SINGULAR SIDE APPROXIMATIONS FOR 2-SPHERES IN E^3 [1]

BY

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Abstract

Bing has proved that each 2-sphere in E^3 can almost be mapped free of itself in the following very nice sense: Suppose that S is a 2-sphere in E^3 and $\varepsilon > 0$; then there is an ε-map

$$f : S \to S \cup \operatorname{Int} S$$

such that $f(S) \cap S$ and $f^{-1}(f(S) \cap S)$ are 0-dimensional and

$$f \mid S - f^{-1}(f(S) \cap S)$$

is a homeomorphism. This paper illustrates how Bing's theorem can be used advantageously as a substitute for Bing's original side approximation theorem. The following are the principal results.

(1) A 2-sphere S is tame if it is (singularly) spanned or capped on tame sets.

(2) A 2-sphere S is tame if each of its points is an inaccessible point of a Sierpiński curve in S which can be pushed by a homotopy into each complementary domain of S.

0. Notation and definitions

We use ρ for the Euclidean metric, Diam for diameter, Cl for closure, Bd for boundary (point set or combinatorial), Int and Ext for interior and exterior, $N(X, \varepsilon)$ for the (open) ε-neighborhood of a set X in Euclidean 3-dimensional space E^3. An ε-set has diameter less than ε; an ε-map or homeomorphism moves no point as far as ε. A loop is a map $f : S^1 \to E^3$ of the 1-sphere or circle S^1 into E^3. A singular disk is a map $g : B^2 \to E^3$ of the circular disk B^2 into E^3. We consider S^1 as the boundary of B^2 and say that the singular disk $g : B^2 \to E^3$ is bounded by the loop $g \mid S^1$. If f (respectively, g) is an embedding, then $f(S^1)$ (respectively, $g(B^2)$) is called a simple closed curve (disk). A set A in E^3 is 1-ULC if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each δ-loop in A bounds a singular ε-disk in A.

1. The singular side approximation theorem

THEOREM 1. *Suppose S is a 2-sphere in E^3 and U is a component of $E^3 - S$. Then for each $\varepsilon > 0$ there exist an ε-map $g : S \to S \cup U$ and disjoint ε-disks $D_1, \cdots, D_n$ in S such that*

(i) $g(S) \cap S$ *and* $g^{-1}(g(S) \cap S)$ *are 0-dimensional subsets of* $\bigcup_{i=1}^{n} \operatorname{Int} D_i$,

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(ii) $g(D_i) \cap g(D_j) = \emptyset$ *for* $i \neq j$,
(iii) $g(S) \cap S = \bigcup_{i=1}^{n} [\operatorname{Int} D_i \cap g(\operatorname{Int} D_i)]$, and
(iv) $g \mid [S - g^{-1}(g(S) \cap S)]$ *is a homeomorphism onto a locally polyhedral set in* U.

Addendum. Let $X_1, X_2, \cdots$ be a sequence of closed subsets of S. Then $g(S) \cap S$ can be required to lie in $S - \bigcup_{i=1}^{\infty} X_i$ for each $\varepsilon > 0$ if and only if $(S \cup U) - X_i$ is 1-ULC for each i. This latter condition on X_i is satisfied, for example, if X_i lies on some tame 2-sphere in E^3 and X_i has no degenerate components.

Proof. Theorem 1 and its addendum are proved piecemeal in [9, Lemma 2.5] (called the "Singular Side Approximation Lemma" in [12]) and [12; Addenda I and II to the Singular Side Approximation Lemma, proof of Case 1 of the Cellularity Lemma].

I discovered Theorem 1 and a form of the addendum during Christmas vacation of 1967 as I read [5]. Theorem 1 has played an important but hidden role in the development of each of the papers [9], [10] and [12]. Only in [12] and in the applications of the present paper have I been unable to avoid its full use But it has proved conceptually important (at least in my work) and seems to deserve explicit statement and examination.

Definition. The map g of Theorem 1 is called an ε-singular side approximation to S from U.

