LOCAL QUASI-ISOMORPHISMS OF TORSION FREE ABELIAN GROUPS

BY

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1. Introduction

In 1945, B. Jonsson [14] gave an example of a torsion free abelian group[2] of finite rank which has two non-isomorphic decompositions into direct sums of indecomposable subgroups. Many later examples have appeared, showing that direct decomposition of finite rank torsion free groups is highly non-unique [8], [13], [15]. In 1959, Jonsson [16] introduced the notion of quasi-isomorphism: two finite rank torsion free groups A and B are quasi-isomorphic if there exist isomorphic subgroups S and T of A and B respectively, such that A/S and B/T are bounded. He showed that if the concept of isomorphism is replaced by that of quasi-isomorphism, one does have a Krull-Schmidt theorem for torsion free groups of finite rank. Further results were obtained by Reid [17] and [18]. In another significant application of this concept, Beaumont and Pierce [4] found complete sets of invariants for torsion free groups of rank two, up to quasi-isomorphism.

The concept of quasi-isomorphism was soon extended to include all abelian groups. One extension was made by deleting the words "finite rank" from the definition above. This concept was utilized extensively by Beaumont and Pierce in their important work on torsion free rings [2], [3], [5], and in investigations of p-groups [6] and [7]. Another extension was made by E. Walker [19], who proved two finite rank torsion free groups are quasi-isomorphic if and only if they are isomorphic in the quotient category $\mathcal{A}/\mathcal{B}$ where $\mathcal{A}$ is the category of abelian groups and $\mathcal{B}$ is the class of all bounded groups. This generalization agrees with that of Beaumont and Pierce for torsion free groups, but weakens the equivalence relation for torsion groups. This concept has the advantage of a natural setting where tools of category theory and homological algebra are available. Many results have been contained concerning isomorphism of groups in $\mathcal{A}/\mathcal{B}$ [9], [12], [19].

In order to generalize theorems concerning decomposability and separability of torsion free abelian groups, a different concept was needed. Both of the generalizations above have the property that there are sequences $\{A_n\}$ and $\{B_n\}$ of groups for which A_n and B_n are quasi-isomorphic for each n, but the direct sums $\sum_{n=1}^{\infty} A_n$ and $\sum_{n=1}^{\infty} B_n$ fail to be quasi-isomorphic. Fuchs and Viljoen [11] have given the following definition, which agrees with the definition of quasi-isomorphism for finite rank torsion free groups.

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[2] In this paper the word *group* will always mean *abelian group*.

DEFINITION (Fuchs and Viljoen). Torsion free groups A and B are *locally quasi-isomorphic* if there is an embedding of A and B as subgroups of a torsion free divisible group D so that for every finite rank summand F of D, there exist non-zero integers m and n such that $m(A \cap F) \subset B \cap F$ and $n(B \cap F) \subset A \cap F$.

Fuchs and Viljoen show that $\sum_{i \in I} A_i$ is locally quasi-isomorphic to $\sum_{i \in I} B_i$ whenever A_i and B_i are locally quasi-isomorphic for all $i \in I$. They define locally quasi-separable groups and prove that for certain types of local quasi-decompositions, summands of locally quasi-separable groups are locally quasi-separable and summands of locally completely decomposable groups are locally completely decomposable. They also obtain isomorphic refinement theorems for these special local quasi-decompositions, and they prove that countable locally quasi-separable groups are completely locally quasi-decomposable. The restriction they place on the decompositions stems from the rather awkward fact that two groups can be locally quasi-equal but have different local quasi-endomorphisms.

The purpose of this note is to remove this barrier, and subsequently remove the restriction on the local quasi-decompositions in Fuchs and Viljoen's theorems. This will be done by constructing an additive category whose objects are torsion free abelian groups, where isomorphism in the category is the same as local quasi-isomorphism. The endomorphism ring of a torsion free abelian group in this category contains Fuchs and Viljoen's local quasi-endomorphisms. Their various decomposition theorems will then follow from general categorical theorems.

2. A categorical setting for local quasi-isomorphism

The definition of local quasi-isomorphism involves an embedding of the groups into a divisible group. We first find a description of local quasi-isomorphism that involves only the groups themselves.

2.1. DEFINITION. Let $A, B \subset H$, with H a torsion free group. Then A is *locally quasi-contained in* B (written $A \mathrel{\dot{\subset}} B$) if for every finite rank subgroup F of A there exists a nonzero integer n for which $nF \subset B$. The subgroups A and B are *locally quasi-equal* if both $A \mathrel{\dot{\subset}} B$ and $B \mathrel{\dot{\subset}} A$ (written $A \doteq B$).

2.2 THEOREM. *Let G and H be torsion free. Then G is locally quasi-isomorphic to H and if and only if there are subgroups A and B of G and H respectively, such that $A \doteq G$, $B \doteq H$ and $A \cong B$.*

Proof. Suppose G is locally quasi-isomorphic to H. Then there is a torsion free divisible group D with embeddings $H \subset D$ and $G \subset D$ satisfying the condition of the definition. Let $A = B = G \cap H$. Let F be a finite rank subgroup of H, and let $\bar{F}$ be the divisible hull of F inside D. Then $\bar{F}$ is a finite rank direct summand of D, so there is a non-zero integer n for which

$$n(\bar{F} \cap H) \subset \bar{F} \cap G.$$

Now $nF \subset H$ and also

$$nF \subset n(\bar{F} \cap H) \subset \bar{F} \cap G \subset G,$$

so we have $nF \subset H \cap G = B$. Now let F be a finite rank subgroup of G, and by the same reasoning as above, there exists a non-zero integer m for which $mF \subset G \cap H = A$.

