POTENTIALS OF STOPPED DISTRIBUTIONS

BY

J. R. Baxter and R. V. Chacon

1. Introduction

Dubins has shown in [2] that if $\{X_n\}$ is an L_2-martingale and if $\{B_t\}$ is Brownian motion with the same initial distribution as the martingale then there exist stopping times $\{s(n)\}$, having finite expectations, such that $0 = s(0) \leq s(1) \leq \cdots$, and such that $\{X_n\}$ and $\{B_{s(n)}\}$ have the same joint distributions. This strengthens a similar result of Skorohod in which the stopping times $s(n)$ were "randomized" in the sense that they depended upon an independent random variable. The essence of the result of Dubins can be stated more concretely: if ν is a probability measure on $\mathbf{R}^1$ with center of mass at the origin and finite second moment and if $\{B_t\}$ is Brownian motion starting at the origin, then there exists a stopping time s having finite expectation such that the distribution of B_s is ν. We note that if s is to have a finite expectation then ν must have a finite second moment (equal to the expectation of s). Doob has shown (see [4]) that if s is not required to have a finite expectation then the problem is trivial in the sense that any probability measure ν can be realized by stopping Brownian motion. Dubins remarks, however, that his construction yields a natural stopping time s for arbitrary ν with center of mass at the origin, even though ν need not have a finite second moment.

Let μ be a probability measure on $\mathbf{R}^p$, $p \geq 1$. Let $S(\mu)$ denote the set of all measures ν obtained as the distributlon of B_s, where $\{B_t\}$ is Brownian motion with initial distribution μ, and s is some stopping time. Let $FS(\mu)$ denote the set of measures ν obtained in the same way using stopping times s with finite expectations. Let $RS(\mu)$ denote those ν obtained using randomized stopping times s, and let $FRS(\mu)$ denote those ν obtained using randomized stopping times s with finite expectations. Clearly $FS(\mu)$ is the smallest of these four sets of measures, and $RS(\mu)$ is the largest.

In the present paper we shall prove the following theorem:

Theorem 1. *Let μ and ν be probability measures on $\mathbf{R}^p$, $p \geq 1$. Let U^μ and U^ν be the potentials of μ and ν (defined in Section 2). Suppose:*

(i) *U^μ, U^ν are finite a.e. with respect to Lebesgue measure,*

(ii) *$U^\mu \geq U^\nu$ on $\mathbf{R}^p$,*

(iii) $\lim_{x\to\infty} [U^\mu(x) - U^\nu(x)] = 0$,

(iv) *U^ν is continuous on $\mathbf{R}^p$.*

Then $\nu \in S(\mu)$. If ν has a finite second moment so does μ, and $\nu \in FS(\mu)$. If $p > 2$, conditions (i) *and* (iii) *can be dropped.*

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We note that condition (i) always holds for $p \geq 3$, and holds for all p when μ and ν have compact support. Also, for any p, if μ and ν have compact support and condition (ii) holds then one can show $U^{\mu} = U^{\nu}$ outside a compact set so condition (iii) is automatically satisfied.

In the case $p = 1$, one can show easily that if the first moment of ν exists then U^{ν} is finite and continuous on $\mathbf{R}^1$. Also, if μ and ν have the same center of mass then condition (iii) holds. Finally if μ is a point mass concentrated at the center of mass of ν then condition (ii) holds, by a trivial argument. Thus the result of Dubins is included in this particular case of the theorem.

It is not clear whether condition (iii) is necessary in the case $p \leq 2$. Condition (iv) also may be unnecessary, although some restriction on U^{ν}, like finiteness, is clearly needed. Continuity appears in the statement of the domination principle (Lemma 3 of Section 2) but this could certainly be avoided if there was a reason to do so. The only actual use of condition (iv) is in the proof in Section 3, where the continuity of U^{ν} ensures the exixtence of an open ball upon which balayage takes place.

Our results have points of contact with the work of Rost [5] and that of Cartier, Fell and Meyer [1]. By combining the work of Dubins with that of Cartier, Fell and Meyer, one obtains the result of our theorem for $p = 1$, but only in the case that μ and ν have compact support.

