ESSENTIALLY (G_1) OPERATORS AND ESSENTIALLY CONVEXOID OPERATORS ON HILBERT SPACE

BY

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Introduction

Let H be a separable Hilbert space and let $\mathscr{B}(H)$ be all operators (continuous linear transformations) from H into H. Let π be the quotient map from $\mathscr{B}(H)$ onto the Calkin algebra $\mathscr{B}(H)/\mathscr{K}$, where $\mathscr{K}$ denotes all compact operators in $\mathscr{B}(H)$. $T \in \mathscr{B}(H)$ is essentially normal, essentially hyponormal, essentially G_1, or essentially convexoid if $\pi(T)$ is normal, hyponormal, G_1, or convexoid in $\mathscr{B}(H)/\mathscr{K}$, respectively. Denote each of the above sets in $\mathscr{B}(H)$ by $e(\mathscr{N})$, $e(\mathscr{H})$, $e(\mathscr{G})$, and $e(\mathscr{C})$, respectively, where $\mathscr{N}$ is the set of all normal operators on H, $\mathscr{H}$, is the set of all hyponormal operators on H, $\mathscr{G}$ is the set of all operators on H satisfying growth condition G_1 (i.e. $\|(T - z)^{-1}\| = 1/d(z, \sigma(T))$ for all $z \notin \sigma(T)$ where $\sigma(T)$ denotes the spectrum of T), and $\mathscr{C}$ is the set of all convexoid operators on H (i.e., the convex hull of the spectrum of T, co $\sigma(T)$, is equal to the closure of the numerical range of T, $\overline{W(T)}$). The spectral properties of essentially G_1 operators and essentially convexoid operators are discussed in [9]. Along with ways of constructing nontrivial examples, section one contains several elementary facts about elements in the Calkin algebra and some of the basic properties of essentially G_1 operators and essentially convexoid operators. The main results of the second section are: (1) $e(\mathscr{N})$ is a closed nowhere dense subset of $e(\mathscr{H})$, (2) $e(\mathscr{H})$ is a closed nowhere dense subset of $e(\mathscr{G})$, (3) $e(\mathscr{G})$ is a closed nowhere dense subset of $e(\mathscr{C})$, and (4) $e(\mathscr{C})$ is a closed nowhere dense subset of $\mathscr{B}(H)$.

I. Basic properties and examples

For each $T \in \mathscr{B}(H)$ let $\sigma_e(T)$ denote the essential spectrum of T, i.e., $\sigma_e(T)$ is the set of all complex numbers λ such that $\pi(T) - \lambda$ is not invertible in the Calkin algebra. The proof of the following remark is straightforward.

Remark 1. If $T = A \oplus B$ on $H \oplus H$, then $\sigma_e(T) = \sigma_e(A) \cup \sigma_e(B)$.

THEOREM 1. *If* $A, B \in \mathscr{B}(H)$, *then* $\|\pi(A \oplus B)\| = \text{Max}\,\{\|\pi(A)\|, \|\pi(B)\|\}$.

Theorem 1 is an immediate consequence of Remark 1 and the fact that the norm of a self-adjoint element of a B^*-algebra is equal to its spectral radius.

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Thus

$$\begin{aligned}\|\pi(A \oplus B)\|^2 &= \|\pi(A \oplus B)^*\pi(A \oplus B)\| \\ &= \|\pi(A^*A \oplus B^*B)\| \\ &= \operatorname{Max}\{|\lambda| : \lambda \in \sigma_e(A^*A) \cup \sigma_e(B^*B)\} \\ &= \operatorname{Max}\{\|\pi(A^*A)\|, \|\pi(B^*B)\|\} \\ &= \operatorname{Max}\{\|(A)\|^2, \|\pi(B)\|^2\}.\end{aligned}$$

COROLLARY. *If A and B are essentially G_1, then $A \oplus B$ is essentially G_1.*

Recall that the essential numerical range of T is equal to the set of all $f(T)$ such that $f \in \mathscr{B}(H)^*$, $\|f\| = 1 = f(1)$, and $f(\mathscr{K}) = 0$. If S is a subset of the complex plane let co (S) denote the convex hull of S.

