MORE NEW INTEGER PAIRS FOR FINITE HJELMSLEV PLANES

BY

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Introduction

Every finite projective Hjelmslev plane H possesses two integer invariants denoted by t and r. (See, e.g., [4].) Every point of H possesses precisely t^2 neighbor points; r denotes the order of the projective plane canonically paired to H. Such an H will be called a (t, r) PH-plane. In [3], Drake and Lenz constructed the first examples of PH-planes with invariants (t, r), t not a power of r. These PH-planes were so constructed as to possess given PH-planes as epimorphic images. The construction methods devised were most successful when the given epimorphic images were taken to be 2-uniform PH-planes; i.e., PH-planes with $t = r$. In this note, we refine the methods of [3] to obtain PH-planes as preimages of strongly n-uniform PH-planes for arbitrary n. (See [1].) PH-planes with invariants $(t, 2)$ are now known to exist for 41 values of t less than or equal to 1,000 (16 of the 41 thanks to the results of this note); they are known not to exist for three values of t but remain in doubt for the other 956 possible values.

1. Preliminaries

Of great importance to our construction is the following familiar lemma couched in unfamiliar terms. (See [5] or [6].)

Theorem 1.1 (König's Lemma). *Let T be a tactical configuration with block size and replication number both equal to $r < \infty$. Let $S = \{n_1, \ldots, n_r\}$ be a labeling set. Then there is a function f from the flags of T to S such that each point and each line of T occurs in r flags labeled by each of the r elements of S.*

We refer the reader to [1] for the definitions of n-uniformity and strong n-uniformity. If H is a strongly n-uniform PH-plane, then the invariants of H are (t, r) where $t = r^{n-1}$. We write $P\,(\sim i)\,Q$ if P and Q are joined by at least r^i lines, $0 \leq i \leq n$; $P\,(\simeq i)\,Q$ if P and Q are joined by exactly r^i lines, $0 \leq i < n$; $P\,(\simeq n)\,Q$ if $P = Q$. By [1, Definition 3.3, Proposition 3.3(2), and Proposition 2.2(2)], we may define $(\sim i)$ and $(\simeq i)$ dually for lines; i.e., $g\,(\sim i)\,h$ if $|g \cap h| \geq r^i$ and $g\,(\simeq i)\,h$ if $|g \cap h| = r^i$. The following result is part of [1, Proposition 2.2] together with [1, Proposition 3.6].

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PROPOSITION 1.2. *Let H be a strongly $(d+1)$-uniform PH-plane with invariants (t, r). Then the following properties hold:*

(1) $t = r^d$.

(2) *If P and Q are distinct points of H, there exists an integer $i \leq d$ such that $P\ (\simeq i)\ Q$.*

(3) *If (P, g) is a flag of H and i is a positive integer $\leq d+1$, then the number of lines through P which meet g in at least r^i points is r^{d+1-i}.*

(4) *$(\sim i)$ is an equivalence relation on the points of H for $i = 0, 1, \ldots, d+1$.*

(5) *Let $|g \cap h| = r^i$, $0 \leq i \leq d$, $P \in g \cap h$. Then*

$$g \cap h = \{X \in g \colon X\ (\sim d + 1 - i)\ P\}.$$

(6) *(Property S.) Suppose that P, Q, g, h are points and lines of H such that $P \in g \cap h$, $Q \in g - h$, $|g \cap h| = r^i$, $P\ (\simeq j)\ Q$, $i + j \leq d$. Then $Q\ (\simeq i+j)\ h$; i.e., there exists a point X on h such that $Q\ (\simeq i + j)\ X$ but not point Y on h such that $Q\ (\sim i + j + 1)\ Y$.*

The dual of a PH-plane H is again a PH-plane with the same invariants (t, r) as H. The following result is then only a slight rewording of [1, Theorem 4.7].