Rost deals with a general Markov process. We shall only describe his results in the context of Brownian motion. Let $\{P_t\}$ denote the semigroup of transition probabilities for Brownian motion, and let $P_t(\alpha) = e^{-\alpha t}P_t$ for all $\alpha \geq 0$. Let $H(\alpha)$ denote the set of bounded excessive functions with respect to the semigroup $\{P_t(\alpha)\}$, where an excessive function f with respect to $\{P_t(\alpha)\}$ is defined to be a non-negative measurable function such that $P_t(\alpha)f \leq f$ for all t and such that $P_t(\alpha)f$ converges point-wise to f as t goes to 0. Rost shows that the following two statements are equivalent:

(1) $\nu \in RS(\mu)$,

(2) $\lim_{\alpha\to\infty} [\sup \{ \int g d\nu - \int g\, d\mu \mid g \in H(\alpha), g \leq 1\}] = 0.$

For $\mathbf{R}^1$ and $\mathbf{R}^2$ equation (2) does not simplify, but for $\mathbf{R}^p$, $p \geq 3$ it becomes

(3) $\int g\, d\mu \geq \int g\, d\nu$ for all g in $H(0)$.

Equation (3) can then be shown to be equivalent to condition (ii) of Theorem 1.

As expressed in equation (1), the stopping time constructed by Rost is randomized. We conjecture that, because of the smoothness of the transition probabilities for Brownian motion, the stopping time constructed by Rost will in fact be a true stopping time, in the special case that μ and ν have disjoint support. However, we have no proof of this conjecture.

Rost does not give conditions under which the expectation of the stopping time which he constructs would be finite. It is possible that the methods used to prove that the stopping time constructed in the present paper has a finite expectation when ν has a finite second moment could also be applied to the stopping time constructed by Rost. We have no proof of this conjecture either.

The next section contains some definitions and lemmas from potential theory. Theorem 1 is proved in Section 3.

2. Potential theory

We define the potential kernel k on $\mathbf{R}^p$ as follows:

$$k(x) = -|x|, p = 1, \qquad k(x) = -\log|x|, p = 2, \qquad k(x) = |x|^{-p+2}, p > 2. \tag{1}$$

For a finite Borel measure μ on $\mathbf{R}^p$ we define the potential U^μ of μ by

$$U^\mu(x) = \int k(x-y)\mu(dy) \tag{2}$$

when the integral exists.

For two finite Borel measure μ and ν, let

$$\langle\mu, \nu\rangle = \int k(x-y)\mu(dy)\nu(dx) \tag{3}$$

when the integral exists with respect to the product measure $\mu \times \nu$. In this case Fubini's theorem applies, and

$$\langle\mu, \nu\rangle = \int U^\mu(x)\nu(dx) = \int U^\nu(y)\mu(dy). \tag{4}$$

$\langle\mu, \nu\rangle$ is called the mutual energy of μ and ν. Let $k^+ = k \vee 0, k^- = -k \wedge 0$. Then $k = k^+$ for $p > 2$. Let

$$^{\pm}U^\mu(x) = \int k^{\pm}(x-x)\mu(dy) \quad \text{and} \quad \langle\mu, \nu\rangle^{\pm} = \int k^{\pm}(x-y)\mu(dy)\nu(dx). \tag{5}$$

All these quantities are either finite or $+\infty$.

For this paper, let us call a function good if it is bounded, measurable, and has compact support. A measure with a good density will be called a good measure. It is easy to see that for any finite measure μ with bounded density f and any constant $c > 0$ we have

$$^{+}U^\mu \le \|f\| \int_{\{k^+ > c\}} k^+(x)\,dx + c\|\mu\| \quad \text{on} \quad \mathbf{R}^p. \tag{6}$$

Here $\|f\|$ is sup norm and $\|\mu\|$ is total mass. It follows that $^{+}U^\mu$ is continuous on $\mathbf{R}^p$ for any finite measure μ with bounded density. Also, for any good measure γ,

$$\lim_{|x|\to\infty} {}^{+}U^\gamma(x) = 0. \tag{7}$$

Hence $\langle \mu, \gamma \rangle^+$ exists and is finite for any finite measure μ. We conclude that ${}^+U^\mu(x)$ is finite a.e. dx.

