ON FINITE LINEAR GROUPS OF DEGREE 16

BY

HARVEY I. BLAU

1. Introduction

The main result of this paper is:

THEOREM 1. *Let G be a finite group with a faithful irreducible complex representation of degree* 16. *Then if P is a Sylow p-subgroup of G for $p \geq 19$ and Z is the center of G, either $P \lhd G$ or $p = 31$ and $G/Z \approx PSL_2(31)$.*

Theorem 1 has several consequences bearing on the situation of a group with a complex representation of degree smaller than a prime dividing its order. We state them here, using the same notation as above.

THEOREM 2. *Let p be a prime. Assume that $|P| = p$, $P \ntriangleleft G$, and G has a faithful irreducible complex representation of degree $d < p - 1$. Let $t|N: C| = p - 1$, where N, C are the normalizer, resp. centralizer, of P in G. If $t \geq 3$ then $t \geq 8$.*

It is known that if $t \geq 3$ then $t \geq 6$ [13]. In view of [2], Theorem 1 eliminates the only remaining numerical case when $t = 6$, namely $p = 19$ and $d = 16$. This case was also listed as unresolved in [1, Section 8] as $p = 19$, $d = 16$, $e = 3$.

THEOREM 3. *Assume $p > 7$. Let G have a faithful irreducible complex representation of degree $d <$ max $\{(7p + 1)/8, p + (3/2) - (p + 5/4)^{1/2}\}$. Then either $P \lhd G$ or $G/Z \approx PSL_2(p)$ and $d = (p \pm 1)/2$.*

For the exceptions to Theorem 3 when $p \leq 7$ see [9, Section 8.5] (or Theorem 4 below for the cases $d < p - 1$).

THEOREM 4. *Assume G has a faithful irreducible complex representation of degree $d \leq 27$. Suppose p is a prime, $p > d + 1$. Then one of the following must occur:*

(i) $P \lhd G$;

(ii) $G/Z \approx PSL_2(p)$, $d = (p \pm 1)/2$;

(iii) $p = 17$, $d = 15$, $G \approx SL_2(16) \times A$ *where A is abelian;*

(iv) $p = 7$, $d = 4$, *and* $G/Z \approx A_7$;

(v) $p = 5$, $d = 3$, *and* $G/Z \approx A_6$.

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In the proof of Theorem 1, the case $p = 19$ is the only one which does not follow quickly from known results. Handling this case involves a fairly straightforward application of the modular-theoretic techniques of [6] and [1], block separation, and a recent result of Walter Feit [10, Theorem 4]. The author would like to thank Professor Feit for informing him of this theorem, and also wishes to acknowledge several useful conversations with Professors Feit and Henry S. Leonard.

2. Notation and preliminary results

Throughout the paper G is a finite group, p a prime, P a Sylow p-subgroup of G. If H is a subgroup, and S a subset, of G, then $N_H(S)$, $C_H(S)$ denote, respectively, the normalizer and centralizer of S in H. $Z(H)$ is the center of H, $N = N_G(P)$, $C = C_G(P)$, and $Z = Z(G)$. $B_0(p)$ is the principal p-block of G.

Fix p and a positive integer $d < p - 1$. We consider two sets of hypotheses:

(*) *G has a faithful irreducible complex representation of degree d.*

(**) *G is not of type $L_2(p)$, $|P| = p$, $G = G'$ and G/Z is simple.*

The following sort of reduction argument, based on the main result of [5], appears in [7, Section 6], [10, Section 4], and [15]. The proof here is essentially that of [7], with a few more details provided.

PROPOSITION 2.1. *Fix p and d. Suppose there is no group satisfying both* (*) *and* (**). *Then if G is any group satisfying* (*), *either $P \lhd G$ or $G/Z \approx PSL_2(p)$ and $d = (p \pm 1)/2$.*

Proof. Suppose (*) holds for G and $P \ntriangleleft G$. Let θ be the given faithful irreducible character of G with $\theta(1) = d$. Since the degree of each irreducible constituent of θ_P is a power of p, it follows that each constituent is linear and hence P is abelian.