THEOREM 1.3. *A PH-plane H is strongly n-uniform if and only if H and the dual of H are both n-uniform PH-planes with the same invariants (t, r).*

As a consequence of Theorem 1.3, all the properties dual to (2)–(6) of Proposition 1.2 hold in a strongly $(d+1)$-uniform PH-plane, and we shall refer to these properties as $(2)^d, \ldots, (6)^d$. We shall refer to all twelve properties throughout the paper without direct mention of Proposition 1.2, referencing, e.g., (3) or $(5)^d$. We now change the notation of [1] and extract the following result from [1, Propositions 3.2 and 3.3]. (The sequences of subscripts on PH-planes and maps below are the reverse of the ones given in [1].)

PROPOSITION 1.4. *Let $\mathscr{H} = \mathscr{H}^{d+1}$ be a strongly $(d+1)$-uniform PH-plane with invariants $(t = r^d, r)$. Then there exist strongly i-uniform PH-planes $\mathscr{H}^i$ with invariants (r^{i-1}, r) for $1 \leq i \leq d+1$ and epimorphisms $\mu_i \colon \mathscr{H}^{d+1} \to \mathscr{H}^i$ such that (1) $\mu_i(P) = \mu_i(Q)$ if and only if $P\ (\sim i)\ Q$, (2) $\mu_i(g) = \mu_i(h)$ if and only if $g\ (\sim i)\ h$ and (3) $\mu_i(P) \in \mu_i(g)$ if and only if there exist incident Q and h in the inverse images (in $\mathscr{H}$) of $\mu_i(P)$ and $\mu_i(g)$. Also when $j < i$, the statement $\mu_i(P)\ (\simeq j)\ \mu_i(Q)$ is equivalent to the statement $P\ (\simeq j)\ Q$; the dual assertion holds (for lines). In particular, every μ_i preserves and reflects the neighbor relation.*

THEOREM 1.5 [1, Theorem 5.4]. *Every finite Desarguesian PH-plane is strongly n-uniform for some n.*

Let $\mathcal{N} = \{N_0, N_1, \ldots, N_q\}$ be a set of square (0, 1)-matrices, each of order s^2, such that the following conditions are satisfied:

$$N_i N_j^T = N_i^T N_j = J \text{ for } i \neq j, \tag{1.1}$$

$$\sum_{i=0}^{q} N_i N_i^T \geq 2J, \quad \sum_{i=0}^{q} N_i^T N_i \geq 2J. \tag{1.2}$$

Then $\mathcal{N}$ is called a set of *auxiliary matrices.* Here J denotes the matrix of all 1's, and we write $[x_{ij}] \geq [y_{ij}]$ if the two matrices are the same size and $x_{ij} \geq y_{ij}$ for all i, j. The proof and statement of [3, Theorem 5.1] together with Theorem 1.5 above yield the following result.

THEOREM 1.6. *Let r be any prime power, d be any nonnegative integer. Then there exists a set of $r + 1$ auxiliary matrices $N_0, \ldots, N_r$ of row size r^{2d}. The concatenated matrix $(N_0 \ldots N_r)$ is the incidence matrix of a point neighborhood $\mathcal{N}^{d+1}$ of a strongly $(d + 1)$-uniform Desarguesian PH-plane $\mathcal{H}^{d+1}$ with invariants $(t, r) = (r^d, r)$. Each N_i corresponds to a single line neighborhood of $\mathcal{H}^{d+1}$. The matrix $(N_0^T \ldots N_r^T)^T$ is the incidence matrix of a line neighborhood of $\mathcal{H}^{d+1}$ with each N_i corresponding to a single point neighborhood.*

In truth, the quoted results only assure that $\mathcal{H} = \mathcal{H}^{d+1}$ is strongly n-uniform for *some n*. Since there are $r + 1$ line neighborhoods which intersect a given point neighborhood of $\mathcal{H}$ nontrivially, the projective plane paired to $\mathcal{H}$ has order r. Since there are r^{2d} points in a point neighborhood of $\mathcal{H}$, the invariant t of $\mathcal{H}$ is r^d. Now Proposition 1.2 (1) yields $n = d + 1$. The following result is [3, Proposition 4.1].