For the converse, suppose A and B are subgroups of G and H respectively, satisfying the condition of the theorem. Let $H \subset D$ with D torsion free divisible and let E be a finite rank direct summand of D. Let $f : G \to D$ be an extension of the isomorphism $A \to B$. Then f is a monomorphism, since $\operatorname{Ker} f \cap A = 0$ and A is an essential subgroup of G. Let H' be the image of f. Then $G \cong H'$. Now $E \cap H$ is a finite rank subgroup of H, so there is a non-zero integer n for which $n(E \cap H) \subset B \subset H'$. Thus

$$n(E \cap H) \subset E \cap H'.$$

Also $f^{-1}(E \cap H')$ is a finite rank subgroup of G, so there is a non-zero integer m for which $m(f^{-1}(E \cap H')) \subset A$, and thus $m(E \cap H') \subset B \subset H$. Thus we have

$$m(E \cap H') \subset E \cap H,$$

and G and H are locally quasi-isomorphic.

For a torsion free group B, let $F(B)$ denote the set of subgroups of B which are locally quasi-equal to B. We need the following facts about $F(B)$ in order to define the categorical setting for local quasi-isomorphism. The proof of the following lemma is very easy.

2.3 Lemma. *Let B be a torsion free group. The set $F(B)$ has the following properties.*

(1) $B \in F(B)$.

(2) *If $A \in F(B)$ and $A \subset C \subset B$, then $C \in F(B)$.*

(3) *If $A, C \in F(B)$, then $A \cap C \in F(B)$.*

(4) *If $f : B \to C$ is a homomorphism, with C torsion free, and $A \in F(C)$, then $f^{-1}(A) \in F(B)$.*

The following construction is similar to the construction of quotient categories using Serre classes. It does not arise from a Serre class however, since the quotient B/A does not determine whether or not $A \doteq B$.

2.4 Definition. Let $\mathcal{L}$ have as objects all torsion free groups. For A, $B \in \mathcal{L}$,

$$\hom_{\mathcal{L}}(A, B) = \operatorname{inj}\lim_{S \in F(A)} \operatorname{Hom}_Z(S, B).$$

If $S \in F(A)$, $f : S \to B$, write $[f]$ for the equivalence class of f in $\hom_{\mathcal{L}}(A, B)$. For $[f] \in \hom_{\mathcal{L}}(A, B)$ and $[g] \in \hom_{\mathcal{L}}(B, C)$ define the composition $[g][f]$ to be $[h]$ where h is obtained as the composition

$$f^{-1}(\operatorname{Dom} g) \xrightarrow{f} \operatorname{Dom} g \xrightarrow{g} C.$$

It follows quickly from Lemma 2.3 that the definition above yields a category. We will show that $\mathcal{L}$ is additive, has infinite sums, kernels, and finite intersections.

2.5 LEMMA. *Let* $[f], [g] \in \hom_{\mathcal{L}}(A, B)$. *Then* $[f] = [g]$ *if and only if* f *and* g *agree on the intersection of their domains.*

Proof. If $[f] = [g]$, there is an $S \subset \operatorname{Dom} f \cap \operatorname{Dom} g$ on which f and g agree, with $S \doteq A$. But A/S is torsion. Thus, letting f' and g' denote the restrictions of f and g, respectively, to $\operatorname{Dom} f \cap \operatorname{Dom} g$, $S \subset \operatorname{Ker}(f' - g')$ implies $\operatorname{Im}(f' - g')$ is torsion. Thus $f' - g' = 0$. The converse is clear.

2.6 THEOREM. *Two torsion free groups are locally quasi-isomorphic if and only if they are isomorphic in* $\mathcal{L}$.

Proof. Suppose G is locally quasi-isomorphic to H. Then by 2.2 there are subgroups A and B of G and H respectively, with $A \doteq G$, $B \doteq H$ and $A \cong B$. Let $f : A \to B$ and $g : B \to A$ be inverses of each other and $i : A \to G$, $j : B \to H$ the inclusions. Then $[jf]$ and $[ig]$ are inverses of each other in $\mathcal{L}$.

Now suppose $G \cong_{\mathcal{L}} H$. Let $[f] : G \to H$ and $[g] : H \to G$ be inverses of each other in $\mathcal{L}$, with $f : A \to H$ and $g : B \to G$. Let $A' = f^{-1}(B)$ and $f' = f \mid A'$. Then $[f'] = [f]$ and it follows from 2.5 that gf' is the inclusion $A' \to G$. Let $B' = \operatorname{Im} f'$. Then $A' \cong B'$ and $A' \doteq G$. Now the restriction g' of g to $g^{-1}(A)$ is one-to-one since the composition $fg' : g^{-1}(A) \to A \to H$ is the inclusion map.

Since $g^{-1}(A)$ is an essential subgroup of G, we also have g one-to-one. It follows that $B' = g^{-1}(A)$ and thus $B' \doteq H$. Thus G is locally quasi-isomorphic to H.