Let G_0 be the subgroup of G generated by all p-elements in G. Thus $G_0 \lhd G$, $P \subseteq G_0$ and $P \ntriangleleft G_0$. The main theorem of [5] says there is a subgroup $P_0 \subseteq P$ with $|P: P_0| = p$ and $P_0 \lhd G$. Thus $P \subseteq C_G(P_0) \lhd G$ implies the normal subgroup generated by P centralizes P_0, i.e. $G_0 \subseteq C_G(P_0)$.

Let G_1 be the p-commutator subgroup of G_0, whence $G_1 \lhd G$. The transfer of G_0 into P has kernel G_1 and image $P \cap Z(N_{G_0}(P)) \supseteq P_0$ [16, Chapter V, Theorem 7]. If $P \cap Z(N_{G_0}(P)) = P$, then G_0 has the normal p-complement G_1, as well as a faithful complex representation of degree $d < p - 1$, a contradiction [15, (2.1)]. Thus the image of the transfer is P_0, and it is easy to see that the transfer maps P_0 onto itself. Hence $G_0 = G_1 \times P_0$. Let $P_1 = P \cap G_1$. Then $P = P_0 \times P_1$ and $P_1 \ntriangleleft G_1$.

Let H be a normal p'-subgroup of G_1. If $P_1 \ntriangleleft P_1H$ then P_1H has a faithful representation of degree d in a field of characteristic p [8, III.3.4] contrary to Theorem B of Hall and Higman [12]. Thus $P_1 \lhd P_1H$ and so $H \subseteq C_{G_1}(P_1)$. Since G_1 is the smallest normal subgroup of G_1 generated by P_1, it follows that

$H \subseteq Z(G_1)$ and $G_1/Z(G_1)$ is simple. If $P_1 \nsubseteq G_1'$ then G_1 has a normal p-complement, a contradiction. So $P_1 \subseteq G_1' \lhd G$ implies $G_1' = G_1$.

Let $\theta_{G_1} = \sum_{i=1}^{s} \omega_i$ where each ω_i is an irreducible character of G_1. Now $\omega_i(1) = \omega_1(1)$ for $i = 1, \ldots, s$, as the ω_i are conjugate under G. So if $s > 1$, $\omega_1(1) < \frac{1}{2}(p - 1)$. Let K_i be the kernel of ω_i. If $P_1 \subseteq K_1$, then $K_1 = G_1 = K_i$, whence G_1 is in the kernel of θ, a contradiction. So K_1 is a p'-group and $K_1 \subseteq Z(G_1)$. Thus $P_1 \lhd P_1K_1$. [11] implies $P_1K_1/K_1 \lhd G_1/K_1$, hence $P_1 \lhd G_1$, a contradiction. So $s = 1$ and θ_{G_1} is irreducible. Thus if $g \in C_G(G_1)$ then g is represented by scalars in the representation of G which affords θ, and so $g \in Z$. Hence $Z(G_1) = Z \cap G_1$ and $G_1C_G(G_1) = G_1Z$.

If G_1 is not of type $L_2(p)$ then G_1 satisfies (*) and (**), which contradicts our hypothesis. So G_1 is of type $L_2(p)$. Then $G_1/Z(G_1) \approx PSL_2(p)$. Thus $G_1 \approx PSL_2(p)$ or $G_1 \approx SL_2(p)$ since the Schur multiplier of $PSL_2(p)$ has order 2. Therefore $d = (p \pm 1)/2$ and θ assumes different values on the two conjugate classes of nontrivial p-elements in G_1 [4, Theorem 71.3], [3, (47b)]. Since any $g \in G$ fixes θ_{G_1} (acting by conjugation), g must fix each conjugate class of p-elements in G_1. It is not hard to see that an automorphism of $SL_2(p)$ or $PSL_2(p)$ which fixes each conjugate class of p-elements must be an inner automorphism. Thus $G = G_1C_G(G_1) = G_1Z$ and $G/Z \approx G_1/Z(G_1) \approx PSL_2(p)$.

The next result seems to be well known.