PROPOSITION 1.7. *Let $A = [A_{ij}]$ be a (0, 1)-matrix where each A_{ij} is square of order t^2. Define a_{ij} to be 0 if A_{ij} is the 0-matrix, 1 otherwise. Suppose $B = [a_{ij}]$ is the incidence matrix of a projective plane of order r. Suppose also that $A_{ik}(A_{jk})^T = J$ when $i \neq j$ and $A_{ik}, A_{jk} \neq 0$; $(A_{ik})^T(A_{ij}) = J$ when $k \neq j$ and $A_{ik}, A_{ij} \neq 0$. Assume also that*

$$\sum_{j=0}^{r^2+r} A_{ij}(A_{ij})^T \geq 2J, \quad \sum_{j=0}^{r^2+r} (A_{ij})^T A_{ij} \geq 2J.$$

Then A is the incidence matrix of a (t, r) PH-plane.

The following easy consequence of Proposition 1.7 was used without explicit statement in [3].

THEOREM 1.8. *The existence of a set $\{N_0, \ldots, N_q\}$ of $q + 1$ auxiliary matrices of row length s^2 implies the existence of a PH-plane with invariants $(t, r) = (s, q)$ provided q is the order of a projective plane.*

Proof. Let $B = [a_{ij}]$ be an incidence matrix for a projective plane of order q. Replace each a_{ij} by a square matrix A_{ij} of order s^2 as follows: if $a_{ij} = 0$, take A_{ij} to be the 0-matrix; if $a_{ij} = 1$, take A_{ij} to be one of the N_i. By König's Lemma, this may be done in such a manner that each row and column of B will receive each N_i as a replacement. The result now follows from Proposition 1.7.

2. Labeling the flags of a point neighborhood of a strongly $(d + 1)$-uniform PH-plane

In this section, we shall frequently write $\{j, \ldots, k\}$ (with only the first and last numbers specified) to mean the set of all positive integers i such that $j \leq i \leq k$. We now take r, d, t, N_i, $\mathcal{N}^{d+1}$, $\mathcal{H}^{d+1}$ to be integers and structures which satisfy all the conditions of Theorem 1.6. Now let $\mathcal{T}^{d+1}$ be the nontrivial incidence structure consisting of a $(\sim d)$-equivalent point class of $\mathcal{N}^{d+1}$ together with a $(\sim d)$-equivalent line class of $\mathcal{N}^{d+1}$. By nontrivial, we mean only that the structure contains at least one flag (Q, g). Let h be an arbitrary line of $\mathcal{N}^{d+1}$. Then $|g \cap h| \geq r^d$. Let $P \in g \cap h$. Either $Q \in h$ or $g \neq h$. In the latter case, $(2)^d$ implies that $|g \cap h| = r^d$. Then (5) and (2) imply that $g \cap h$ consists of all neighbors of P which lie in g, hence that $Q \ (\simeq 0)\ P$. Then (6) implies the existence of a point R on h such that $R \ (\simeq d)\ Q$. In every case, h contains a point of $\mathcal{T}^{d+1}$. By duality, every point R of $\mathcal{T}^{d+1}$ lies on a line h of $\mathcal{T}^{d+1}$. Now (3) and $(3)^d$ imply that $\mathcal{T}^{d+1}$ is a tactical configuration with block size and replication number both equal to r. We take $S = \{0, \ldots, r - 1\}$; applying (4), $(4)^d$, and Theorem 1.1, we obtain the existence of a function f_{d+1}, defined on the flags of all of $\mathcal{N}^{d+1}$ and satisfying the conclusion of Theorem 1.1 on each nontrivial $\mathcal{T}^d$.

For $2 \leq i \leq d + 1$, let $\mathcal{N}^i = \mu_i(\mathcal{N}^d)$ where the μ_i are the epimorphisms of Proposition 1.4. Then each $\mathcal{N}^i$ is the point neighborhood of a strongly i-uniform PH-plane with invariants $(t = r^{i-1}, r)$. As above we obtain the existence of a function f_i, defined on the flags of $\mathcal{N}^i$ and satisfying the conclusion of Theorem 1.1 on each nontrivial $\mathcal{T}^i$; for this function, we take S to be the set

$$\{0, r^{d+1-i}, 2r^{d+1-i}, \ldots, (r - 1)r^{d+1-i}\}. \tag{2.1}$$

(In case $i = d + 1$, this merely repeats the definition of the preceding paragraph.) Each flag (P, g) of $\mathcal{N}^{d+1}$ defines a sequence of flags