An additive category is a category $\mathcal{C}$ together with an abelian group structure on each of its morphism sets, subject to the following condition:

(i) Composition is bilinear, i.e., if $\alpha, \beta \in \hom_{\mathcal{C}}(A, B)$ and $\gamma, \omega \in \hom_{\mathcal{C}}(B, C)$, then $\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$ and $(\gamma + \omega)\alpha = \gamma\alpha + \omega\alpha$.

2.7 PROPOSITION. *The category* $\mathcal{L}$ *is additive.*

2.8 PROPOSITION. *A map* $[f] \in \hom_{\mathcal{L}}(A, B)$ *is monic if and only if* f *is one-to-one. The map* $[f]$ *is epic if and only if* $\operatorname{Cok} f$ *is torsion.*

Proof. Suppose f is one-to-one and $f : S \to B$, $g : T \to A$. Let

$$T' = g^{-1}(S), \quad \text{and} \quad g' = g \mid T'.$$

Then $[fg'] = [0]$ implies $fg' = 0$. Thus $g' = 0$ and $[g] = [g'] = [0]$, implying $[f]$ is monic.

Suppose $\operatorname{Ker} f \neq 0$. Let $g : \operatorname{Ker} f \to A$ be the inclusion. Then $[g] \neq [0]$ but $[f][g] = [0]$, implying $[f]$ is not monic.

Now suppose $\operatorname{Cok} f$ is torsion and $[g][f] = [0]$ with $g : K \to C$, $K \doteq B$. Let $S' = f^{-1}(K)$, and $f' = f \mid S'$. Then $gf' = 0$. Thus g can be factored through $K/\mathrm{Im}\, f'$. But since both S/S' and $\mathrm{Cok}\, f$ are torsion, we see that $K/\mathrm{Im}\, f'$ is also torsion. Since C is torsion free, we have $g = 0$ and hence $[g] = [0]$. Thus $[f]$ is epic.

Now suppose $\mathrm{Cok}\, f$ is not torsion. Let $T/\mathrm{Im}\, f = t(B/\mathrm{Im}\, f)$. Then B/T is non-zero and torsion-free. Let $g : B \to B/T$ be the quotient map. Then $[g] \neq [0]$ but $[g][f] = [0]$. Thus $[f]$ is not epic.

We can see easily from this result that the inclusion $Z \to Q$ is both monic and epic in $\mathcal{L}$. It is not an isomorphism in $\mathcal{L}$, however, since Z and Q are not quasi-isomorphic and hence not locally quasi-isomorphic. In particular, the category $\mathcal{L}$ is not abelian.

2.9 PROPOSITION. *The category $\mathcal{L}$ has kernels, in fact* $\ker[f] = [\ker f]$.

Proof. Let $[f] \in \hom_{\mathcal{L}}(A, B), f : S \to B$ with $S \doteq A$. Let $\ker f : \mathrm{Ker}\, f \to A$ be the inclusion map. Then $[\ker f]$ is monic, by 2.8. Suppose

$$[g] \in \hom_{\mathcal{L}}(G, A)$$

with $[f][g] = [0]$. Then $g : T \to A$ with $T \doteq G$. Let $T' = g^{-1}(S)$ and $g' = g \mid T'$. Then $fg' = 0$. Thus there exists a homomorphism $h : T' \to \mathrm{Ker}\, f$ for which $(\ker f)h = g'$. But $T' \doteq G$, so we have

$$[h] \in \hom\mathcal{L}(G, \mathrm{Ker}\, f) \quad \text{and} \quad [g] = [g'] = [(\ker f)h] = [\ker f][h],$$

as desired.

2.10 PROPOSITION. *The category $\mathcal{L}$ has intersections.*

Proof. Let $[f] : A \to C$ and $[g] : B \to C$ be monics in $\mathcal{L}$. Then $f : A' \to C$ and $g : B' \to C$ with $A' \doteq A$ and $B' \doteq B$. Let $I = \mathrm{Im}\, f \cap \mathrm{Im}\, g$ with $i : I \to C$ the inclusion. Then there are homomorphisms f and g for which the diagram

$$\begin{tikzcd} A' \arrow[r, "f"] & C \\ I \arrow[u] \arrow[r] \arrow[ur, "i"] & B' \arrow[u, "g"'] \end{tikzcd}$$

commutes. Suppose there is a commutative diagram in $\mathcal{L}$ of subobjects

$$\begin{tikzcd} A \arrow[r, "{[f]}"] & C \\ E \arrow[u, "{[h]}"] \arrow[r, "{[k]}"] & B \arrow[u, "{[g]}"'] \end{tikzcd} .$$

Let $E' = h^{-1}(A') \cap k^{-1}(B') \cap \mathrm{Dom}\ k \cap \mathrm{Dom}\ h$. Then $E' \doteq E$ and if

$h' = h \mid E'$ and $k' = k \mid E'$, we have $fh' = gk'$. It follows that

$$\operatorname{Im} fh' = \operatorname{Im} gk' \subset \operatorname{Im} f \cap \operatorname{Im} g = I,$$

so there is a homomorphism $s : E' \to I$ for which $is = fh' = gk'$, i.e., I is the greatest lower bound of A and B, or $I = A \cap_{\mathfrak{L}} B$.