THEOREM 2. *If $T = A \oplus B$ on $H \oplus H$, then $W_e(T) = \operatorname{co}(W_e(A) \cup W_e(B))$.*

Proof. From [4, corollary, p. 189], $\lambda \in W_e(S)$ if and only if there exists an orthonormal sequence $\{x_n\}$ such that $\lambda = \lim_{n\to\infty}(Sx_n, x_n)$. Since $W_e(T)$ is convex, it follows that co $(W_e(A) \cup W_e(B)) \subseteq W_e(T)$.

Let $f \in B(H_1 \oplus H_2)$ with $f(1) = 1 = \|f\|$, $f(\mathscr{K}) = 0$. Then there are $g_i \in B(H_i)$ $(i = 1, 2)$ such that $g_1(X_1) = f(X_1 \oplus 0)$ and $g_2(X_2) = f(0 \oplus X_2)$ where $X_i \in B(H_i)$. Let $f_i = g_i/\|g_i\|$ if $g_i \neq 0$; otherwise put $f_i = 0$. Then $f(X_1 \oplus X_2) = \|g_1\|f_1(X_1) + \|g_2\|f_2(X_2)$. In particular,

$$f(T) = \|g_1\|f_1(A) + \|g_2\|f_2(B) \in \operatorname{co}(W_e(A) \cup W_e(B))$$

because $\|g_1\| + \|g_2\| = g_1(1) + g_2(1) = f(1) = 1.$ and $f_1(K_1) = 0 = f_2(K_2)$ if K_i is a compact operator on H_i. Therefore $W_e(T) \subseteq$ co $(W_e(A) \cup W_e(B)$ and the proof is complete.

COROLLARY. *If A and B are essentially convexoid, then $A \oplus B$ is essentially convexoid.*

The next few theorems give ways of generating nontrivial examples that will be used in the sequel.

THEOREM 3. *If $T = A \oplus B$ on $H \oplus H$ where B is essentially G_1 with $\sigma_e(B) \supseteq W_e(A)$, then T is essentially G_1.*

Proof. Let $z \notin \sigma_e(T) = \sigma_e(A) \cup \sigma_e(B) = \sigma_e(B)$, since $\sigma_e(A) \subseteq W_e(A) \subseteq \sigma_e(B)$. From [10, Lemma 1, p. 418],

$$\|(\pi(A) - z)^{-1}\| \leq \frac{1}{d(z, W_e(A))} \leq \frac{1}{d(z, \sigma_e(T))}.$$

Therefore,

$$\begin{aligned}\|(\pi(T) - z)^{-1}\| &= \|(\pi(A) - z)^{-1} \oplus (\pi(B) - z)^{-1}\| \\ &= \operatorname{Max}\{\|(\pi(A) - z)^{-1}\|, \|(\pi(B) - z)^{-1}\|\} \\ &= \operatorname{Max}\left\{\|(\pi(A) - z)^{-1}\|, \frac{1}{d(z, \sigma_e(B))}\right\} \\ &= \frac{1}{d(z, \sigma_e(T))}.\end{aligned}$$

Therefore T is essentially G_1 and the proof is complete.

From [8] we know that $T = A \oplus B$ is a G_1 operator if B is a G_1 operator with $\sigma(B) \supseteq W(A)$. An easy way to construct an essentially G_1 operator or G_1 operator with $\sigma_e(B) \supseteq W_e(A)$ or $\sigma(B) \supseteq W(A)$ is to take B to be any normal operator whose essential spectrum is $W_e(A)$ or whose spectrum is $\overline{W(A)}$, respectively.

In an analogous fashion, we can construct essentially convexoid and convexoid operators.

THEOREM 4. *If $T = A \oplus B$ on $H \oplus H$ and if B is (essentially) convexoid with* (co $\sigma_e(B) \supseteq W_e(A)$) co $\sigma(B) \supseteq W(A)$, *then T is (essentially) convexoid.*

Proof. First observe that co $\sigma_e(B) \supseteq W_e(A) \supseteq$ co $\sigma_e(A)$.

Therefore,

$$\operatorname{co} \sigma_e(T) = \operatorname{co} (\sigma_e(A) \cup \sigma_e(B)) = \operatorname{co} \sigma_e(B) = W_e(B).$$

Thus,

$$W_e(T) = \operatorname{co} (W_e(A) \cup W_e(B)) = \operatorname{co} W_e(B) = W_e(B) = \operatorname{co} \sigma_e(T)$$

so that T is essentially convexoid. The proof when B is convexoid with co $\sigma(B) \supseteq W(A)$ is similar.