It is easy to check that

$$(8) \qquad | k^-(x) - k^-(y) | \le | x - y | \quad \text{for} \quad x, y \text{ in } \mathbf{R}^p.$$

Hence

$$(9) \quad {}^-U^\mu(y) - | x - y | \mu(\mathbf{R}^p) \le {}^-U^\mu(x) \le {}^-U^\mu(y) + | x - y | \mu(\mathbf{R}^p)$$

Thus ${}^-U$ is either finite for all x or infinite for all x, and if it is finite it is continuous. Let M denote the collection of finite measures μ for which ${}^-U^\mu$ is finite. For $p > 2$, M contains all finite measures. For any p, M contains all finite measures with compact support. For any μ in M and every x in $\mathbf{R}^p$, $U^\mu(x)$ exists as a finite number or $+\infty$. It is a straightforward matter to verify that U^μ is lower semicontinuous and superharmonic.

Let μ be in M and let ν be a finite measure with compact support. Since clearly $\langle \mu, \nu \rangle^- < \infty$, we see that $\langle \mu, \nu \rangle$ exists as a finite number or $+\infty$. Hence by (7), $\langle \mu, \gamma \rangle$ exists and is finite for any measure μ in M and any good measure γ.

Let μ_n be a uniformly bounded sequence in M such that

$$(10) \qquad \lim_{n\to\infty} \langle \mu_n, \gamma \rangle \text{ exists for all good measures } \gamma.$$

Then there exists a finite measure μ such that

$$(11) \qquad \lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu \quad \text{for all } f \text{ in } \mathcal{C}_0(\mathbf{R}^p)$$

where $\mathcal{C}_0(\mathbf{R}^p)$ is the collection of all continuous functions on $\mathbf{R}^p$ with compact support. This is clear because any smooth function in $\mathcal{C}_0(\mathbf{R}^p)$ can be written as $U^{\gamma_1} - U^{\gamma_2}$.

(11) does not quite imply (10). If (11) holds, and μ_n is a uniformly bounded sequence in M, then

$$(12) \qquad \lim_{n\to\infty} \langle \mu_n, \gamma \rangle^+ = \langle \mu, \gamma \rangle^+ \quad \text{for any good measure } \gamma.$$

This is clear by (7). It is easy to see from (8) that if $\langle \mu_n, \nu \rangle^-$ is bounded in n for one non-zero measure ν then ${}^-U^{\mu_n}$ is uniformly bounded on any bounded set in $\mathbf{R}^p$ for all n, and μ is in M. One can show that if (11) holds and μ_n is a uniformly bounded sequence in M and if $\langle \mu_n, \nu \rangle^-$ converges for one non-zero measure ν with compact support then ${}^-U^{\mu_n}$ converges uniformly on any bounded set in $\mathbf{R}^p$. Furthermore there exists a constant $c \ge 0$ such that

$$(13) \qquad \lim_{n\to\infty} {}^-U^{\mu_n} = {}^-U^\mu + c \quad \text{on} \quad \mathbf{R}^p.$$

LEMMA 1. *Let μ and ν be two probability measures in M. Suppose $U^\nu + c \le U^\mu$ on $\mathbf{R}^p$ where $c \ge 0$. Then $c = 0$.*

Proof. Let D be a ball of radius r centered at the origin. Let $\chi_D \mu = \mu_1$, $\chi_D \nu = \nu_1$, $\chi_D c\mu = \mu_2$, $\chi_D c\nu = \nu_2$. Let λ and σ be the harmonic measures on

∂D for μ_1 and ν_1 respectively. Then

$$U^{\sigma}(0) + U^{\nu_2}(0) + c \leq U^{\lambda}(0) + U^{\mu_2}(0)$$

or

$$c \leq U^{\lambda}(0) - U^{\sigma}(0) + U^{\mu_2}(0) - U^{\nu_2}(0)$$

or

$$c \leq k_r(\mu(D) - \nu(D)) + U^{\mu_2}(0) - U^{\nu_2}(0),$$

where $k_r = k(x)$, $|x| = r$. Since μ and ν are in M, $U^{\mu_2}(0) \to 0$ and $U^{\nu_2}(0) \to 0$ as $r \to \infty$. For $p > 2$, $k_r \to 0$ as $r \to \infty$. Thus $c = 0$ in this case. (We note that the fact that μ and ν have the same total mass is not needed for $p > 2$.) For $p \leq 2$,

$$| k_r(\mu(D) - \nu(D)) |$$
$$= | k_r(\nu(D^c) - \mu(D^c)) | \leq | U^{\mu_2}(0) | + | U^{\nu_2}(0) |, \quad r \geq 1.$$

Thus again $c = 0$ so Lemma 4 is proved.