2.11 Lemma. *Let $\{B_\alpha\}_{\alpha \in I}$ be a family of torsion free groups, $A_\alpha \subset B_\alpha$, and $A_\alpha \doteq B_\alpha$ for each $\alpha \in I$. Then $\sum_{\alpha \in I} A_\alpha \doteq \sum_{\alpha \in I} B_\alpha$.*

Proof. Let F be a finite rank subgroup of $\sum_{\alpha \in I} B_\alpha$. Then there is a finite subset J of I with $F \subset \sum_{\alpha \in J} B_\alpha$. Let F_α be the image of the projection of F into B_α. Then $F \subset \sum_{\alpha \in J} F_\alpha$. There is a non-zero integer n_α for which $n_\alpha F_\alpha \subset A_\alpha$. Let $n = \prod_{\alpha \in J} n_\alpha$. Then

$$nF \subset n\left(\sum_{\alpha \in J} F_\alpha\right) \subset \sum_{\alpha \in J} A_\alpha \subset \sum_{\alpha \in I} A_\alpha.$$

2.12 Proposition. *The category $\mathfrak{L}$ has arbitrary direct sums.*

Proof. Let $\{A_\alpha\}_{\alpha \in I}$ be a family of torsion free groups, $\sum_{\alpha \in I} A_\alpha$ their direct sum, with injection maps $i_\beta : A_\beta \to \sum_{\alpha \in I} A_\alpha$. We will show $\sum_{\alpha \in I} A_\alpha$ with the maps

$$[i_\beta] : A_\beta \to \sum_{\alpha \in I} A_\alpha$$

is a direct sum in $\mathfrak{L}$. Let B be a torsion-free group and $[f_\alpha] : A_\alpha \to B$ a family of maps. Let $S_\alpha = \operatorname{Dom} f_\alpha$. Then $\sum_{\alpha \in I} S_\alpha \doteq \sum_{\alpha \in I} A_\alpha$ by 2.11. There exists a homomorphism $f : \sum_{\alpha \in I} S_\alpha \to B$ such that $fi'_\alpha = f_\alpha$ for each $\alpha \in I$, where i'_α denotes the restriction of i_α to S_α. This gives us

$$[f][i_\alpha] = [f][i'_\alpha] = [f_\alpha] \quad \text{for each} \quad \alpha \in I.$$

It follows easily from 2.5 that $[f]$ is unique. Thus we have a direct sum in $\mathfrak{L}$.

The definition of S-separable objects made in the next section requires the notion of a small object. See [20] or any book on category theory for further discussion of this notion.

2.13 Definition. An object S in a category $\mathfrak{C}$ is *small* if every map $S \to \sum_{\alpha \in I} A_\alpha$ into a direct sum factors through a finite direct sum

$$\sum_{i=1}^{n} A_{\alpha_i} \subset \sum_{\alpha \in I} A_\alpha.$$

This is equivalent to saying that the functor $\operatorname{Hom}_{\mathfrak{C}}(S, \ )$ commutes with arbitrary direct sums. Note that in the category $\mathfrak{L}$, a group is small if and only if it has finite rank.

2.14 Proposition. *For every object $G \in \mathfrak{L}$ there is a map $\sum_{\alpha \in I} P_\alpha \to G$ which is both monic and epic, with each P_α small and projective in $\mathfrak{L}$.*

Proof. The additive group Z of integers is small since it has finite rank. Let $[f] : A \to B$ be epic. Then $f : A' \to B$ with $A' \doteq A$, and $B/f(A')$ is torsion. Let $[g] : Z \to B$. Then $g : nZ \to B$ for some positive integer n.

Moreover, $mg(nZ) \subset f(A')$ for some positive integer m. Since

$$g : mnZ \to f(A'),$$

and mnZ is projective as a group, there is a map $h : mnZ \to A'$ for which $fh = g$ on mnZ. This gets $[h] : Z \to A$ for which $[f][h] = [g]$. Thus Z is projective in $\mathcal{L}$. It follows that all free groups are projective in $\mathcal{L}$. The proposition now follows immediately from 2.8, since every torsion free group contains a fiee subgroup for which the quotient is torsion.

The following definition is due to Warfield [20]. It is akin to the idea of an object being the least upper bound of its small subobjects.

2.15 Definition. An object M in an additive category is *finitely approximable* if for any object L and any morphism $f : L \to M$, f is an isomorphism if and only if (i) f is monic, and (ii) for any small object S and any morphism $g : S \to M$, there is a morphism $h : S \to L$ such that $g = fh$.

2.16 Proposition. *Every object in $\mathcal{L}$ is finitely approximable. If A and B are small and $f : A \to M$, $g : B \to M$ are maps, there is a small object C with maps $h : C \to M$, $j : A \to C$, $k : B \to C$ such that $hj = f$ and $hk = g$. If $A \subset B$ and B is small, then A is small.*

Proof. Let $M \in \mathcal{L}$, $[f] : L \to M$, and suppose (i) and (ii) hold for $[f]$. Let F be a finite rank subgroup of M. Then there is a map $[h] : F \to L$ such that $[g] = [f][h]$ where $g : F \to M$ is the inclusion. It follows that $nF \subset \operatorname{Im} f$ for some positive integer n and thus that $\operatorname{Im} f \doteq M$. Thus $[f]$ is an isomorphism. The remainder of the proposition is clear.