The next theorem gives an easy method of constructing many operators that are not essentially convexoid or are not convexoid operators. For the proof we need to introduce some notation and terminology that will be used several times in the sequel. Recall that $\mathscr{B}(H)/\mathscr{K}$ is a C^*-algebra and hence there exists a Hilbert space H_0 such that $\mathscr{B}(H)/\mathscr{K}$ is isometrically isomorphic to a closed, self-adjoint subalgebra of $\mathscr{B}(H_0)$. Let $\nu: \mathscr{B}(H)/\mathscr{K} \to \mathscr{B}(H_0)$ be this isometric isomorphism. $T \in \mathscr{B}(H)$ is essentially invertible if and only if $\pi(T)$ is invertible in the Calkin algebra. By Atkinson's theorem T is essentially invertible if and only if T is Fredholm, i.e., T has closed range with finite nullity and finite co-rank.

THEOREM 5. *If*

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

on $H \oplus H$ where C is (essentially) invertible, then T is not (essentially) convexoid.

Proof. Part I. Suppose C is invertible and show T is not convexoid: If $A - z$ and $B - z$ are invertible, then $T - z$ is invertible and

$$T_z^{-1} = \begin{pmatrix} A_z^{-1} & -A_z^{-1}CB_z^{-1} \\ 0 & B_z^{-1} \end{pmatrix},$$

where $S_z = S - z$ for all operators S. It follows from this that $\sigma(T) \subseteq \sigma(A) \cup \sigma(B)$. Let $z \notin \sigma(B) \cup \sigma(B)$. It follows from the matrix representation of T_z^{-1} that

$$\|T_z^{-1}\| > \|B_z^{-1}\|.$$

Since $\|B_z^{-1}\| \geq 1/d(z, \sigma(B))$, $\|(T - z)^{-1}\| > 1/d(z, \sigma(B))$.

Observe that T^* is unitarily equivalent to

$$\begin{pmatrix} B^* & C^* \\ 0 & A^* \end{pmatrix}$$

via the unitary operator

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $z \notin \sigma(A) \cup \sigma(B)$ implies $\bar{z} \notin \sigma(A^*) \cup \sigma(B^*)$, we may apply the above argument to

$$\begin{pmatrix} B^* & C^* \\ 0 & A^* \end{pmatrix},$$

which is unitarily equivalent to T^*, to conclude that for all $z \notin \sigma(A) \cup \sigma(B)$,

$$\|(T - z)^{-1}\| > \operatorname{Max}\left\{\frac{1}{d(z, \sigma(A))}, \frac{1}{d(z, \sigma(B))}\right\}.$$

Suppose T is convexoid, i.e., co $\sigma(T) = \overline{W(T)}$. Since

$$\left(T\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}\right) = \left(\begin{pmatrix} Ax \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}\right) = (Ax, x),$$

$W(A) \subseteq W(T)$. Since

$$\left(T\begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right) = \left(\begin{pmatrix} Cy \\ By \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right) = (By, y),$$

$W(B) \subseteq W(T)$. Therefore, since $\sigma(T) \subseteq \sigma(A) \cup \sigma(B)$, we have

$$\begin{aligned} \text{co } (\sigma(A) \cup \sigma(B)) &\subseteq \text{co } (\overline{W(A)} \cup \overline{W(B)}) \subseteq \overline{W(T)} = \text{co } \sigma(T) \\ &\subseteq \text{co } (\sigma(A) \cup \sigma(B)). \end{aligned}$$

Consequently, co $\sigma(T) =$ co $(\sigma(A) \cup \sigma(B))$. Now pick $z \notin$ co $\sigma(T)$ such that $d(z, \text{co } \sigma(T)) = d(z, \sigma(A))$ (if no such z exists, then there does exist $z \notin$ co $\sigma(T)$

such that $d(z, \text{co}\, \sigma(T)) = d(z, \sigma(B))$ and this case is handled similarly). From above,

$$\|(T - z)^{-1}\| > \frac{1}{d(z, \sigma(A))} = \frac{1}{d(z, \text{co}\, \sigma(T))} = \|(T - z)^{-1}\|,$$

contradiction. Therefore T is not convexoid and Part I is shown.

Part II. Suppose C is essentially invertible and show that T is not essentially convexoid: The idea is to reduce this part of the proof to the previous one.