$$(P^i = \mu_i(P), g^i = \mu_i(g)), \quad i = 2, 3, \ldots, d + 1.$$

We define f on the flags of $\mathcal{N}^{d+1}$ by the rule

$$f(P, g) = 1 + \sum_{i=2}^{d+1} f_i(P^i, g^i). \tag{2.2}$$

Clearly f is a map into the set $\{1, \ldots, r^d\}$, and $f(P, g) = f(Q, h)$ is equivalent to the assertion that $f_i(P^i, g^i) = f_i(Q^i, h^i)$ for all i. Let $\mathcal{N}(d + 1, i)$, $0 \leq i \leq r$, denote the incidence structure consisting of the points of $\mathcal{N}^{d+1}$ and the lines of a single neighbor class of lines of $\mathcal{H}^{d+1}$; we may choose each $\mathcal{N}(d + 1, i)$ so that it is represented by the incidence matrix N_i. Suppose now that $f(P, g) = f(P, h)$ where P, g, h are point and lines from some $\mathcal{N}(d + 1, i)$. Then $(2)^d$

implies that $g\ (\sim 1)\ h$, and we make the induction assumption that $g\ (\sim i - 1)\ h$. By Proposition 1.4, $g^i\ (\sim i - 1)\ h^i$. The two assertions, $g^i\ (\sim i - 1)\ h^i$ and $f_i(P^i, g^i) = f_i(P^i, h^i)$, together imply that $g^i = h^i$; i.e., that $g\ (\sim i)\ h$. By induction, $g\ (\sim d + 1)\ h$; i.e., $g = h$. We have proved the following result:

(2.3) Let P be an arbitrary fixed point of an arbitrary $\mathcal{N}(d + 1, i) = \mathcal{N}$. Then the restriction of f to the flags (P, x) of $\mathcal{N}$ is a one-to-one mapping onto $\{1, \ldots, r^d\}$.

Dualizing the last few lines of the above argument yields:

(2.4) The restriction of f to the flags (X, g) of $\mathcal{N}$, g a fixed line of $\mathcal{N}$, is a one-to-one mapping onto $\{1, \ldots, r^d\}$.

Next we prove:

(2.5) Let $P\ (\simeq i)\ Q$ for some $P \in \mathcal{N}$, some positive integer $i \leq d$. Then there exists an integer k such that

$$\{f(P, x): P, Q \in x\} = \{kr^i + 1, \ldots, (k + 1)r^i\}.$$

We begin by labeling the lines which join P to Q by g_j, $1 \leq j \leq r^i$. Then $(5)^d$ implies that $g_j\ (\sim d + 1 - i)\ g_1 = g$ for all j; i.e., $(g_j)^b = g^b$ for all $b \leq d + 1 - i$. This means that $\sum_{b=2}^{d+1-i} f_b(P^b, (g_j)^b)$ is independent of j and divisible by r^i. Since

$$\sum_{b=d-i}^{d+1} f_b(P^b, (g_j)^b) < r^i,$$

there exists an integer k such that $\{f(P, g)\} \subset \{kr^i + 1, \ldots, (k + 1)r^i\}$. The truth of (2.5) now follows from (2.3), and the dual argument yields the following conclusion:

(2.6) Let $g\ (\simeq d + 1 - i)\ h$ for some $h \in \mathcal{N}$, some positive integer $d + 1 - i \leq d$. Then there exists an integer k such that

$$\{f(X, g): X \in g \cap h\} = \{kr^{d+1-i} + 1, \ldots, (k + 1)r^{d+1-i}\}.$$

Now let $P\ (\simeq i)\ Q$ for some positive $i \leq d$; $P, Q \in g$. Then $(5)^d$, (3) and (2) imply the existence of a line h such that $P, Q \in h$ and $h\ (\simeq d + 1 - i)\ g$. By (2.6), there exists a k such that

$$f(P, g), f(Q, g) \in \{kr^{d+1-i} + 1, \ldots, (k + 1)r^{d+1-i}\}.$$

If $X \in g$ such that $P\ (\sim i + 1)\ X$, the same argument yields the existence of a k' such that

$$f(P, g), f(X, g) \in \{k'r^{d-i} + 1, \ldots, (k' + 1)r^{d-i}\} = D.$$

By $(3)^d$, there are precisely r^{d-i} such X; hence (2.4) implies that $f(Q, g) \notin D$.