2. Singularities in singular side approximations

A result of Eaton [16, Theorem 17] implies that if a 2-sphere S in E^3 is wild from Int S at more than one point and Int S is an open 3-cell, then, for all sufficiently small $\varepsilon > 0$, any ϵ-singular side approximation to S from Int S must indeed have singularities. We note in this section that Eaton's argument taken in conjunction with Theorem 1 also establishes the following result.

Theorem 2. *Suppose that* S *is a 2-sphere in* E^3 *satisfying:*

(i) *For each* $\varepsilon > 0$ *there is an* ε*-homeomorphism* $g : S \to S \cup \operatorname{Int} S$ *such that* $g(S) \cap S$ *is* 0*-dimensional; and*
(ii) *if* C *is a closed and* 0*-dimensional subset of* S, *then* $(S \cup \operatorname{Int} S) - C$ *is* 1-ULC.
Then $S \cup \operatorname{Int} S$ *is a* 3*-cell.*

Proof. The proof is exactly like that of Eaton's Theorem 13 [16, pp. 713–715] except that one must find a substitute for Eaton's Lemma 12.

Eaton's idea as translated into our setting would be the following: take a nonsingular ε-singular-side-approximation $g : S \to S \cup \operatorname{Int} S$ to S from Int S with associated disks $D_1, \cdots, D_n$ in S as outlined in Theorem 1 (the existence is ensured by condition (i) of Theorem 2 and by a disk collecting argument like that in [20]); (2) use, for each i, a singular disk

$$f_i : D_i \to S \cup \text{Int } S$$

supplied by the following lemma to cut the disk $g(D_i)$ off inside S (in the application of the lemma, one takes δ very small and $C_i = g(S) \cap \text{Int } D_i$). For (2), follow a procedure similar to the argument given by Eaton in Cases 1 and 2 on page 714 of [16]. The argument establishes that S can be homeomorphically ε-approximated in Int S for each $\varepsilon > 0$; thus $S \cup \text{Int } S$ is a 3-cell by [2] (or [14]).

LEMMA (corresponding to [16, Lemma 12]). *If S is a 2-sphere, D_i is a disk in S, and C_i is a closed 0-dimensional subset of* Int D_i *such that*

$$(S \cup \text{Int } S) - C_i$$

is 1-ULC, *and* $\delta > 0$, *then there is a singular disk* $f_i : D_i \to S \cup \text{Int } S$ *such that*

(1) $f_i(D_i) \cap S$ *and* $f_i^{-1}[f_i(D_i) \cap S]$ *are 0-dimensional subsets of* Int $D_i - C_i$, *and*

(2) $f_i \mid \{D_i - f_i^{-1}[f_i(D_i) \cap S]\}$ *is a δ-homeomorphism onto a locally polyhedral subset of* Int S.

Proof. This is an immediate corollary to Theorem 1 and its addendum.

Remark. The need for condition (i) in Theorem 2 is obvious from examples; indeed, condition (ii) is satisfied from the wild side of the examples in [3], [17], and [1]. The precise point where hypothesis (i) is used in the proof of Theorem 2 or the corresponding hypothesis is used in [16, Theorem 13] can easily escape notice and lead to one of the most standard errors in the subject. The difficulty is at the root of many of the false proofs of the "Free Sphere Conjecture" [7. p. 302] that I have seen or constructed myself.

3. Spheres which are singularly capped or spanned

Let S denote a 2-sphere in E^3, and choose some homeomorphism $h : S \to S^2$, where

$$S^2 = \{x \in E^3 \mid \rho(x, 0) = 1\}$$

is the standard 2-sphere. If $X \subset S$, define $a(X) = \{h^{-1}(-h(x)) \mid x \in X\}$ to be the *antipodal set* of X. This somewhat arbitrary assignment of antipodal sets of S will prove a technical convenience. It will also be convenient to assign to each pair $\{x, a(x)\}$ of antipodal points in S a simple closed curve $J(x) = J(a(x))$ such that $J(x) \cap S = \{x, a(x)\}$ and $J(x)$ *pierces* S at x and $a(x)$.