We observed above that a torsion free group has finite rank if and only if it is a small object in $\mathcal{L}$. Actually a much stronger statement can be made. Rank is an invariant in $\mathcal{L}$, that is, if G and H are locally quasi-isomorphic torsion free groups, then they have the same rank. To see this, recall that locally quasi-isomorphic groups G and H have isomorphic subgroups A and B with $A \doteq G$ and $B \doteq H$. Since, in particular, G/A and H/B are torsion, one has rank G = rank A = rank B = rank H.

The finitely generated groups are distinguished in $\mathcal{L}$ by the fact that they are precisely the small projectives. It is of interest to describe the class of projectives. A group is $\aleph_1$*-free* if every subgroup having fewer than $\aleph_1$ elements is free.

2.17 Proposition. *A torsion free group G is locally quasi-isomorphic to a free group if and only if G is $\aleph_1$-free. A group is projective in $\mathcal{L}$ if and only if it is $\aleph_1$-free.*

Proof. Suppose G is locally quasi-isomorphic to a free group H. Then G and H have isomorphic subgroups A and B with $A \doteq G$ and $B \doteq H$. Now A, being isomorphic to a subgroup of a free group, is free. Let F be a finite rank subgroup of G. Then there is a nonzero integer n for which $nF \subset A$. Thus nF is free. It follows that F is free, and thus that G is $\aleph_1$-free.

Now suppose that G is $\aleph_1$-free. Let S be a free subgroup of G for which G/S is torsion. Let F be a finite rank subgroup of G. Then F is free and hence finitely generated. Thus there is a nonzero integer n for which $nF \subset S$. Thus $S \doteq G$.

Suppose G is projective in $\mathfrak{L}$. Let $f : F \to G$ be an inclusion map with F free and G/F torsion. Then $[f]$ being epic, there is a map $[g]: G \to F$ with $[f][g] = [1_G]$. Now $g : G' \to F$ with $G' \doteq G$, and $fg = 1_{G'}$ implies g is a one-to-one and hence that G' is free. It follows that G is $\aleph_1$-free. The fact that $\aleph_1$-free groups are projective follows from the facts that Z is projective, and direct sums of projectives are projective, together with the first statement of this proposition.

The following elementary properties of local quasi-decompositions are useful.

2.18 Proposition. *If* $G \subset \sum_{\alpha \in I} A_\alpha$ *and* $G \doteq \sum_{\alpha\in I} A_\alpha$ *then*

$$\sum_{\alpha \in I} (G \cap A_\alpha) \doteq \sum_{\alpha \in I} A_\alpha.$$

Proof. Let F be a finite rank subgroup of A_α. There is a nonzero integer n for which $nF \subset G$, and thus $nF \subset G \cap A_\alpha$. Thus $G \cap A_\alpha \doteq A_\alpha$. The proposition now follows from 2.11.

2.19 Proposition. *Let* $\mathcal{C}$ *be an additive category. If* $A \oplus B = A \oplus C$ *in* $\mathcal{C}$, *then* B *and* C *are isomorphic in the category* $\mathcal{C}$.

Proof. Let

$$A \underset{i_A}{\overset{p_A}{\leftrightarrows}} A \oplus B \underset{i_B}{\overset{p_B}{\rightleftarrows}} B \quad \text{and} \quad A \underset{e_A}{\overset{\pi_A}{\leftrightarrows}} A \oplus C \underset{e_C}{\overset{\pi_C}{\rightleftarrows}} C$$

be the injection and projection maps, and

$$f : A \oplus B \to A \oplus C \quad \text{and} \quad g : A \oplus C \to A \oplus B$$

identity maps, so $fi_A = e_A$ and $ge_A = i_A$. Now $p_B\, ge_A = p_B\, gfi_A = p_B\, i_A = 0$. Thus

$$\begin{aligned}(p_B\, ge_C)(\pi_C\, fi_B) &= p_B\, ge_C\, \pi_C\, fi_B + p_B\, ge_A\, \pi_A\, fi_B \\ &= p_B\, g(e_C\, \pi_C + e_A\, \pi_A) fi_B = p_B\, gfi_B = p_B i_B = 1_B.\end{aligned}$$

Similarly $(\pi_C\, fi_B)(p_B\, ge_C) = 1_C$.

Before leaving this section, we look at local quasi-endomorphisms. Let G be a torsion-free group and let D be a divisible hull of G. Fuchs and Viljoen define the local quasi-endomorphisms of a torsion free group G to be the set

$$LQE(G) = \{\alpha \in \operatorname{End}(D) \mid G \cap \alpha(G) \doteq \alpha(G)\}.$$

To aid in the comparison we find a similar description for $\hom_{\mathcal{L}}(G, G)$. If $G' \subset G$ and $G' \doteq G$, then G' is an essential subgroup of G and hence also of D. Since D is also torsion free, each homomorphism $G' \to G \subset D$ has a unique extension to an endomorphism of D. Thus to $[\alpha] \in \hom_{\mathcal{L}}(G, G)$ we may associate the unique extension $\alpha : D \to D$. Now $G' \subset \alpha^{-1}(G) \cap G \subset G$ and $G' \doteq G$ imply $\alpha^{-1}(G) \cap G \doteq G$. On the other hand, if $\alpha : D \to D$ is any endomorphism of D for which $\alpha^{-1}(G) \cap G \doteq G$, we can associate with α the restriction of α to $G' = \alpha^{-1}(G) \cap G$ to get $[\alpha] \in \hom_{\mathcal{L}}(G, G)$. This shows that $\hom_{\mathcal{L}}(G, G)$ may be identified with the set

$$L(G) = \{\alpha \in \operatorname{End}(D) \mid \alpha^{-1}(G) \cap G \doteq G\}.$$