COROLLARY 1. *The same proof shows that U^{μ} and U^{ν} must be equal outside any ball containing their supports. (Since then $U^{\sigma}(0) = U^{\lambda}(0)$, and hence $U^{\sigma} = U^{\lambda}$.)*

COROLLARY 2. *The same proof also shows that if μ and ν are any measures in M with $U^{\mu} \geq U^{\nu}$ on $\mathbf{R}^p$, $p > 2$, then $\mu(\mathbf{R}^p) \geq \nu(\mathbf{R}^p)$.*

By combining Lemma 1 with the previous remarks on convergence we see that if μ_n and ν are probability measures in M such that (10) holds, and also $U^{\mu_n} \geq U^{\nu}$ on $\mathbf{R}^p$, then there exists a probability measure μ in M such that

$$\lim_{n\to\infty} \langle \mu_n, \gamma \rangle = \langle \mu, \gamma \rangle \quad \text{for all good } \gamma. \tag{14}$$

LEMMA 2. *Let μ_n, ν be probability measures in M. Suppose $U^{\mu_n} \geq U^{\nu}$ on $\mathbf{R}^p$ and $\lim_{n\to\infty} U^{\mu_n}(x)$ exists* a.e. dx. *Then there exists a probability measure μ in M such that* (14) *holds, and such that* $\lim_{n\to\infty} U^{\mu_n}(x) = U^{\mu}(x)$ a.e. dx.

Proof. A good measure $\bar{\gamma}$ will be called an especially good measure if there exists a set A such that U^{μ_n} is uniformly bounded on A and $\bar{\gamma}(A^c) = 0$. For any $\varepsilon > 0$, and any good measure γ, we can find an especially good measure $\bar{\gamma}$ such that $\| \gamma - \bar{\gamma} \| < \varepsilon$ and (using (6)) such that $| {}^+U^{\gamma} - {}^+U^{\bar{\gamma}} | < \varepsilon$ on $\mathbf{R}^p$. The lemma then follows easily.

Lemma 2 is the convergence result we will need later. We now note a "domination principle".

LEMMA 3. *Let λ be a measure in M with U^{λ} continuous on $\mathbf{R}^p$. Let W be a continuous superharmonic function on $\mathbf{R}^p$ such that*

(i) *$W \geq U^{\lambda}$ on the support of λ and*

(ii) $\liminf_{|x|\to\infty} (W(x) - U^{\lambda}(x)) \geq 0$.

Then $W \geq U^{\lambda}$ on $\mathbf{R}^p$. If $p > 2$ then (ii) *can be replaced by*

(iii) $\liminf_{|x|\to\infty} W(x) \geq 0$.

Proof. The first statement is a trivial consequence of the minimum principle. As to the second statement, we note first that λ can be assumed to have compact support. But then the first statement applies. (For a stronger domination theorem, cf. [3, page 184].)

We will require a well-known result, connecting stopping times and second moments, which we state as:

LEMMA 4. *Let μ be a probability measure in M. Let $\{B_t\}$ be Brownian motion with initial distribution μ. Let s be a stopping time with finite expectation. Let ν be the distribution of B_s. Then $\int x^2\nu(dx) = \int x^2\mu(dx) + E(s)$. Also ν is in M and $U^\nu \leq U^\mu$ on $\mathbf{R}^p$.*

LEMMA 5. *Let μ and ν be measures in M. Suppose $U^\mu \geq U^\nu$ on $\mathbf{R}^p$ and $\mu(\mathbf{R}^p) = \nu(\mathbf{R}^p)$. Then*

$$\int x^2\nu\,(dx) = \int x^2\mu\,(dx) + c_p \int (U^\mu(x) - U^\nu(x))\,dx, \tag{15}$$

where c_p is a constant (= Laplacian x^2/*flux of a unit mass*).