Recall that the standard 1-sphere S^1 and disk B^2 in the plane E^2 are defined by

$$S^1 = \{x \in E^2 \mid \rho(x, 0) = 1\} \quad \text{and} \quad B^2 = \{x \in E^2 \mid \rho(x,0) \leq 1\}.$$

More generally, if t is a positive real number, then

$$tS^1 = \{tx \mid x \in S^1\} \quad \text{and} \quad tB^2 = \{tx \mid x \in B^2\}.$$

DEFINITION 1. A map $f : \operatorname{Int} B^2 \to \operatorname{Int} S$ is said to *cap* S at $p \in S$ from $\operatorname{Int} S$ if the following conditions are satisfied.

(1) $\rho(f(x), S) \to 0$ as $\rho(x, S^1) \to 0$.

(2) $\operatorname{Cl}(f(\operatorname{Int} B^2)) - f(\operatorname{Int} B^2) \subset S - \{p, a(p)\}$.

(3) For all positive numbers sufficiently close to but less than 1, the loop $f \mid tS^1$ and the simple closed curve $J(p)$ link (cf. [14, §1]).

If, in addition,

(4) $\operatorname{Cl}(f(\operatorname{Int} B^2)) \subset N(p, \varepsilon)$,

then f is called an *ε-cap* at p.

The set $\operatorname{Cl}(f(\operatorname{Int} B^2)) - f(\operatorname{Int} B^2)$ is called the *boundary of f* and is denoted by $\operatorname{Bd} f$. If $\operatorname{Bd} f$ lies on a tame 2-sphere in E^3, then f is said to have *tame boundary*. Similarly, we define $\operatorname{Int} f = f(\operatorname{Int} B^2)$.

DEFINITION 2. The 2-sphere S is said to be (singularly) *capped from* $\operatorname{Int} S$ if, for each $p \in S$ and each $\varepsilon > 0$, there is a map which ε-caps S at p from $\operatorname{Int} S$. If, for each $p \in S$ and each $\varepsilon > 0$, the corresponding ε-cap can be chosen to have tame boundary, then S is said to be *tamely capped from* $\operatorname{Int} S$.

DEFINITION 1′. Suppose K is a finite dimensional compact metric continuum. A map $f : K \times I \to S \cup \operatorname{Int} S$ is said to *span* S at $p \in S$ from $\operatorname{Int} S$ if the following conditions are satisfied.

(1) $f(K \times (0, 1]) \subset \operatorname{Int} S$.

(2) $f \mid K \times \{1\}$ is a constant map.

(3) $f \mid K \times \{0\}$ is an essential map into $S - \{p, a(p)\}$.

If, in addition,

(4) $f(K \times I) \subset N(p, \varepsilon)$,

then f is called an *ε-span* at p.

The set $f(K \times \{0\})$ is called the *boundary* of f and is denoted by $\operatorname{Bd} f$. If $\operatorname{Bd} f$ lies on a tame 2-sphere in E^3, then f is said to have *tame boundary*. Similarly, we define $\operatorname{Int} f = f(K \times (0, 1])$.

DEFINITION 2′. The 2-sphere S is said to be (singularly) *spanned* from $\operatorname{Int} S$ if, for each $p \in S$ and each $\varepsilon > 0$, there is a map which ε-spans S at p from $\operatorname{Int} S$. *Tamely spanned* is defined in the obvious way.

Remark. Although Definitions 1 and 1′ reflect the arbitrary choices made in assigning antipodal sets and curves $J(p)$, it is easy to see that Definitions 2 and 2′ are independent of those choices and are properties of S itself.

The following theorem generalizes results of White [24] and is related to results in [6], [13, §5], [21], and [22].

THEOREM 3. *If S is a 2-sphere in E^3 and S is either tamely capped or tamely spanned from* $\operatorname{Int} S$, *then* $S \cup \operatorname{Int} S$ *is a 3-cell.*

Remark. Theorem 3 is almost surely true with the word "tamely" omitted.

Proof. By a compactness argument, there is a sequence $f_1, f_2, \cdots$ of maps, each of which either caps or spans S at some point of S from Int S and each of which has tame boundary, such that for each $p \in S$ and each $\delta > 0$ there is an integer i such that f_i is a δ-cap or δ-span at p.

For each i, Bd f_i is a nondegenerate continuum in S (by conditions (1), (2), and (3) of Definition 1 or by condition (3) of Definition 1′) and lies on a tame 2-sphere in E^3 (since Bd f_i is tame).