In the case G has finite rank, it is easy to see that $LQE(G) = L(G)$, but in general we have only $LQE(G) \subset L(G)$. To verify this inclusion, let $\alpha \in LQE(G)$. Then $\alpha(G) \cap G \doteq \alpha(G)$. If F is a finite rank subgroup of G, $\alpha(F)$ is a finite rank subgroup of $\alpha(G)$ so there is a positive integer n with $n\alpha(F) \subset \alpha(G) \cap G$. Then

$$nF \subset \alpha^{-1}(n\alpha(F)) \subset \alpha^{-1}(\alpha(G) \cap G) \subset \alpha^{-1}(G),$$

and thus $nF \subset \alpha^{-1}(G) \cap G$. This shows $\alpha^{-1}(G) \cap G \doteq G$ and thus that $LQE(G) \subset L(G)$. An example of Fuchs' and Viljoen's shows it may be a proper subset.

In the case G has finite rank, however, all definitions agree. In fact,

$$\begin{aligned} L(G) = LQE(G) = QE(G) &\cong \hom_{\mathcal{L}}(G, G) \\ &= \hom_{\mathcal{A}/\mathcal{B}}(G, G) \cong Q \otimes \operatorname{Hom}_Z(G, G). \end{aligned}$$

(The final isomorphism in this sequence is proved by E. A. Walker in [19].) In the general case, it is true that

$$L(G) = \bigcup_{S \doteq G} LQE(S) = \operatorname{inj}\lim_{S \doteq G} LQE(S).$$

The definition of local quasi-isomorphism can easily be extended to a category with objects all abelian groups. Using the word *rank* to mean *torsion free rank,* simply eliminate the words "torsion free" wherever they appear in 2.1, 2.3 and 2.4. This gets an additive category with kernels. This category fails to have infinite sums. It is interesting to note, however, that two p-groups are isomorphic in this category if and only if they are quasi-isomorphic in the sense of Beaumont and Pierce [6].

3. $\mathcal{S}$-separable objects and locally quasi-separable groups

A torsion free group is *completely decomposable* if it is isomorphic to a direct sum of rank one groups. A torsion free group is *separable* if every finite subset of elements is contained in a completely decomposable summand. The two most interesting facts about separable groups are that countable separable groups are completely decomposable (Baer [1]) and summands of separable groups are separable (Fuchs [10]). Fuchs and Viljoen [11] give a definition of locally quasi-separable groups and prove the analogs of these two theorems. We will make the definition of "$\mathcal{S}$-separable" in an additive category, and prove these theorems in a rather general setting. The notion of $\mathcal{S}$-separable specializes to those of separable and locally quasi-separable for the appropriate categories and classes $\mathcal{S}$ which will be described later.

Let $\mathcal{C}$ be an additive category which satisfies the following axioms.

(i) $\mathcal{C}$ has kernels and infinite direct sums.

(ii) For every object A there is a map $\sum_{\alpha\in I} P_\alpha \to A$ which is epic, with each P_α a small projective. For each small object S there is an epic $P \to S$ with P a small projective.

(iii) Every object in $\mathcal{C}$ is finitely approximable.

(iv) If A and B are small, and $f : A \to M$, $g : B \to M$ for some M in $\mathcal{C}$, then there is a small object S and maps $h : S \to M$, $j : A \to S$, $k : B \to S$ with $hj = f$ and $hk = g$.

(v) Subobjects of small objects are small.

3.1 DEFINITION. Let $\mathcal{S}$ be a class of objects of $\mathcal{C}$. An object M of $\mathcal{C}$ is $\mathcal{S}$-*separable* if every map $P \to M$, with P a small projective, factors through a direct summand of M which belongs to $\mathcal{S}$.

Consider the full (additive) subcategory of the category of abelian groups with objects the torsion free groups, and let $\mathcal{S}$ be the completely decomposable groups. The $\mathcal{S}$-separable objects in this setting are the separable torsion free groups.

3.2 DEFINITION. An object $\mathcal{S}$ of $\mathcal{C}$ is *countable* if there is a map $\sum_{n=1}^{\infty} P_n \to S$ which is epic, with each P_n a small projective.

3.3 LEMMA. *Let*

$$Q \xrightarrow{f} G \quad \textit{and} \quad P \xrightarrow{g} G$$

be maps with Q and P small projectives. Then there is a small projective R and maps

$$R \xrightarrow{h} G, \quad Q \xrightarrow{j} R \quad \text{and} \quad P \xrightarrow{k} R$$

with $hj = f$ and $hk = g$.

Proof. By axiom (iv) there is a small object S with maps

$$S \xrightarrow{h'} G, \quad Q \xrightarrow{j'} S \quad \text{and} \quad P \xrightarrow{k'} S$$

such that $h'j' = f$ and $j'k' = g$. By axiom (ii) there is a small projective R with an epic $t : R \to S$. Since t is epic and Q and P are projective there are maps $j : Q \to R$ and $k : P \to R$ for which $tj = j'$ and $tk = k'$. Let $h = h't$ Then $hj = h'tj = h'j' = f$, and similarly, $hk = g$.