In a manner similar to above one shows that $\sigma_e(T) \subseteq \sigma_e(A) \cup \sigma_e(B)$. Let P be the orthogonal projection

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

onto $H \oplus 0$. Then $\nu \circ \pi(P)$ is a nontrivial orthogonal projection in $\mathscr{B}(H_0)$ (see the comments before the statement of proof for the definition of ν and H_0). Let M = range $(\nu \circ \pi(P))$. Then relative to $H_0 = M \oplus M^{\perp}$,

$$\nu \circ \pi(T) = \begin{pmatrix} A_1 & C_1 \\ D_1 & B_1 \end{pmatrix}.$$

Since $(1 - P)TP = 0$, $D_1 = 0$. Since $PT|_{0 \oplus H} = C$ is essentially invertible, C_1 is invertible. Since $\sigma_e(T) = \sigma(\nu \circ \pi(T))$ [2, Theorem 4.28], and since ν is an isometric (algebra) isomorphism, T is essentially convexoid if and only if $\nu \circ \pi(T)$ is convexoid. By Part I, $\nu \circ \pi(T)$ is not convexoid. Therefore T is not essentially convexoid and the proof is complete.

COROLLARY. *If $A, B, C \in \mathscr{B}(H)$ and C is* (*essentially*) *invertible then*

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

is not (*essentially*) G_1.

It is easily seen that $\mathscr{N} + \mathscr{K} \subseteq e(\mathscr{N})$ and $\mathscr{H} + \mathscr{K} \subseteq e(\mathscr{H})$; both containments are actually proper. However, the above type of relationship is not true for the essentially G_1 operators nor for the essentially convexoid operators.

THEOREM 6. *$\mathscr{G}$ is not a subset of $e(\mathscr{G})$, and $e(\mathscr{G})$ is not a subset of $\mathscr{G}$.*

Proof. Write $H = M_1 \oplus M_2 \oplus M_3$ where each M_i has infinite dimension. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathscr{B}(M_1 \oplus M_2).$$

By [7, Theorem 7] there exists a compact normal operator $N \in \mathscr{B}(M_3)$ such that $T = A \oplus N \in \mathscr{G}$. However, since A is not essentially G_1 and N is compact, T is not essentially G_1.

Let

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus 0 \quad \text{on } H = M \oplus M^{\perp}$$

where the dimension of M is two. Then $\pi(T) = 0$ so T is essentially G_1. Since $\sigma(T) = (0)$ and $W(T)$ is $\{z: |z| \leq \frac{1}{2}\}$, T is not convexoid and hence not G_1.

THEOREM 7. *$\mathscr{C}$ is not a subset of $e(\mathscr{C})$, and $e(\mathscr{C})$ is not a subset of $\mathscr{C}$.*

Proof. To see that $e(\mathscr{C})$ is not a subset of $\mathscr{C}$ take a compact $T \notin \mathscr{C}$, then $\pi(T) = 0$ so T is in $e(\mathscr{C})$. For example, take

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus 0 \quad \text{on } M \oplus M^{\perp}$$

where M has dimension 2.

To see that $\mathscr{C}$ is not a subset of $e(\mathscr{C})$ let $H = M_1 \oplus M_2 \oplus M_3$ where M_1 and M_2 have infinite dimension and M_3 has dimension equal to 3. Let

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus N \quad \text{on } (M_1 \oplus M_2) \oplus M_3$$

where

$$N = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

and a, b, and c are complex numbers chosen so that

$$\operatorname{co}\{a, b, c\} \supseteq W\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

By Theorem 4, T is convexoid. Since N has finite rank,

$$\pi(T) = \pi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus 0\right).$$

It is easily seen that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus 0$$

is not essentially convexoid (its essential spectrum is $\{0\}$. and its essential numerical range is a disc of radius $\frac{1}{2}$ about the origin). Hence, T is not essentially convexoid and this completes the proof.

If T is essentially normal, then it is not necessarily true that there exists a compact operator K such that $T + K$ is normal (let T be a unilateral shift of finite multiplicity). By taking the adjoint of a unilateral shift of finite multiplicity, we have an example of an essentially hyponormal operator, T, such that there does not exist a compact operator K such that $T + K$ is hyponormal (compute the Fredholm index of $T + K$) [4, Remark 2, p. 186]. It is not known what happens in the essentially G_1 and essentially convexoid cases:

Conjecture 1. If T is essentially G_1, then there exists a compact operator K such that $T + K$ is G_1.