Proof. The idea of the proof is to approximate $-x^2$ by functions each of which differs from a potential by at most a constant. This is done by considering the potential of a uniform mass density on a sequence of bounded balls. We give the proof for $p = 2$. The other cases are similar and simpler.

Choose $a > 0$. Let $f_a(x) = 2/\pi$ for $|x| \leq a$, $f_a(x) = 0$ for $|x| > a$. Let $\sigma_a = f_a\,dx$. Let $V_a = U^{\sigma_a} - a^2 + 2a^2 \log a$. We find $V_a(x) = -x^2$ for $|x| \leq a$ and $V_a(x) = -2a^2 \log|x| - a^2 + 2a^2 \log a$ for $|x| > a$. It follows that $V_a(x)$ is a decreasing function of a for fixed x, converging to $-x^2$ as a goes to ∞.

$$\begin{aligned}\int V_a\,d\nu &= \int V_a\,d\mu + \int V_a\,d\nu - \int V_a\,d\mu \\ &= \int V_a\,d\mu + \langle \nu, \sigma_a\rangle - \langle \mu, \sigma_a\rangle \\ &= \int V_a\,d\mu + \int (U^\nu(x) - U^\mu(x))\sigma_a\,(dx).\end{aligned}$$

Letting a go to ∞ the lemma follows, with $c_p = 2/\pi\,(= 4/2\pi)$.

LEMMA 6. *Let μ be a probability measure in M. Let $\{B_t\}$ be Brownian motion with initial distribution μ. Let s be a stopping time with finite expectation. Let ν be the distribution of B_s. Then*

$$E(s) = c_p \int (U^\mu(x) - U^\nu(x))\,dx.$$

Proof. Follows readily from Lemmas 8 and 9.

3. Proof of the theorem

Let μ and ν be probability measures on $\mathbf{R}^p$. If $p > 2$ let (ii) and (iv) of Theorem 1 hold. If $p \leq 2$ let (i), (ii), (iii) and (iv) of Theorem 1 hold. Let $\{B_t\}$ be Brownian motion with initial distribution μ. We shall construct a stopping time r such that $\nu =$ distribution B_r, and such that

$$E(r) = c_p \int (U^\mu(x) - U^\nu(x))\, dx.$$

This will prove the theorem.

r will be constructed by a method of exhaustion, as a limit of stopping times $s(n)$. First let g be a fixed continuous function on $\mathbf{R}^p$, such that $g > 0$ on $\mathbf{R}^p$, such that if $m = g\, dx$ then $\langle \sigma, m \rangle$ exists for all σ in M, and such that if $\langle \sigma(n), \gamma \rangle$ converges for some uniformly bounded sequence $\sigma(n)$ in M and all good measures γ, then $\langle \sigma(n), m \rangle$ converges. Such a function g can be seen to exist, by considering (7) and (8). The measure m will serve as a gauge, to measure the progress made in constructing r.

Let $\mathcal{S}$ denote the collection of stopping times s having finite expectation such that if $\sigma =$ distribution B_s then $U^\sigma \geq U^\nu$ on $\mathbf{R}$.

Let $s(0) = 0, \sigma(0) = \mu$. Having defined $s(0), \cdots, s(n), \sigma(0), \cdots, \sigma(n), \alpha(1), \cdots, \alpha(n)$, let

$$\alpha(n+1) = \sup \{ \langle \sigma(n), m \rangle - \langle \sigma, m \rangle \} \tag{1}$$

where the supremum is over all $\sigma =$ distribution B_s for s in $\mathcal{S}$ with $s \geq s(n)$. Choose $s(n+1)$ in $\mathcal{S}$, $s(n+1) \geq s(n)$, such that if $\sigma(n+1) =$ distribution $B_{s(n+1)}$ then

$$\langle \sigma(n), m \rangle - \langle \sigma(n+1), m \rangle \geq \alpha(n+1) - 1/(n+1). \tag{2}$$