Suppose $\varepsilon > 0$ given. By Theorem 1 and its addendum, there exist an ε-map

$$g : S \rightarrow (S \cup \text{Int } S) - \bigcup_{i=1}^{\infty} \text{Bd } f_i$$

and disjoint ε-disks $D_1, \cdots, D_n$ in S such that g and the disks satisfy conditions (i)–(iv) of Theorem 1 and such that

$$\text{Diam } (D_i \cup g(D_i)) < \varepsilon \quad (i = 1, \cdots, n).$$

Let $M_1, \cdots, M_n$ be disjoint ε-neighborhoods of

$$\text{Int } D_1 \cup g(\text{Int } D_1), \cdots, \text{Int } D_n \cup g\ (\text{Int } D_n),$$

respectively, in

$$E^3 - [(S - \bigcup_{i=1}^{n} \text{Int } D_i) \cup g(S - \bigcup_{i=1}^{n} \text{Int } D_i)].$$

Let $N_i = M_i \cap \text{Int } S$.

In order to show that $S \cup \text{Int } S$ is a 3-cell, it suffices to show that, for each i, $g(\text{Bd } D_i)$ bounds a polyhedral disk E_i in $N_i \cup g(\text{Bd } D_i)$. For then,

$$g(S - \bigcup_{i=1}^{n} D_i) \cup (\bigcup_{i=1}^{n} E_i)$$

will be a polyhedral 2-sphere in Int S which is homeomorphically within ε of S; and will give the desired result.

We work with each singular disk $g(D_i)$ separately, hence, for notational simplicity, set $N = N_i$, $M = M_i$, and $D = D_i$. We may clearly assume that the antipodal set $a(D)$ of D is disjoint from D and that there is an arc α in

$$(a(D) \cup \text{Int } S) - N$$

that is irreducible from $g(\text{Bd } D)$ to $a(D)$.

Consider the compact 0-dimensional subset $C = g(D) \cap D$ of Int D. By our choice of the maps $f_1, f_2, \cdots$ and the compactness of C, there are finitely many points $x_1, \cdots, x_j$ in C and corresponding integers $n_1, \cdots, n_j$ such that f_{n_i} is a span or cap for S at x_i $(i = 1, \cdots, j)$,

$$\text{Int } f_{n_i} \cup \text{Bd } f_{n_i} \subset M \quad (i = 1, \cdots, j),$$

and for each $x \in C$ there is an x_i such that x and x_i are in the same component of $S - \text{Bd } f_{n_i}$.

Since each $\operatorname{Bd} f_{n_i}$ misses C (by our choice of g) and since each $\operatorname{Int} f_{n_i}$ lies in N, the sets $g^{-1}(C)$ and $g^{-1}[g(D) \cap (\bigcup_{i=1}^{j} \operatorname{Int} f_{n_i})]$ are disjoint compact subsets of Int D. Since $g^{-1}(C)$ is 0-dimensional, there are finitely many disjoint disks $F_1, \cdots, F_k$ in $\operatorname{Int} D - g^{-1}[g(D) \cap (\bigcup_{i=1}^{j} \operatorname{Int} f_{n_i})]$ such that each contains a point of $g^{-1}(C)$ and such that $g^{-1}(C) \subset \bigcup_{i=1}^{k} \operatorname{Int} F_1$.

Let D_0 denote the polyhedral disk-with-holes $g(D - \bigcup_{i=1}^{k} \operatorname{Int} F_i)$ (which lies in Int S). Let $N_0 = N - \bigcup_{i=1}^{k} g(F_i)$. The Loop Theorem [23] implies (by an argument that is by now standard; e.g. [13; Proof of (3.4), 7th paragraph] or, in more detail, [26, Proof of Theorem 1]) that $g(\operatorname{Bd} D)$ bounds, together with some other finite number of the curves from Bd D_0, a polyhedral disk-with-holes E_0 in $N_0 \cup \operatorname{Bd} D_0$ such that the homomorphism $\pi_1(\operatorname{Int} E_0) \to \pi_1(N_0)$ induced by the inclusion $\operatorname{Int} E_0 \subset N_0$ is 1-1; for otherwise, since Int D_0 separates N_0, the Loop Theorem applies directly to Int D_0 together with one component of $N_0 - \operatorname{Int} D_0$ and supplies disks which cap over holes in D_0. We claim that E_0 is actually a disk, in fact, precisely the disk E needed to complete the proof of Theorem 3 since $E_0 \subset N \cup g(\operatorname{Bd} D)$.