Conjecture 2. If T is essentially convexoid, then there exists a compact operator K such that $T + K$ is convexoid.

II. Topological properties of the sets $e(\mathcal{N})$, $e(\mathcal{H})$, $e(\mathcal{G})$, and $e(\mathcal{C})$

THEOREM 8. *$e(\mathcal{N})$ is a closed, nowhere dense subset of the closed set $e(\mathcal{H})$.*

Proof. Let $\mathcal{N}_c$ and $\mathcal{H}_c$ denote the normal and hyponormal elements in the Calkin algebra $\mathcal{B}(H)/\mathcal{K}$. It is easily seen that both of these sets are closed in $\mathcal{B}(H)/\mathcal{K}$. Since the quotient map, π, is continuous and since $e(\mathcal{N}) = \pi^{-1}(\mathcal{N}_c)$ and $e(\mathcal{H}) = \pi^{-1}(\mathcal{H}_c)$, $e(\mathcal{N})$ and $e(\mathcal{H})$ are (norm) closed subsets of $\mathcal{B}(H)$. Therefore, to show that $e(\mathcal{N})$ is a nowhere dense subset of $e(\mathcal{H})$ it suffices to show that $e(\mathcal{N})$ has empty interior in $e(\mathcal{H})$, i.e., for each $T \in e(\mathcal{N})$, there exists $T_n \to T$ such that $T_n \in e(\mathcal{H}) \sim e(\mathcal{N})$.

Let T be essentially normal. Then [4] there exists an infinite rank projection P and a complex number λ such that $P(T - \lambda)$ and $(T - \lambda)P$ are compact. Hence if M is the range of P then $T - \lambda = (0 \oplus T_4) + K$ relative to $H = M \oplus M^{\perp}$ where K is compact. Let U be an essentially hyponormal operator in $\mathcal{B}(M)$ that is not essentially normal (for example take U to be a unilateral shift of infinite multiplicity). Let $T_n = \lambda + K + ((1/n)U) \oplus T_4$. Then each T_n is clearly essentially hyponormal and not essentially normal because $(1/n)U$ is not essentially normal. Furthermore, $\|T - T_n\| = \|(1/n)U\| \to 0$. Therefore the proof is complete.

THEOREM 9. *$e(\mathcal{H})$ is a closed nowhere dense subset of the closed set $e(\mathcal{G})$.*

Proof. We already know (Theorem 8) that $e(\mathcal{H})$ is closed. First recall that ν is the isometric (algebra) embedding of $\mathcal{B}(H)/\mathcal{K}$ into $\mathcal{B}(H_0)$, where H_0 is a Hilbert space. Also recall that $\sigma_e(T) = \sigma(\nu \circ \pi(T))$ for all $T \in \mathcal{B}(H)$. Let $T_n \to T$ where each T_n is essentially G_1. We need to show that T is also in $e(\mathcal{G})$. First observe that $S \in e(\mathcal{G})$ if and only if $\nu \circ \pi(S)$ is G_1 in $\mathcal{B}(H_0)$. Therefore, $\nu \circ \pi(T_n) \to \nu \circ \pi(T)$ and each $\nu \circ \pi(T_n)$ is G_1. Since [8, Theorem 2.2] the G_1 operators in $\mathcal{B}(H_0)$ is a closed set, $\nu \circ \pi(T)$ is G_1. Hence T is essentially G_1 and $e(\mathcal{G})$ is (norm) closed. Therefore, to complete the proof, we need to show that for each $T \in e(\mathcal{H})$ there exists $T_n \to T$ such that $T_n \in e(\mathcal{G}) \sim e(\mathcal{H})$.

Let $T \in e(\mathcal{H})$. Since $\pi(T)$ is normaloid [4, p. 187] there exists $\lambda \in \sigma_e(T)$ such that $|\lambda| = \|\pi(T)\|$. Now proceed exactly as in the proof of Theorem 8 only now let U be an essentially G_1 operator that is not essentially hyponormal. For example, take

$$U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus N \quad \text{on } H = (M_1 \oplus M_2) \oplus M_3$$

where each M_i has infinite dimension and N is a normal operator on M_3 with

$$\sigma(N) = \sigma_e(N) = W_e \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus we obtain the desired sequence $T_n \to T$ such that $T_n \in e(\mathscr{G}) \sim e(\mathscr{H})$. This completes the proof.