PROPOSITION 2.2. *Let p be a prime such that $p \mid |G|$. Let χ be an irreducible character in $B_0(p)$ such that χ is rational on all p-elements, and no p-element (except* 1) *is in the kernel of χ. Let $v_p(\chi(1)) = m$. Then $\chi(1) \geq p^m(p - 1)$.*

Proof. Let x be an element of order p in $Z(P)$. Then K, the conjugate class of x, has order prime to p. Since $\chi(x)$ is rational, $\chi(x)$ and $\chi(x)|K|/\chi(1)$ are rational integers and $v_p(\chi(x)) \geq v_p(\chi(1))$. $\chi \in B_0(p)$ implies $\chi(x)|K|/\chi(1) \equiv |K|$ (mod p) (see [4, Theorem 61.2]). Let $q = p^m$. Then

$$\frac{(\chi(x)/q)|K|}{\chi(1)/q} \equiv |K| \pmod p \quad \text{implies} \quad \frac{\chi(x)}{q} \equiv \frac{\chi(1)}{q} \pmod p.$$

Let γ be a faithful linear character of $\langle x \rangle$, and let n be its multiplicity as a constituent of $\chi_{\langle x \rangle}$. Then $\chi(x) \neq \chi(1)$ and $\chi(x)$ rational imply $n > 0$ and each of the $p - 1$ algebraic conjugates of γ occurs in $\chi_{\langle x \rangle}$ with multiplicity n. Hence

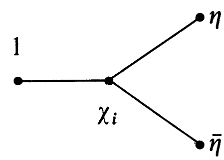
$$\chi(x) = n(-1) + (\chi(1) - n(p - 1)) = \chi(1) - np.$$

Therefore $(\chi(1) - np)/q \equiv \chi(1)/q$ (mod p) implies $q \mid n$. Now $\chi(1) \geq n(p - 1)$ yields the result.

PROPOSITION 2.3. *Assume that $|P| = p$ and $|N\colon C| = 3$. If the Brauer tree corresponding to $B_0(p)$ is not an open polygon then $p \equiv 1$ (mod 4).*

Proof. Since $|N\colon C| \mid p - 1$, we have $p \equiv 1$ (mod 3). The discussion in [14, Section 5] shows that the map sending each ordinary or modular irreducible character to its complex conjugate reflects the tree across a unique real stem.

The exceptional vertex lies on the stem. Thus if the graph is not an open polygon, it must have the form



where 1 is the principal character, $\eta(1) \equiv 1 \pmod p$, $\bar{\eta}$ is the complex conjugate of η, the χ_i are exceptional characters, and $\chi_i(1) \equiv 3 \pmod p$ for $i = 1, \ldots, (p-1)/3$. Hence $\chi_i(1)\eta(1)\bar{\eta}(1) \equiv 3 \pmod p$. Now $\chi_i(1)\eta(1)\bar{\eta}(1)$ is the square of a rational integer [10, Theorem 4], so that 3 is a quadratic residue mod p. Quadratic reciprocity implies $p \equiv 1 \pmod 4$.

3. Proof of Theorem 1

By Proposition 2.1, it suffices to assume G satisfies (**) and then show such a group cannot exist.

Now $C_G(P) = P \times Z$ (so that N/P is abelian) and $z = |Z| \mid 16$ [7, (2.1)]. Then the situation of [1, (4.3)] holds. (**) and [14] imply $16 > (p+1)/2$, whence $16 = p - e$, where $e = |N: C| \mid p - 1$. Since $e \leq (p-1)/3$ (so that $16 \geq (2p+1)/3$), it follows that $p = 19$, $e = 3$.

So the theorem is proved for all primes $q > 19$. If $q \mid |G|$ for some prime $q > 19$ then no Sylow q-subgroup is normal, since G/Z is simple. Hence $q = 31$ and $G/Z \approx PSL_2(31)$, a contradiction. Thus no prime larger than 19 divides $|G|$.

Let R be the ring of integers in a 19-adic number field F so that both F and $K = R/I$ are splitting fields for all subgroups of G, where I is the maximal ideal of R. Let X be an R-free RG-module affording a faithful irreducible character θ of degree 16 such that $L = X/XI$ is an indecomposable KG-module. Then L is faithful, as G has no proper normal 19-subgroup.