We have proved:

(2.7) If $P\ (\simeq i)\ Q$ and $P, Q \in g$, then for some integer k,

$$f(P, g), f(Q, g) \in \{kr^{d+1-i} + 1, \ldots, (k+1)r^{d+1-i}\};$$

but for no integer k, is it the case that

$$f(P, g), f(Q, g) \in \{kr^{d-i} + 1, \ldots, (k+1)r^{d-i}\}.$$

3. The constructions

To state our main result, we must consider the following inequality:

$$\left(\sum_{j=0}^{b} r^j\right)(r+1) \leq q + 1 \leq r^{2b}(r+1). \tag{3.1}$$

THEOREM 3.1. *Let $b \geq 0$, $s \geq 1$, $r \geq 2$, q be integers such that* (3.1) *holds, r is a prime power, and there exists a set of $q + 1$ auxiliary matrices of row length s^2. Then there exists a set of $r + 1$ auxiliary matrices of row length $(r^{2b}s)^2$ and (consequently) an $(r^{2b}s, r)$ PH-plane.*

Hint to the reader. In Theorem 3.2 we shall state the comparable result for the odd power case (with r^{2b+1}). The proofs of the two cases are essentially the same, and we leave the second to the reader. The conscientious reader will likely save time by supplying his proof of Theorem 3.2 simultaneously with his reading of the proof of Theorem 3.1.

Proof of Theorem 3.1. If $b = 0$, Theorem 1.8 yields the desired conclusion, so we assume henceforth that $b \geq 1$. Throughout the proof, we use the results of Section 2 with $2b$ substituted for d. For the proof of Theorem 3.2, one must, of course, substitute $2b + 1$ for d. By Theorem 1.6, there exists a set $\mathscr{M} = \{M_0, \ldots, M_r\}$ of auxiliary matrices of row size r^{4b} which satisfy:

$$M_i M_j^T = M_i^T M_j = J \text{ for } i \neq j; \tag{3.2}$$

$$\sum_{i=0}^{r} M_i M_i^T \geq 2J, \quad \sum_{i=0}^{r} M_i^T M_i \geq 2J; \tag{3.3}$$

(3.4) $(M_0 \ldots M_r)$ is the incidence matrix of a point neighborhood $\mathscr{M}^{2b+1}$ of a strongly $(2b+1)$-uniform PH-plane $\mathscr{H}^{2b+1}$ with invariants (r^{2b}, r); each M_i represents a single neighbor class of lines. In addition, $(M_0^T \ldots M_r^T)^T$ is the incidence matrix of a line neighborhood of $\mathscr{H}^{2b+1}$, so represented that each M_i corresponds to a single neighbor class of points.

By hypothesis, there exists a set $\mathscr{L} = \{L_0, \ldots, L_q\}$ of auxiliary matrices of row size s^2 which satisfy:

$$L_i L_j^T = L_i^T L_j = J \text{ for } i \neq j; \tag{3.5}$$

$$\sum_{i=0}^{q} L_i L_i^T \geq 2J, \quad \sum_{i=0}^{q} L_i^T L_i \geq 2J. \tag{3.6}$$