THEOREM 10. *$e(\mathscr{G})$ is a closed, nowhere dense subset of the closed set $e(\mathscr{C})$.*

Proof. From Theorem 9, $e(\mathscr{G})$ is closed. To see that $e(\mathscr{C})$ is closed, let $T_n \to T$ where each T_n is essentially convexoid. Thus $\nu \circ \pi(T_n)$ is convexoid in $\mathscr{B}(H_0)$ and $\nu \circ \pi(T_n) \to \nu \circ \pi(T)$. Since the set of convexoid operators in $\mathscr{B}(H_0)$ is closed [8, Theorem 2.7], $\nu \circ \pi(T)$ is convexoid. Thus $\pi(T)$ is convexoid so that T is essentially convexoid.

Since $e(\mathscr{G})$ is closed, in order to show $e(\mathscr{G})$ is a nowhere dense subset of $e(\mathscr{C})$ it suffices to show $e(\mathscr{G})$ has empty interior in $e(\mathscr{C})$. Let $T \in e(\mathscr{C})$. After a translation and rotation, if necessary, we may assume that $0 \in \sigma_e(T) \cap \partial W_e(T)$ and $\operatorname{Re} W_e(T) \geq 0$. Since $0 \in \sigma_e(T) \cap \partial W_e(T)$, we may apply a theorem of Joel Anderson [1] to obtain an orthonormal sequence $\{x_n\}$ such that

$$\|Tx_n\| + \|T^*x_n\| \to 0.$$

By J. G. Stampfli's corollary to Theorem 2 [11], T is unitarily equivalent (under, say U) to $(T \oplus 0) + K$ on $H \oplus H_1$, where H_1 is a separable Hilbert space and K is a compact operator on $H \oplus H_1$. Let

$$S_n = \left(T \oplus \left(\frac{1}{n}\right) A\right) + K \quad \text{on } H \oplus H_1,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus N \quad \text{on } H_1 = (M_1 \oplus M_2) \oplus M_3,$$

each M_i has infinite dimension, and N is a normal operator on M_3 with

$$\sigma(N) = \sigma_e(N) = \partial W_e \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easily seen that

$$W_e \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is the same as the numerical range of

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

on a two dimensional Hilbert space. Furthermore, by Donoghue [2], the numerical range of

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is a disc about the origin of radius $\frac{1}{2}$. Thus $\sigma(N) = \sigma_e(N)$ is a circle of radius $\frac{1}{2}$ about the origin. By Theorem 4, A is essentially convexoid so by the Corollary to Theorem 3, S_n is essentially convexoid. Observe that

$$\|T - US_nU^{-1}\| = \|U^{-1}TU - S_n\| = \|(T \oplus 0) + K - S_n\|$$

$$= \left\| \left(\frac{1}{n}\right) A \right\| = \frac{1}{n}.$$

Therefore to complete the proof it suffices to show that S_n is not essentially G_1. To carry this out, we first observe that for all $z \neq 0$.

$$\left\| \left(\begin{pmatrix} 0 & 1/n \\ 0 & 0 \end{pmatrix} - z \right)^{-1} \right\|^2 = \left\| \begin{pmatrix} -z & 1/n \\ 0 & -z \end{pmatrix}^{-1} \right\|^2$$

$$= \left\| \begin{pmatrix} -1/z & -1/(nz^2) \\ 0 & -1/z \end{pmatrix} \right\|^2$$

$$\geq \sup_{\|y\|=1} \left\| \begin{pmatrix} -1/z & -1(nz^2) \\ 0 & -1/z \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} \right\|^2$$

$$= \sup_{\|y\|=1} \left\| \frac{1}{(nz^2)y} \right\|^2 + \left\| \left(\frac{1}{z}\right) y \right\|^2$$

$$= \frac{1}{n^2|z|^4} + \frac{1}{|z|^2} > \frac{1}{n^2|z|^4}.$$

Therefore

$$\left\| \left(\begin{pmatrix} 0 & 1/n \\ 0 & 0 \end{pmatrix} - z \right)^{-1} \right\| > \frac{1}{n|z|^2}.$$