Suppose E_0 is not a disk and that $g(\operatorname{Bd} F_i) \subset \operatorname{Bd} D_0$ is a boundary curve of E_0 distinct from $g(\operatorname{Bd} D)$. Let β denote an arc in $E_0 \cup g(F_i)$ irreducible from $\alpha \cap g(\operatorname{Bd} D)$ to S. Let γ be an arc from $\alpha \cap a(D)$ to $\beta \cap D$ such that $\operatorname{Int} \gamma \subset \operatorname{Ext} S$. Then $\alpha \cup \beta \cup \gamma$ forms a loop L in a natural way. We shall complete the proof of Theorem 3 by showing that there is a loop near one of the sets $\operatorname{Int} f_{n_r}$ which links L and yet does not link L, a contradiction arising from the false assumption that E_0 has more than one boundary component.

By the choice of $x_1, \cdots, x_j$ and $n_1, \cdots, n_j$, there is an integer r such that $\beta \cap D$ and x_r lie in the same component of $S - \operatorname{Bd} f_{n_r}$. We consider separately the case where f_{n_r} is a cap and a span.

Case 1. The map f_{n_r} is a cap. Since the sets

$$(S \cup \operatorname{Int} S) - \operatorname{Bd} f_{n_r} \quad \text{and} \quad (S \cup \operatorname{Ext} S) - \operatorname{Bd} f_{n_r}$$

have trival first homology, (cf. [25, Chapter 10, §3]) it follows that L and $J(x_r)$ are homologous in $E^3 - \operatorname{Bd} f_{n_r}$. This, together with condition (3) of Definition 1, implies that, for all positive numbers t sufficiently close to but less than 1, $f_{n_r} \mid tS^1$ links L (cf. [14, §1]).

Let t be a positive number chosen so as to satisfy the requirements of the preceding paragraph and also so that $f_{n_r}(tS^1) \cap E_0 = \emptyset$. We may put the singular disk $f_{n_r}(tB^2)$ (which lies in N_0) and the open disk-with-holes Int E_0 in general position by a slight adjustment of $f_{n_r} \mid \operatorname{Int}\ (tB^2)$. If J is any component of $f_{n_r}^{-1}(f_{n_r}(tB^2) \cap E_0)$, then $f_{n_r} \mid J$ must be a loop that is nullhomotopic in Int E_0 since $\pi_1(\operatorname{Int} E_0) \to \pi_1(N_0)$ is 1-1. But this implies that $f_{n_r} \mid tB^2$ may be adjusted in N_0 in such a manner that $f_{n_r} \mid tS^1$ is kept fixed but the adjusted $f_{n_r}(tB^2)$ misses E_0. Since $L \cap N_0 \subset E_0$, this implies that $f_{n_r} \mid tS^1$ and L do not link, our desired contradiction. This completes the proof of Theorem 3 in Case 1.

Case 2. The map f_{n_r} is a span defined on a product $K \times I$, where K is ite dimensional compact metric continuum. Since $L \cap D$ and x_r lie in the same component of $S - \text{Bd } f_{n_r}$ as do $L \cap a(D)$ and $a(x_r)$, it follows that $f_{n_r} \mid K \times \{0\}$ is an essential map into $S - L$ (cf. condition (3) of Definition 1′). We shall complete the proof by showing that $f_{n_r} \mid K \times \{0\}$ is inessential in $S - L$, a contradiction which again arises from the false assumption that E_0 has more than one boundary component.