Let $L_N = V_{16}(\lambda)$ in the notation of [1]. L is irreducible [1, Proposition 6.1]. We have [1, (5.2), (5.3)]

$$(L \otimes L^*)_N = V_1(1) \oplus V_3(\alpha) \oplus V_5(\alpha^2) \oplus \sum_{i=3}^{15} V_{19}(\alpha^i)$$

where $\alpha\colon N \to K$, a linear character of order 3, is defined in [1, Section 2].

$$L \otimes L^* = L_0 \oplus L_1 \oplus L_2 \oplus Q$$

where L_0 is the one-dimensional trivial KG-module, Q is projective, and

$$L_{1_N} = V_3(\alpha) \oplus \sum_{j \in \mathscr{S}_1} V_{19}(\alpha^j), \quad L_{2_N} = V_5(\alpha^2) \oplus \sum_{j \in \mathscr{S}_2} V_{19}(\alpha^j)$$

where $\mathscr{S}_1$ and $\mathscr{S}_2$ are sets of integers with $|\mathscr{S}_1| + |\mathscr{S}_2| \leq 13$. Let $m_i = |\mathscr{S}_i|$, $i = 1, 2$. Then

$$\dim L_i = 2i + 1 + 19m_i, \quad m_i > 0, m_1 + m_2 \leq 13.$$

If χ is an exceptional character in $B_0(19)$ then $\chi(1) \equiv 3 \pmod{19}$ by [7, (2.1)] and [7, (4.1)] applied to θ and its complex conjugate. Then Proposition 2.3 implies there are only two possibilities for the graph of $B_0(19)$:

$$\begin{array}{ccccccc} & L_0 & & M_2 & & M_{18} & \\ \bullet & \text{———} & \bullet & \text{———} & \bullet & \text{———} & \bullet \\ 1 & & \chi_i & & \xi & & \eta \end{array} \tag{i}$$

where the χ_i are exceptional characters, $1 \leq i \leq 6$, ξ and η are nonexceptional characters with $\xi(1) \equiv 1 \pmod{19}$, $\eta(1) \equiv -1 \pmod{19}$, and M_2 and M_{18} are irreducible KG-modules. Since L_0 and M_2 are the only constituents of a KG-module with socle L_0 [8, I.17.12], [1, Proposition 4.5] implies M_2 has Green correspondent $V_2(\alpha^2)$. Similarly, M_2 and M_{18} are the only constituents of a KG-module with socle M_2, so M_{18} has Green correspondent $V_{18}(\alpha)$.

$$\begin{array}{ccccccc} & L_0 & & M_{17} & & M_3 & \\ \bullet & \text{———} & \bullet & \text{———} & \bullet & \text{———} & \bullet \\ 1 & & \eta & & \xi & & \chi_i \end{array} \tag{ii}$$

where again $\xi(1) \equiv 1 \pmod{19}$, $\eta(1) \equiv -1 \pmod{19}$, and M_{17}, M_3 have Green correspondents $V_{17}(\alpha^2)$, $V_3(\alpha)$ respectively. (So $M_3 = L_1$.)

(3.1) $\chi_i(1) = 22$, $\xi(1) = 77$, $\eta(1) = 56$ *in either* (i) *or* (ii).

Proof. Suppose (i) holds. By [1, Lemma 2.4], the npmv's of $V_2(\alpha^2) \otimes V_3(\alpha)$ are $\alpha^3 = 1$ and α^{-1}. Hence $M_2 \otimes L_1$ contains a nonzero invariant (as a KG-module) [1, Theorem 4.1]. Since $M_2 \approx M_2^*$ and $L_1 \approx L_1^*$, we see that $M_2 \subseteq \text{socle}(L_1)$ and $M_2 \subseteq L_1/\text{rad}(L_1)$. Thus M_2 is a constituent of L_1 with multiplicity at least two.

Now $V_{18}(\alpha) \otimes V_3(\alpha)$ has 1 as a npmv [1, Lemma 2.6]. Then as above, M_{18} is a constituent of L_1 with multiplicity at least two. Similarly, M_2 occurs at least twice as a constituent of L_2. Let $\dim M_2 = 2 + 19a$, $a > 0$, and $\dim M_{18} = 18 + 19b$, $b \geq 0$. Then

$$4 \dim M_2 + 2 \dim M_{18} \leq \dim L_1 + \dim L_2 \leq 8 + 13 \cdot 19$$

implies

$$4a + 2b + 2 \leq 13. \tag{3.2}$$

Now $\chi_1(1)\xi(1)\eta(1) = (3 + 19a)(1 + 19(a + b + 1))(18 + 19b)$ is the square of a rational integer [10, Theorem 4]. But the only values of $a > 0$, $b \geq 0$ satisfying (3.2) for which this is true are $a = 1$, $b = 2$. Hence $\chi_1(1) = 22$, $\xi(1) = 77$, $\eta(1) = 56$.