This defines $s(n)$, $\sigma(n)$ for $n \geq 0$ and $\alpha(n)$ for $n \geq 1$. Clearly

$$\begin{aligned} \langle \mu, m \rangle - \langle \nu, m \rangle &\geq \langle \mu, m \rangle - \langle \sigma(n), m \rangle \\ &\geq \alpha(1) + \cdots + \alpha(n) - (1 + \cdots + 1/n). \end{aligned}$$

Hence

$$\lim_{n\to\infty} \alpha(n) = 0. \tag{3}$$

Let $r = \lim_{n\to\infty} s(n)$. Let $\lambda =$ distribution B_r. For $p > 2$ it is clear that

$$\lim_{n\to\infty} \int f\, d\sigma(n) = \int f\, d\lambda \quad \text{for all } f \in \mathcal{C}_0(\mathbf{R}^p). \tag{4}$$

If $p \leq 2$, we note that $\langle \sigma(n), \gamma \rangle$ converges and hence is bounded for any good measure γ. If $r = \infty$ held with positive probability a sequence of stopping times $\bar{s}(n)$ would exist such that $\bar{s}(n) \leq s(n)$ and $|B_{\bar{s}(n)}| \to \infty$ with positive probability. But then $\langle \sigma(n), \gamma \rangle \leq \langle \bar{\sigma}(n), \gamma \rangle \to -\infty$, a contradiction. Thus r is finite almost everywhere. Hence (4) holds for $p \leq 2$ also.

It follows from Lemma 2 that λ is a probability measure on $\mathbf{R}^p$ and $\lim_{n\to\infty} U^{\sigma(n)}(x) = U^{\lambda}(x)$ a.e. dx. Hence $U^{\lambda} \geq U^{\nu}$ on $\mathbf{R}^p$.

Let $G = \{x \mid U^{\lambda}(x) > U^{\nu}(x)\}$. Since $U^{\lambda} - U^{\nu}$ is lower semicontinuous G is open. We claim

$$\lambda(G) = 0. \tag{5}$$

If (5) does not hold then there must exist an open ball $Q \subseteq G$ and a number c such that $U^{\lambda} > c > U^{\nu}$ on a neighborhood of $\bar{Q}$, and $\lambda(Q) > 0$. Let $\bar{s}(n)$ = first time of exit from Q after $s(n)$. Let $\bar{\sigma}(n)$ = distribution $B_{\bar{s}(n)}$. Let s = first time of exit from Q after r. Let σ = distribution B_s. Clearly

$$\langle \lambda, m \rangle > \langle \sigma, m \rangle. \tag{6}$$

But also $U^{\bar{\sigma}(n)} \geq U^{\nu}$ on $\mathbf{R}^p$, so $\bar{s}(n)$ is in $\mathcal{S}$. Hence

$$\alpha(n+1) \geq \langle \sigma(n), m \rangle - \langle \bar{\sigma}(k), m \rangle \quad \text{for all } k \geq n.$$

Thus

$$\alpha(n+1) \geq \langle \sigma(n), m \rangle - \langle \sigma, m \rangle \quad \text{for all} \quad n. \tag{7}$$

Since $\langle \sigma(n), m \rangle \geq \langle \lambda, m \rangle$ for all n, (6) and (7) contradict (3). Thus (5) must hold.

Let $F = \mathbf{R}^p - G$. F supports λ and $U^{\lambda}|_F = U^{\nu}|_F$ is continuous. It follows easily from the theorem of Evans and Vasilesco (cf. [4, page 117]) that U^{λ} is everywhere continuous. By Lemma 3 we can now conclude that $U^{\nu} \geq U^{\lambda}$ on $\mathbf{R}^p$. Hence $U^{\nu} = U^{\lambda}$, so $\nu = \lambda$. Since

$$\begin{aligned} E(r) = \lim_{n\to\infty} E(s(n)) &= \lim_{n\to\infty} c_p \int (U^{\mu}(x) - U^{\sigma(n)}(x))\, dx \\ &= c_p \int (U^{\mu}(x) - U^{\nu}(x))\, dx \end{aligned}$$

the theorem is proved.

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University of Minnesota
Minneapolis, Minnesota