By a classical result, we may assume that K lies in a high dimensional Euclidean space E^m. We shall extend f_{n_r} to a neighborhood of $K \times I$ in $E^m \times I$ as follows. Consider the open set $M_0 = M - (\bigcup_{i=1}^k g(F_i) \cup \gamma)$. Note that $M_0 \cap \text{Int } S = N_0$. The set M_0, $M_0 \cap S$, and $f_{n_r}(K \times \{1\})$ are all absolute neighborhood retracts. Thus there is a neighborhood K_0 of K in E^m and an extension f^* of f_{n_r} to $K_0 \times I$ such that

(1) $f^* : K_0 \times I \to M_0$
(2) $f^*(K_0 \times \{1\}) = f_{n_r}(K \times \{1\}) =$ single point
(3) $f^*(K_0 \times \{0\}) \subset M_0 \cap S$.

There are a polyhedral neighborhood K_1 of K in K_0 and a positive number t, $0 < t < 1$, such that

(4) $f^*(K_1 \times [0, t]) \subset E^3 - (L \cup E_0)$
(5) $f^*(K_1 \times [t, 1]) \subset N_0$.

Let J be any loop in K_1. Then $f^* \mid J \times \{t\}$ is a loop in $N_0 - E_0$ which bounds a singular disk in $f^*(K_1 \times [t, 1]) \subset N_0$ (by conditions (2) and (5) of this proof). By the argument employed in the second paragraph of Case 1 above, $f^* \mid J \times \{t\}$ is nullhomotopic in $N_0 - E_0$, hence in $E^3 - L$. But $f^* \mid J \times \{t\}$ and $f^* \mid J \times \{0\}$ are homotopic in $E^3 - L$; hence neither links L. This implies that $f^* \mid J \times \{0\}$ is nullhomotopic in $S - L$. Thus the 1-skeleton of any triangulation of $K_1 \times \{0\}$ is mapped trivially into $S - L$ by f^*. Since $S - L$ has trivial higher homotopy groups $(\pi_2, \pi_3, \cdots)$, it follows that $f^* \mid K_1 \times \{0\}$ and therefore $f_{n_r} \mid K \times \{0\}$ are nullhomotopic in $S - L$, our desired contradiction.

4. Collared Sierpiński curves

Let S denote a 2-sphere in E^3. A *Sierpiński curve* in S is the continuum which remains in S after one removes from S the interiors of a null sequence of disjoint disks whose union is everywhere dense in S. The points of a Sierpiński curve which do not belong to any of these disks are called *inaccesible* points.

DEFINITION. Let S denote a 2-sphere in E^3 and K a subset of S. We say that K can be *singularly collared* from Int S or that K can be *pushed into* Int S *with a homotopy* if there exists a map f of $K \times I$ into $S \cup \text{Int } S$ such that $f \mid K \times \{0\}$ is the identity homeomorphism on K and $f(K \times (0, 1])$ us a subset of Int S.

The following theorem answers in the affirmative a question raised by Bing in [5] and by Burgess in [6]. A partial result in this direction was obtained by White [24].

THEOREM 4.1. *A 1-sphere S in E^3 is tame from* Int S *if each point of S is an inaccessible point of a Sierpiński curve on S that can be pushed into* Int S *with a homotopy.*

Proof. By the Hosay-Lininger Theorem [15], [18], [19], we may assume that S is tame from Ext S. We shall show that, under this additional assumption, S is tamely spanned from Int S. The desired result will then follow from Theorem 3.

In order to show that S is tamely spanned from Int S, let $p \in S$ and $\varepsilon > 0$. By hypothesis, there exist a Sierpiński curve X in S and a map

$$f : X \times I \to S \cup \operatorname{Int} S$$

such that p is an inaccessible point of X, $f \mid X \times \{0\}$ is the identity homeomorphism on X, and $f(X \times (0, 1]) \subset \operatorname{Int} S$. Burgess [6, proof of Theorem 14] has shown under these conditions that there exist a simple closed curve J in X and a map $f^* : J \times I \to S \cup \operatorname{Int} S$ which ε-spans S at p such that $f^* \mid J \times \{0\}$ is the identity homeomorphism. It is easy to see from Burgess' proof and from the structure of a Sierpiński curve that we may take J to lie in the inaccessible part of X; hence J is tame by Theorem 4.2. This completes the proof.