We now divide $\mathscr{L}$ into $r + 1$ disjoint subsets $\mathscr{L}_i$, $0 \leq i \leq r$, each with at least $1 + r + \cdots + r^b$ and at most r^{2b} elements. We denote $|\mathscr{L}_i|$ by $n(i) = n$. Then there exists a set S of $n + 1$ integers r_k such that $0 = r_0 < r_1 < \cdots < r_n = r^{2b}$ and $S - \{0\}$ contains all of the following integers: jr^b for $1 \leq j \leq r^b$; $jr^{b+h} + r^{b-h}$ for $1 \leq h \leq b$, $0 \leq j < r^{b-h}$. (For the proof of Theorem 3.2, one asks that S contain the following integers: jr^{b+1} for $1 \leq j \leq r^b$; $jr^{b+h} + r^{b+1-h}$ for $1 \leq h \leq b + 1$, $0 \leq j < r^{b-h+1}$.) There exists a *surjective* mapping

$$F_i\colon \{1, \ldots, r^{2b}\} \to \mathscr{L}_i$$

such that $F_i(\mu) = F_i(v)$ when $r_{k-1} < \mu, v \leq r_k$, $1 \leq k \leq n$. Then:

$$F_i(\mu) \neq F_i(v) \quad \text{if } \mu \leq jr^b < v,\ 1 \leq j < r^b; \tag{3.7}$$

$$F_i(\mu) \neq F_i(v) \quad \text{if } \mu \leq jr^{b+h} + r^{b-h} < v, \quad 1 \leq h \leq b,\ 0 \leq j < r^{b-h}. \tag{3.8}$$

Now we replace each element m^i_{jk} of M_i by a square matrix N^i_{jk} of order s^2 according to the following rule: if $m^i_{jk} = 0$, set N^i_{jk} equal to the zero matrix. If $m^i_{jk} = 1$, let (P_j, g_k) denote the flag of $\mathscr{M}^{2b+1}$ associated with m^i_{jk}, and we set

$$N^i_{jk} = F_i(f(P_j, g_k)). \tag{3.9}$$

Next we set

$$N_i = (N^i_{jk}),\ 1 \leq j, k \leq r^{4b}. \tag{3.10}$$

We obtain $N_i N_l^T = (S_{jk})$, where

$$S_{jk} = \sum_{x=1}^{r^{4b}} N^i_{jx}(N^l_{kx})^T. \tag{3.11}$$

If $i \neq l$, then (3.2) and (3.5) imply the existence of an integer c such that

$$\sum_{x=1}^{r^{4b}} N^i_{jx}(N^l_{kx})^T = N^i_{jc}(N^l_{kc})^T = J.$$

By the dual argument, (1.1) is satisfied.

Next we consider $N_i N_i^T = (S^i_{jk})$, and compute

$$\sum_{i=0}^{r} N_i N_i^T = (T_{jk}). \tag{3.12}$$

We have

$$T_{jj} = \sum_{i=0}^{r} S^i_{jj} = \sum_{i=0}^{r} \sum_{x=1}^{r^{4b}} N^i_{jx}(N^i_{jx})^T.$$

By (3.9) and (2.3), this sum is $\sum_{i=0}^{r} \sum_{y=1}^{r^{2b}} F_i(y) \cdot (F_i(y))^T$. Since the F_i's are all surjective, (3.6) yields

$$T_{jj} \geq \sum_{k=0}^{q} L_k L_k^T \geq 2J. \tag{3.13}$$

We next examine T_{jk} with $j \neq k$. Distinct points of a PH-plane are joined by lines from only one neighbor class. Then (3.4) implies the existence of an integer c such that $T_{jk} = \sum_{i=0}^{r} S_{jk}^{i} = S_{jk}^{c}$. We have

$$T_{jk} = \sum_{x=1}^{r^{4b}} N_{jx}^{c}(N_{kx}^{c})^{T}. \tag{3.14}$$

Now let P_j, P_k be the points of $\mathscr{M}^{2b+1}$ represented by the jth and kth rows of $(M_0 \ldots M_r)$.

We consider two cases. First suppose that $P_j\ (\simeq i)\ P_k$ where $i \leq b$, and let g be any line of $\mathscr{H}^{2b+1}$ which joins P_j and P_k. (For the proof of Theorem 3.2, we also assume $i \leq b$.) Then (2.7) implies the existence of an integer h such that

$$\mu = f(P_j, g) \leq hr^{2b-i} < \nu = f(P_k, g). \tag{3.15}$$

Suppose that g is represented by the eth column of M_c. Then (3.9) yields $N_{je}^{c} = F_c(\mu)$ and $N_{ke}^{c} = F_c(\nu)$. Then (3.15) and (3.7) imply that N_{je}^{c} and N_{ke}^{c} are distinct elements of $\mathscr{L}$. By (3.5),