Suppose (ii) holds. As above, we see that M_3 and M_{17} are both constituents of L_2 with multiplicity at least two. Let $\dim L_1 = 3 + 19c$, $c > 0$, and $\dim M_{17} = 17 + 19f$, $f \geq 0$. Then

$$3 \dim L_1 + 2 \dim M_{17} \leq \dim L_1 + \dim L_2 \leq 8 + 13 \cdot 19$$

implies $3c + 2f \leq 11$. Since

$$\chi_1(1)\eta(1)\xi(1) = (3 + 19c)(18 + 19f)(1 + 19(c + f + 1))$$

is the square of an integer, it follows that $c = 1, f = 2$. (3.1) is established.

(3.3) $4 \mid z$.

Proof. [7, Theorem 1] implies $z > 1$. Suppose $z = 2$. Then there are two 19-blocks of positive defect, say $B_0(19)$ and B. L and L^* both lie in B. Since L separates three vertices from the exceptional vertex, we see that $L \approx L^*$ (see [1, Section 4]). Then [1, Lemma 3.3] implies $m_1 \leq 7, m_2 \leq 6$. But in case (i), we have

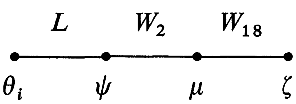
$$2(2 + 19) + 2(3 \cdot 19 - 1) = 2(\dim M_2 + \dim M_{18}) \leq \dim L_1 \leq 3 + 19 \cdot 7,$$

a contradiction. In case (ii),

$$2(3 + 19) + 2(17 + 2 \cdot 19) = 2(\dim M_3 + \dim M_{17}) \leq \dim L_2 \leq 5 + 19 \cdot 6,$$

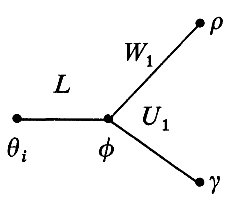
again a contradiction. Since $z \mid 16$, the result follows.

There are two possible configurations for the block B which contains L:



(a)

where $\theta_i(1) = 16, 1 \leq i \leq 6, \psi(1) \equiv -1 \equiv \zeta(1) \pmod{19}, \mu(1) \equiv 1 \pmod{19}$. Since L and W_2 are the only constituents of a KG-module with socle L, [1, Proposition 4.5] implies the Green correspondent of W_2 is $V_2(\lambda\alpha^2)$. Similarly, $W_{18} \leftrightarrow V_{18}(\lambda\alpha)$.



(b)

where $\theta_i = 16$, $\phi(1) \equiv -1 \pmod{19}$, $\rho(1) \equiv \gamma(1) \equiv 1 \pmod{19}$, and $W_1 \leftrightarrow V_1(\lambda\alpha)$, $U_1 \leftrightarrow V_1(\lambda\alpha^2)$.

(3.4) *If* (a) *holds, then* $\psi(1) = \zeta(1) = 56$ *and* $\mu(1) = 96$. *If* (b) *holds, then one of* $\phi(1) = 132, \rho(1) = 96, \gamma(1) = 20$; $\phi(1) = 132, \rho(1) = 20, \gamma(1) = 96$; *or* $\phi(1) = 56, \rho(1) = 20 = \gamma(1)$ *is true.*

Proof. Suppose $B_0(19)$ satisfies (i). Then $M_{2_N} = V_2(\alpha^2) \oplus V_{19}(\sigma)$. Since Z is in the kernel of all ordinary and Brauer characters in $B_0(19)$, $\sigma \in \langle\alpha\rangle$. Then [1, Lemma 2.3] and the fact that $G = G'$ forces the action of any element of G, on any KG-module, to have determinant 1 imply $\sigma = 1$. Now by [1, Lemma 2.4, Lemma 2.5],

$$(L \otimes M_2)_N = V_{17}(\lambda\alpha^2) \oplus V_{15}(\lambda\alpha) \oplus \sum_{i=0}^{15} V_{19}(\lambda\alpha^{-i}). \tag{3.5}$$