THEOREM 4.2. *Suppose that S is a 2-sphere in E^3, that S is tame from* Ext S, *that X is a Sierpiński curve in S, that K is a compact subset of the inaccessible part of X, and that there is a map*

$$f : X \times I \to S \cup \operatorname{Int} S$$

such that $f \mid X \times \{0\}$ is the identity homeomorphism on X and

$$f(X \times (0, 1]) \subset (S \cup \operatorname{Int} S) - K.$$

Then K is tame.

Proof. We first show that we may assume that S is locally tame at each point of $S - K$ and that $f(X \times (0, 1]) \subset \operatorname{Int} S$. Since S is tame from Ext S, there is an embedding $g : S \times [0, 1] \to S \cup \operatorname{Ext} S$ such that $g \mid S \times \{0\}$ is the identity homeomorphism on S. Since K is closed in S, there is a continuous function $h : S \to [0, 1)$ such that $K = h^{-1}(0)$. Let

$$k : S \to g(S \times [0, 1))$$

be the embedding defined by $k(x) = g(x, h(x))$. Let $S' = g(S)$ and $X' = k(X)$. It is obvious that S' is tame from Ext S' and is locally tame at each point of $S' - K$; and it is an easy matter to use the map f and the collar g to construct a map $f' : X' \times I \to S' \cup \operatorname{Int} S'$ such that $f' \mid X' \times \{0\}$ is the identity homeomorphism of X' and such that $f'(X' \times (0, 1]) \subset \operatorname{Int} S'$. Having established the existence of S', X', and f', we assume that S, X, and f themselves orginally satisfied the additional desired properties.

We next show that S, with the additional properties of the preceding paragraph assumed for S, X, and f, is tamely spanned from Int S. By Theorem 3, the proof will be complete.

In order to show that S is tamely spanned from Int S, let $p \in S$ and $\varepsilon > 0$. If $p \in S - K$, then the desired result is clear since S is locally tame at each point of $S - K$. Assume therefore that $p \in K$. The same argument of Burgess referred to in the proof of Theorem 4.1 shows that if L is any sufficiently small loop in X, then there is a map

$$g : B^2 \to S \cup \operatorname{Int} S$$

such that $g \mid \operatorname{Bd} B^2 = L$, $g(\operatorname{Int} B^2) \subset \operatorname{Int} S$, and $\operatorname{Diam} g(B^2) < \varepsilon$. Hence it suffices to find an arbitrarily small tame loop L in $X - \{p\}$ such that L links $J(p)$ (cf. §3). The existence of such a tame loop is, however, established by Theorem 4.3; and so the proof of Theorem 4.2 is complete.

THEOREM 4.3. *Suppose that S is a 2-sphere in E^3, that X is a Sierpiński curve in S, that each component of $S - X$ has tame boundary, and that p is an inaccessible point of X. Then, for each $\varepsilon > 0$, there is a tame loop L in $X \cap N(p, \epsilon)$ such that L links $J(p)$. (Recall that $J(p)$ is defined in the first paragraph of* §3.)

Proof. By [4], there is a tame simple closed curve J in $S - J(p)$ such that J links $J(p)$ and J lies in an $\varepsilon/3$-neighborhood of p. Since p is an inaccessible point of X, we may require that J intersects no component of $S - X$ which has diameter as large as $\varepsilon/3$. Let $U_1, U_2, \cdots$ be the components of $S - X$ which J intersects. Then $M = J \cup \operatorname{Bd} U_1 \cup \operatorname{Bd} U_2 \cup \cdots$ is a closed subset of S which is the union of countably many nondegenerate tame continua, hence is tame by [8, Theorem 4.2]. We may certainly require that $a(p)$ not lie in $\operatorname{Cl} U_1 \cup \operatorname{Cl} U_2 \cup \cdots$. There is a homotopy of J in

$$J \cup \operatorname{Cl} U_1 \cup \operatorname{Cl} U_2 \cup \cdots \subset S - J(p)$$

which takes J into $M \cap X$. The image loop is the desired tame loop L.

Remark. There exists a 2-sphere S in E^3 and a Sierpiński curve X in S such that no nondegenerate subcontinuum of X is tame [11, Example 2]. Therefore, the hypothesis that each component of $S - X$ has tame boundary in Theorem 4.3 is an important one.

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