Since $0 \in \sigma_e(T)$,

$$\sigma_e(S_n) = \sigma_e\left(T \oplus \left(\frac{1}{n} A\right)\right) = \sigma_e(T) \cup \sigma_e\left(\frac{1}{n} A\right)$$

$$= \sigma_e(T) \cup \left\{z : |z| = \frac{1}{2n}\right\}.$$

Recall that Re $\sigma_e(T) \geq 0$ so that we may choose (for each fixed $n \geq 1$) z so that $-1/(4n) < z < 0$ and $z \notin \sigma_e(S_n)$. Then

$$\begin{aligned}
\|(\pi(S_n) - z)^{-1}\| &= \left\| \left(\pi(T) \oplus \left(\frac{1}{n} A \right) - z \right)^{-1} \right\| \\
&= \left\| (\pi(T) - z)^{-1} \oplus \left(\pi\left(\frac{1}{n} A \right) - z \right)^{-1} \right\| \\
&= \operatorname{Max} \left\{ \|(\pi(T) - z)^{-1}\|, \left\| \left(\pi\left(\frac{1}{n} A \right) - z \right)^{-1} \right\| \right\} \\
&= \operatorname{Max} \left\{ \|(\pi(T) - z)^{-1}\|, \left\| \left(\pi\begin{pmatrix} 0 & 1/n \\ 0 & 0 \end{pmatrix} - z \right)^{-1} \right\|, \right. \\
&\qquad\qquad \left. \left\| \left(\pi\left(\frac{1}{n} N \right) - z \right)^{-1} \right\| \right\} \\
&\geq \left\| \left(\pi\begin{pmatrix} 0 & 1/n \\ 0 & 0 \end{pmatrix} - z \right)^{-1} \right\| \\
&= \left\| \pi\begin{pmatrix} -1/z & -1/(nz^2) \\ 0 & -1/z \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} -1/z & 1/(nz^2) \\ 0 & -1/z \end{pmatrix} \right\| \\
&= \left\| \left(\begin{pmatrix} 0 & 1/n \\ 0 & 0 \end{pmatrix} - z \right)^{-1} \right\| \\
&> \frac{1}{n|z|^2}.
\end{aligned}$$

The second to last equality holds since the matrix is Toeplitz (i.e., if A is Toeplitz, then $\|A + K\| \geq \|A\|$ for all compact operators K [3, p. 180]). Therefore for all $z < 0$ close enough to 0,

$$\|(\pi(S_n) - z)^{-1}\| = \left\| \left(\begin{pmatrix} 0 & 1/n \\ 0 & 0 \end{pmatrix} - z \right)^{-1} \right\| > \frac{1}{n|z|^2}.$$

Hence S_n is not essentially G_1 and the theorem is proved.

THEOREM 11. *$e(\mathscr{C})$ is a closed, nowhere dense subset of $\mathscr{B}(H)$.*

Proof. $e(\mathscr{C})$ is closed from Theorem 10. Therefore to complete the proof it suffices to show that $e(\mathscr{C})$ has empty interior. Let T be essentially convexoid. After a translation and rotation, if necessary, we may assume $0 \in \sigma_e(T) \cap \partial W_e(T)$ and Re $W_e(T) \geq 0$. By a result of Joel Anderson [1], there exists an orthonormal sequence $\{x_n\}$ such that $\|Tx_n\| + \|T^*x_n\| \to 0$. Proceeding as in the proof of Theorem 10, we apply the Corollary to Theorem 2 of Stampfli [11] to conclude that T is unitarily equivalent (under, say, U) to $(T \oplus 0) + K$ on $H \oplus H_1$, where H_1 is a separable Hilbert space and K is a compact operator on $H \oplus H_1$. Let $\varepsilon > 0$ and define

$$S = \begin{pmatrix} T & \varepsilon \\ 0 & 0 \end{pmatrix} + K \quad \text{on } H \oplus H_1.$$

By Theorem 5,

$$\begin{pmatrix} T & \varepsilon \\ 0 & 0 \end{pmatrix}$$

is not essentially convexoid; hence S is not essentially convexoid so that USU^{-1} is not essentially convexoid. Furthermore

$$\|T - USU^{-1}\| = \|U^{-1}TU - S\| = \|(T \oplus 0) + K - S\| = \left\| \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix} \right\| = \varepsilon.$$

Therefore the set of essentially convexoid operators has empty interior in $\mathscr{B}(H)$. Thus $e(\mathscr{C})$ is nowhere dense and the theorem is proved.

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