Let L_{17}, L_{15} be the indecomposable KG-modules such that $L_{17} \leftrightarrow V_{17}(\lambda\alpha^2)$, $L_{15} \leftrightarrow V_{15}(\lambda\alpha)$. (3.5) implies $\dim L_{17} + \dim L_{15} \leq 32 + 19 \cdot 16$. [1, Lemma 2.3] says that $L^* \leftrightarrow V_{16}(\lambda^{-1})$, $W_2^* \leftrightarrow V_2(\lambda^{-1}\alpha^2)$, $W_{18}^* \leftrightarrow V_{18}(\lambda^{-1}\alpha)$, $L_{17}^* \leftrightarrow V_{17}(\lambda^{-1}\alpha^2)$, $L_{15}^* \leftrightarrow V_{15}(\lambda^{-1}\alpha)$.

[1, Lemma 2.6] implies 1 is a npmv of both $V_{16}(\lambda^{-1}) \otimes V_{17}(\lambda\alpha^2)$ and $V_{17}(\lambda^{-1}\alpha^2) \otimes V_{16}(\lambda)$. Hence, both $L^* \otimes L_{17}$ and $L_{17}^* \otimes L$ have a nonzero invariant [1, Theorem 4.1], so that $L \subseteq L_{17}$ and $L \subseteq L_{17}/\mathrm{rad}\, L_{17}$. Thus the multiplicity of L as a constituent of L_{17} is at least two.

Suppose (a) holds. By the method used above, we see that the multiplicity of W_2 as a constituent of L_{17}, and the multiplicities of L, W_2, and W_{18} as constituents of L_{15}, are all at least two.

Let $\dim W_2 = 2 + 19a$, $a > 0$, and $\dim W_{18} = 18 + 19b$, $b \geq 0$. Then

$$\begin{aligned} 4(16 + 2 + 19a) + 2(18 + 19b) &= 4(\dim L + \dim W_2) + 2 \dim W_{18} \\ &\leq \dim L_{15} + \dim L_{17} \leq 32 + 19 \cdot 16 \end{aligned}$$

implies

$$2a + b \leq 6. \tag{3.6}$$

Since W_2 and W_{18} are constituents of L_{15},

$$W_{2_N} = V_2(\lambda\alpha^2) \oplus \sum_{i(a\text{ terms})} V_{19}(\lambda\alpha^{-i}), \quad W_{18_N} = V_{18}(\lambda\alpha) \oplus \sum_{i(b\text{ terms})} V_{19}(\lambda\alpha^{-i}).$$

We set determinants equal to 1 (as $G = G'$) and apply [1, Lemma 2.3] to obtain

$$1 = \lambda^{2+19a}\alpha^n = \lambda^{18+19b}\alpha^m$$

for some integers n and m. Since α is trivial on Z, $1 = (\lambda^{2+19a})_Z = (\lambda^{18+19b})_Z$. Since L is faithful, λ is faithful on Z and Z is cyclic [1, Proposition 5.1]. Thus by (3.3), $4 | z | 2 + 19a$, $4 | z | 18 + 19b$; hence $a \equiv b \equiv 2 \pmod 4$. Then (3.6) implies $a = b = 2$. So $\dim W_2 = 40$, $\dim W_{18} = 56$, and the result follows if (i) and (a) are true.

Still assuming $B_0(19)$ satisfies (i), suppose (b) holds. As above, we see that $W_1 \subseteq L_{15}$, $U_1 \subseteq L_{15}/\mathrm{rad}\, L_{15}$, $U_1 \subseteq L_{17}$, $W_1 \subseteq L_{17}/\mathrm{rad}\, L_{17}$, $L \subseteq L_{15}$, and $L \subseteq L_{15}/\mathrm{rad}\, L_{15}$. Let $\dim W_1 = 1 + 19w$, $\dim U_1 = 1 + 19u$. Then

$$4 \dim L + 2(\dim W_1 + \dim U_1) \leq \dim L_{15} + \dim L_{17} \leq 32 + 19 \cdot 16$$

implies $u + w \leq 7$. As before, $G = G'$, [1, Lemma 2.3], [1, Proposition 5.1] and (3.3) imply $4 \mid 1 + 19w$, $4 \mid 1 + 19u$. Then $u \equiv w \equiv 1 \pmod 4$. It follows that either $w = 1$ and $u = 5$ (hence $\phi(1) = 132$, $\rho(1) = 20$, $\gamma(1) = 96$), $w = 5$ and $u = 1$ (hence $\phi(1) = 132$, $\gamma(1) = 20$, $\rho(1) = 96$), or $w = 1 = u$ (and $\phi(1) = 56$, $\rho(1) = 20 = \gamma(1)$).

