$

and define $H : M \times \partial B^{s+1} \to Q \times \partial B^{s+1}$ by $H(x, t) = ((h(t))(x), t)$. Then H does not represent an element of $\pi_s PL(M, Q; f)$ since it is not PL. But by Lemma 6.1 there is a level-preserving isotopy K which takes H to a PL embedding

$$G : M \times \partial B^{s+1} \to Q \times \partial B^{s+1}$$

and $K_t H_y$ is PL for all $(t, y) \in I \times \partial B^{s-1}$, since H_y is PL and $K_0 = 1$. Therefore we can define a homotopy $k : \partial B^{s+1} \times I \to \mathrm{Hom}_{PL}(M, Q)$ by $k_t = \phi_2(K_t)$ and so get a homotopy of h to $\phi_2(G)$. A similar argument shows that ϕ_{2*} is one-to-one if $m + s \leq q - 4$.

We have already observed that ϕ_{1*} is an isomorphism, which by commutativity of the diagram establishes the claims about j_{2*}. This completes the proof of Theorem 2.4.

COROLLARY 6.2. *$\pi_s TOP(S^m, S^q; f) = 0$ for any f if $m + s \leq q - 3$.*

Proof. By Irwin's embedding theorem [I] and Theorem 3.2, $\pi_s PL(S^m, S^q; f) = 0$; and by Theorem 2.4 j_{1*} is onto.

In the proof of Theorem 2.4 we did not use the fact that the isotopy constructed in Lemma 6.1 could be made arbitrarily short. This additional information provides us with a proof of Theorem 2.5.

Proof of Theorem 2.5. Given $h \in \mathrm{Hom}_{TOP}(M, Q)$ and $\varepsilon > 0$, we must find

δ so that if $f : \partial B^{s+1} \to \mathrm{Hom}_{TOP}(M, Q)$ is such that $d(f(y), h) < \delta$ for all $y \in \partial B^{s+1}$, then there is an extension $k : B^{s+1} \to \mathrm{Hom}_{TOP}(M, Q)$ of f such that $d(k(y), h) < \varepsilon$ for all $y \in B^{s+1}$. Furthermore, if h is PL and

$$f(\partial B^{s+1}) \subset \mathrm{Hom}_{PL}(M, Q),$$

then we require that $k(B^{s+1}) \subset \mathrm{Hom}_{PL}(M, Q)$. We choose $\delta = \min (\varepsilon/2\mathrm{m}\ \delta(\varepsilon/2, h \times 1))$, where $\delta(\varepsilon/2, h \times 1)$ comes from Theorem 4.2 (m, s). Then as in the proof of Lemma 6.1 there is a level-preserving $(\varepsilon/2)$-isotopy

$$H : M \times \partial B^{s+1} \times I \to Q \times \partial B^{s+1} \times I$$

such that $\phi_1(H_0) = f$ and $H_1 = h \times 1$. If we regard $\partial B^{s+1} \times I$ as a collar of ∂B^{s+1} in B^{s+1}, then we can extend H by $h \times 1$ on

$$M \times (B^{s+1} - (\partial B^{s-1} \times I))$$

to a level-preserving embedding $K : M \times B^{s+1} \to Q \times B^{s+1}$ such that $K \mid M \times \partial B^{s+1} = H_0$ and $d(K_y, h) < \varepsilon$ for all $y \in B^{s+1}$. Then if we define $k : B^{s+1} \to \mathrm{Hom}_{TOP}(M, Q)$ by $k(y) = K_y$, k is the desired extension of f.

7. Locally flat embeddings of topological manifolds

Theorem 2.3 can be used to extend results on the homotopy of spaces of PL embeddings of PL manifolds to spaces of locally flat embeddings of topological manifolds. We give an example. Let $LF(M, Q)$ and $C(M, Q)$ be the semi-simplicial complexes (or Δ-sets) whose s-simplices are locally flat s-simplices of embeddings as defined in Section 5 and level-preserving maps from $M \times B^s \to Q \times B^s$ respectively.

THEOREM 7.1. *Let M be a closed $(2m + s + 1 - q)$-connected topological m-manifold and let Q be a $(2m + s + 2 - q)$-connected topological q-manifold without boundary. Then if $m + s \leq q - 3$ the homomorphism $\psi : \pi_s LF(M, Q) \to C(M, Q)$ induced by the inclusion is an isomorphism. If the connectivities of M and Q are each lowered by one, then ψ is an epimorphism.*

The proof of this theorem in the PL case [H4] involves deforming a level-preserving map along levels to an embedding. The techniques used are level-preserving engulfing and general position. In order to apply these techniques in topological manifolds one needs a level-preserving taming theorem so that an embedding of $V \times N$ onto $Q \times N$, where V is a coordinate neighborhood of M and N is a subset of B^s, can be extended to $U \times N$, where U is a coordinate neighborhood which overlaps V, by sliding the embedding of $V \times N$ along levels to an embedding which is PL with respect to the PL structure of $U \times N$. Such a result is provided by Theorem 2.3.

Theorems 2.1 to 2.5 can also be extended to the category of topological manifolds and locally flat embeddings. The methods are the same as those for $s = 0$, which can be found in [D]. As noted in the introduction, such extensions, without restriction on s, have been announced by Cernavskii [C1, C2].

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