3.4 LEMMA. *Let $G = T \oplus H$ with T small, and let $P \to G$ be a map with P a small projective. Suppose G is $\mathcal{S}$-separable. Then G has a summand S in $\mathcal{S}$ containing T, with the map $P \to G$ factoring through S.*

Proof. Let $Q \to T$ be epic with Q a small projective. There is a small projective R with a commutative diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[dr] & G & P \arrow[l] \arrow[dl] \\ & R & \end{tikzcd} .$$

Then G has a summand S in $\mathcal{S}$ with $R \to G$ factoring through S, say

$$G = T \oplus H = S \oplus K.$$

Let $\pi_K : G \to K$ be the projection with kernel S, and $\pi_T : G \to T$ the projection with kernel H. Now

$$Q \to R \xrightarrow{\pi_T} T$$

is epic, implying $R \to T$ is epic, thus the composition

$$R \xrightarrow{\pi_T} T \xrightarrow{\pi_K} K$$

being 0 implies

$$T \xrightarrow{\pi_K} K$$

is the 0 map. It follows that $T \subset S$.

3.5 THEOREM. *Let $\mathcal{S}$ be a class of small objects of $\mathcal{C}$ which is closed under summands. A countable $\mathcal{S}$-separable object S of $\mathcal{C}$ can be written as a direct sum of objects in $\mathcal{S}$, i.e., is completely $\mathcal{S}$-decomposable.*

Proof. Let $\sum_{n=1}^{\infty} P_n \to S$ be epic with each P_n a small projective. Applying 3.4 repeatedly obtains a chain $T_1 \subset T_2 \subset \cdots$ of summands of S with $P_n \to S$ factoring through the inclusion $T_n \to S$ and with each T_n in $\mathcal{S}$. Let $S_1 = T_1$, and write $T_{n+1} = T_n \oplus S_{n+1}$ for $n = 1, 2, \cdots$. Then $\sum_{n=1}^{\infty} S_n \subset S$, and S_n is in $\mathcal{S}$ for each n. Let K be small and $g : K \to S$. Then there is a small projective P with an epic $P \to K$. We get a commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & K \arrow[d, "g"] \\ \sum_{n=1}^{\infty} P_n \arrow[r] & S \end{tikzcd} ,$$

since P is projective and the lower horizontal map is epic. Since P is small, the map $P \to \sum_{n=1}^{\infty} P_n$ factors through a finite sum $\sum_{n=1}^{m} P_n$. The map

$\sum_{n=1}^{m} P_n \to S$ factors through T_m. Let $\pi_H : S \to H$ be a projection with kernel T_m. Then

$$\left(P \to K \xrightarrow{\pi_H g} H\right) = 0$$

implies $\pi_H g = 0$ since $P \to K$ is epic. It follows that g factors through

$$T_m = \sum_{n=1}^{m} S_n \subset \sum_{n=1}^{\infty} S_n,$$

and thus $\sum_{n=1}^{\infty} S_n = S$, applying axiom (iii).

3.6 COROLLARY. *A countable separable torsion free abelian group is completely decomposable.*

Proof. Let $\mathcal{S}$ be the completely decomposable finite rank groups and $\mathcal{C}$ the category of torsion free groups. See [10] for a proof that $\mathcal{S}$ is closed under summands.

3.7 DEFINITION. An object B in $\mathcal{C}$ has the *exchange property* if the following condition holds.

If $G = B \oplus C = \sum_{i \in I} A_i$, then there exist $A_i' \subset A_i$ such that

$$G = B \oplus \sum_{i \in I} A_i'.$$

The following lemmas appear in [20].

3.8 LEMMA. *Let $\pi_B : B \oplus C \to B$ be the projection map and let*

$$f : S \to B \oplus C$$

be monic. Then $\operatorname{Ker} \pi_B f = S \cap C$. *If $B \subset S$, then $S = B \oplus (S \cap C)$.*

3.9 LEMMA. *If $G \in \mathcal{C}$ and $e \in \operatorname{Hom}_{\mathcal{C}}(G, G)$ is an idempotent, then*

$$G = \operatorname{Ker} e \oplus \operatorname{Ker}(1 - e).$$

3.10. LEMMA. *If S is a small object in $\mathcal{C}$ and the endomorphism ring of S is a local ring, then S has the exchange property.*

3.11 THEOREM. *Let $\mathcal{S}$ be a class of small objects of $\mathcal{C}$ which is closed under summands, and suppose the objects of $\mathcal{S}$ have the exchange property. Then, in the category $\mathcal{C}$, summands of $\mathcal{S}$-separable objects are $\mathcal{S}$-separable.*

Proof. Let A be $\mathcal{S}$-separable, and $A = B \oplus C$. Let $P \to B$ be a map with P a small projective. There is a summand S of A with $S \in \mathcal{S}$ and the map $P \to B \to A$ factoring through S. Say $A = S \oplus T$. Now S has the exchange property, hence there are subobjects $B' \subset B$ and $C' \subset C$ such that

$$A = S \oplus B' \oplus C'.$$

Then $B = B' \oplus ((S \oplus C') \cap B)$. Now let π_S be the projection of A onto S

with kernel $B' \oplus C'$, and let π'_S be the restriction of this map to $B \cap (S \oplus C')$. Then

$$\operatorname{Ker} \pi'_S = [B \cap (S \oplus C')] \cap (C' \oplus B') = B \cap [(S \oplus C') \cap (C' \oplus B')] = B \cap C' = 0.$$

Thus $B \cap (S \oplus C') \subset S$, so that $B \cap (S \oplus C')$ is a summand of S and hence in $\mathcal{S}$. Since $P \to B \to A$ factors through both $B \to A$ and $S \oplus C' \to A$, it factors through their intersection. It follows that B is $\mathcal{S}$-separable.