Now suppose (ii) holds for $B_0(19)$. By the method applied to M_2 in case (i), we see that $M_{3_N} = V_3(\alpha) \oplus V_{19}(1)$. By [1, Lemma 2.4, Lemma 2.5],

$$(L \otimes M_3)_N = V_{18}(\lambda\alpha) \oplus V_{16}(\lambda) \oplus V_{14}(\lambda\alpha^2) \oplus \sum_{i=0}^{15} V_{19}(\lambda\alpha^{-i}). \tag{3.7}$$

Let $L_{18} \leftrightarrow V_{18}(\lambda\alpha)$, $L_{14} \leftrightarrow V_{14}(\lambda\alpha^2)$ under the Green correspondence. Of course, $L \leftrightarrow V_{16}(\lambda)$. Then (3.7) implies $\dim L_{18} + \dim L_{14} \leq 32 + 16 \cdot 19$.

Suppose (a) holds. Then $L_{18} = W_{18}$. We see, by the method used above, that both L and W_2 occur as constituents of L_{14} with multiplicity at least two. Let $\dim W_2 = 2 + 19a$, $\dim W_{18} = 18 + 19b$. Then

$$\begin{aligned} 18 + 19b + 2(16 + 2 + 19a) &= \dim L_{18} + 2(\dim L + \dim W_2) \\ &\leq \dim L_{18} + \dim L_{14} \leq 32 + 16 \cdot 19 \end{aligned}$$

implies $2a + b \leq 14$. As before, $G = G'$, [1, Lemma 2.3], [1, Proposition 5.1] and (3.3) imply $4 \mid 2 + 19a$, $4 \mid 18 + 19b$. Then $a \equiv b \equiv 2 \pmod 4$. It follows that one of $a = b = 2$, $a = 6$ and $b = 2$, or $a = 2$ and $b = 6$ must hold. But the last two cases imply $\mu(1) = \dim W_2 + \dim W_{18} = 172 = 4 \cdot 43$. Hence $43 \mid |G|$, a contradiction. Therefore $a = b = 2$, and (3.4) is true if (a) holds.

Suppose (b) holds. As before, L is a constituent of L_{14} with multiplicity at least two, and each of W_1, U_1 are constituents of both L_{18} and L_{14}. Again, let $\dim W_1 = 1 + 19w$, $\dim U_1 = 1 + 19u$. Then

$$\begin{aligned} 2(16 + 1 + 19w + 1 + 19u) &= 2(\dim L + \dim W_1 + \dim U_1) \\ &\leq \dim L_{18} + \dim L_{14} \\ &\leq 32 + 19 \cdot 16 \end{aligned}$$

implies $u + w \leq 7$. Since $u \equiv w \equiv 1 \pmod 4$, we again have one of $w = 1$ and $u = 5$, $u = 1$ and $w = 5$, or $w = u = 1$. Thus (3.4) holds in all cases.

We use 19–11 block separation to complete the proof. Since $\xi(1) = 77$, $11 \mid |G|$. Because the centralizer of a nontrivial 19-element has order $19z$, $z \mid 16$, the centralizer of a nontrivial 11-element has order prime to 19. Since the exceptional characters in a 19-block of positive defect agree on $19'$-elements, they must agree on the centralizer of a nontrivial 11-element. If they are zero on all 11-singular elements, then each is in its own 11-block of defect zero [8; IV.3.13, IV.4.20]. Otherwise, they are all in the same 11-block by Brauer's second main theorem.

Since ξ is the only character of degree 77 in $B_0(19)$, which is invariant under algebraic conjugation, it follows that ξ is rational. Since G/Z is simple, the kernel of ξ is precisely Z. Then Proposition 2.2 implies $\xi \notin B_0(11)$.

Now block separation [8, IV.4.23] says that $\sum \tau(1)\tau(x) \equiv 0 \pmod{11^m}$ where $\langle x \rangle = P$, $m = \nu_{11}(|G|)$, and τ ranges over all irreducible characters in

$B_0(19) \cap B_0(11)$. Since $\sum_i \chi_i(x) = -1$, the only possibilities for $\sum \tau(1)\tau(x)$ divisible by 11 are 1–56 and 1–56–22. Hence $11^2 \nmid |G|$.

Now for any 11-block of G of positive defect, there is an integer $n \not\equiv 0 \pmod{11}$ and an integer $r \mid 10$ such that all degrees of irreducible characters in the block are congruent (mod 11) to $\pm n$ or $\pm rn$ [3]. Let B' be the 11-block of the θ_i.

If (a) holds, then (3.4) and block separation imply $B \subseteq B'$. But $\theta(1) \equiv 5 \pmod{11}$, $\mu(1) \equiv -3 \pmod{11}$, and $\zeta(1) = 56 \not\equiv \pm 5$ or $\pm 3 \pmod{11}$, a contradiction.

If (b) holds and $\phi(1) = 132$, then $\phi \notin B'$. Then block separation implies ρ, γ (and all the θ_i) are in $B' \cap B$. However, $\theta(1) \equiv 5 \pmod{11}$, and $\rho(1)$, $\gamma(1)$ are congruent (mod 11) to 8, 9 in some order, a contradiction.

So (3.4) implies the character degrees of B are 16 (6 of them), 20 (2), and 56. Then block separation forces $B \subseteq B'$. But $20 \not\equiv \pm 56$ or $\pm 16 \pmod{11}$, a final contradiction.

4. Proofs of the consequences

Proof of Theorem 2. Either $d > (p + 1)/2$ or $G/Z \approx PSL_2(p)$ [14]. But the latter implies $t = 2$, a contradiction. Since G has a faithful indecomposable representation of degree d in characteristic p, we see that $d = p - (p - 1)/t$ [1, (4.3)]. Assume $3 \leq t < 8$. By Proposition 2.1 there is a group G_1, not of type $L_2(p)$, with a faithful irreducible complex representation of degree d, a Sylow p-subgroup of order p, $G_1 = G_1'$, and $G_1/Z(G_1)$ simple. Since d, and hence t (again by [1, (4.3)]) are the same for G and G_1, we may assume $G = G_1$.

[2] implies $p \leq t^2 - 3t + 1$. So $t > 3$. If $t = 4$ then $p = 5$ and $e = 1$, whence $d = p - 1$, a contradiction. If $t = 5$ then $p \leq 11$. Since $e \geq 2$, we must have $p = 11, e = 2, d = 9$. This contradicts [9, 8.3.4.iii], [10, Theorem 2] (and was eliminated in [13]). If $t = 6$ then $p = 19$ and $d = 16$, contradicting Theorem 1. If $t = 7$ then e is even. [7, (2.1)] implies $|Z|$ is odd. This contradicts [7, Theorem 1].

Proof of Theorem 3. If G exists satisfying the hypothesis but not the conclusion, then Proposition 2.1 implies we may assume G is not of type $L_2(p)$ and $|P| = p$. Then [14] and [1, (4.3)] imply $d = p - e$, where $e = |N{:}\ C| \leq (p - 1)/3$. Theorem 2 yields $t \geq 8$, so $d \geq (7p + 1)/8$. [2] implies $d \geq p + (3/2) - (p + 5/4)^{1/2}$, a contradiction.

Proof of Theorem 4. If $p = 3$ then G is abelian. If $p = 5$ the result follows by [15]. So assume $p > 5$. Then we may suppose $d < p - 2$ [9, 8.3.4.iii], [10]. Hence we may assume $p > 7$ [15]. It suffices to show (i) or (ii) must hold. By Proposition 2.1, [14], and Theorem 2, we may assume that $|P| = p$, G is not of type $L_2(p)$, and $d = p - (p - 1)/t$ where $t|N{:}\ C| = p - 1$, $t \geq 8$. Then $p \geq 31$. If $p = 31$, $d \geq 31 - 30/10 = 28$. If $p \geq 37$, $d \geq (7p + 1)/8 > 32$.

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NORTHERN ILLINOIS UNIVERSITY
DE KALB, ILLINOIS