We now compare these definitions and theorems applied to the category $\mathcal{L}$ with Fuchs and Viljoen's notion of locally quasi-separable.

3.12 Definition. A torsion free group is *locally quasi-separable* if every finitely generated subgroup is quasi-contained in a local quasi-summand of finite rank.

A torsion free group is then locally quasi-separable if and only if it is $\mathcal{S}$-separable in the category $\mathcal{L}$ for the class $\mathcal{S}$ of finite rank groups.

3.13 Definition. A torsion free group G is (*locally*) *quasi-decomposable* if there are nonzero torsion free groups A and B, with G (locally) quasi-isomorphic to $A \oplus B$. If G is not (locally) quasi-decomposable, G is (*locally*) *strongly indecomposable.* A torsion free group G is *completely* (*locally*) *quasi-decomposable* if G is (locally) quasi-isomorphic to a direct sum of strongly indecomposable finite rank torsion free groups.

B. Jonsson showed, among other things, that every finite rank torsion free group is completely quasi-decomposable. Thus a torsion free group is completely locally quasi-decomposable if and only if it is locally quasi-isomorphic to a direct sum of finite rank groups. It is proved in [19] that the quasi-endomorphism ring of a strongly indecomposable finite rank torsion free group is local. It follows from these two facts that all small objects of $\mathcal{L}$ have the exchange property. It follows that both theorems of this section apply to the category $\mathcal{L}$, with $\mathcal{S}$ any class of finite rank groups closed under quasi-summands. In particular:

3.14 Theorem. *A countable locally quasi-separable group is completely locally quasi-decomposable.*

3.15 Theorem. *A local quasi-summand of a locally quasi-separable group is locally quasi-separable.*

If G is locally quasi-isomorphic to a separable torsion free group, then G is locally quasi-separable. Also, of course, completely locally quasi-decomposable groups are locally quasi-separable.

More interesting examples are elusive.

4. A Krull-Schmidt theorem

Our last theorem follows from a theorem of Warfield [20], which we quote.

THEOREM A. *Let $\mathcal{A}$ be an additive category satisfying the following conditions.*

(i) *Any morphism has a kernel.*

(ii) *Infinite direct sums (coproducts) exist.*

(iii) *If $M = \sum_{i \in I} M_i$ with associated injections e_i, and $f : M \to X$ is a map such that $\operatorname{Ker}(\sum_{j \in J} fe_j) = 0$ for all finite $J \subset I$, then f is monic.*

Let M be an object in $\mathcal{A}$ with $M = \sum_{i \in I} M_i$, where each of the M_i has a local ring as its endomorphism ring, and each of the M_i is small. Then:

(a) *Any indecomposable summand of M is isomorphic to one of the M_i.*

(b) *If $M = \sum_{j \in J} N_j$ where the N_j are indecomposable, then there is a bijective map $\phi : I \to J$ such that $M_i \cong N_{\phi(i)}$.*

(c) *If N is any summand of M, N is a direct sum of indecomposable objects.*

(d) *Any two direct decompositions of M have isomorphic refinements.*

Since every object in $\mathcal{L}$ is finitely approximable, the category $\mathcal{L}$ satisfies condition (iii) of Theorem A. The other conditions are satisfied also, by 2.7, 2.9 and 2.12. The finite rank groups which are indecomposable in $\mathcal{L}$ have local endomorphism rings [19]. Thus Theorem A applies to the category $\mathcal{L}$. We state the theorem in the language of local quasi-isomorphisms. See 3.13 and the remarks immediately following that definition.

4.1 THEOREM. *Let M be completely locally quasi-decomposable, that is, M is locally quasi-isomorphic to a direct sum $\sum_{\alpha \in I} M_\alpha$ with each M_α strongly indecomposable of finite rank. Then any strongly indecomposable finite rank local quasi-summand of M is quasi-isomorphic to one of the M_α. If M is locally quasi-isomorphic to $\sum_{\beta \in J} N_\beta$ where each N_β is strongly indecomposable of finite rank, then there is a bijective map $\phi : I \to J$ such that M_α is quasi-isomorphic to $N_{\phi(\alpha)}$. If N is any local quasi-summand of M, N is locally quasi-isomorphic to a direct sum of strongly indecomposable finite rank groups, i.e., N is completely locally quasi-decomposable. Moreover, any two local quasi-decompositions of M have local quasi-isomorphic refinements.*

Examples have been given to show that a summand of a direct sum of finite rank torsion free groups may not be again a direct sum of finite rank groups. By this theorem it must, however, at least contain a locally quasi-equal subgroup which is a direct sum of finite rank groups.

Theorem 4.1 can be generalized by replacing the finite rank summands by groups of countable rank whose $\mathcal{L}$-endomorphism rings are local. The first part of the theorem generalizes to groups of arbitrary rank whose $\mathcal{L}$-endomorphism rings are local. We have no examples of infinite rank groups with local $\mathcal{L}$-endomorphism rings, however.

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