$$N_{je}^{c}(N_{ke}^{c})^{T} = J. \tag{3.16}$$

Since there are at least two lines g joining P_j to P_k, there are at least two values of e for which (3.16) holds. Then (3.14) implies that

$$T_{jk} \geq 2J. \tag{3.17}$$

We now handle the remaining case; i.e., the case $P_j\ (\simeq b + i)\ P_k$, $1 \leq i \leq b$. (The case $P_j = P_k$ is trivial and is left to the reader.) (For the proof of Theorem 3.2, take $1 \leq i \leq b + 1$.) The assumptions on P_j and P_k and assertion (2.5) yield the existence of an integer h such that

$$\{f(P_j, x)\colon P_j, P_k \in x\} = \{hr^{b+i} + 1, \ldots, (h + 1)r^{b+i}\}.$$

Then there exists a line g_1 joining P_j to P_k such that $f(P_j, g_1) \leq hr^{b+i} + r^{b-i}$. (For Theorem 3.2, we demand that $f(P_j, g_1) \leq hr^{b+i} + r^{b+1-i}$.) Now (2.7) implies that $hr^{b+i} + r^{b-1} < f(P_k, g_1)$. Similarly, there exist a line g_2 through P_j and P_k and an integer h' such that

$$f(P_k, g_2) \leq h'r^{b+i} + r^{b-i} < f(P_j, g_2).$$

From (3.8), we obtain $F_c(f(P_j, g_x)) \neq F_c(f(P_k, g_x))$ for $x = 1, 2$. As in the proof of the first case, we obtain

$$T_{jk} \geq 2J. \tag{3.18}$$

From (3.12), (3.13), (3.17), and (3.18), we conclude that $\sum_{i=0}^{r} N_i N_i^T \geq 2J$. By duality, (1.2) is fulfilled by the set $\mathscr{N} = \{N_0, \ldots, N_r\}$; i.e., $\mathscr{N}$ is a set of $r + 1$ auxiliary matrices of order $(r^{2b}s)^2$. The truth of Theorem 3.1 now follows immediately from Theorem 1.8.

Next, consider the condition

$$\left(r^b + \sum_{j=0}^{b} r^j\right)(r + 1) \leq q + 1 \leq r^{2b+1}(r + 1). \tag{3.19}$$

THEOREM 3.2. *Let $b \geq 0$, $s \geq 1$, $r \geq 2$, q be integers such that* (3.19) *holds, r is a prime power, and there exists a set of $q + 1$ auxiliary matrices of row length s^2. Then there exist a set of $r + 1$ auxiliary matrices of row length $(r^{2b+1}s)^2$ and an $(r^{2b+1}s, r)$ PH-plane.*

This theorem was proved in the case $b = 0$ in [3]. The general proof is similar to the proof of Theorem 3.1 and will be left to the reader.

4. Concluding comments

In [3] the following theorem was proved.

THEOREM 4.1. *Assume the existence of a (t, r) PH-plane and a prime power q such that $q + 1 = t(r + 1)$. Then for an arbitrary positive integer b, there exists a (tq^b, r) PH-plane.*

As an indication of the progress made to date on the existence problem for finite *PH*-planes, we now survey the values of t under 1,000 for which $(t, 2)$ *PH*-planes are known to exist. Theorem 1.6 yields existence for all powers of 2; namely, $2^0 = 1, 2, \ldots, 2^9 = 512$. Theorem 4.1 with $(t, r) = (2, 2), (4, 2), (8, 2), (16, 2)$ yields the seven additional t-values: $2 \cdot 5^b$, $1 \leq b \leq 3$; $4 \cdot 11^b$, $1 \leq b \leq 2$; $8 \cdot 23$; $16 \cdot 47$. Next we apply Theorem 3.2 with $b = 0$, $r = 5$. By Theorems 3.2 and 1.6, one may take s to be an arbitrary power of q, q to be any prime power between 11 and 29. Then there exist sets of 6 auxiliary matrices whose row lengths are $(5x)^2$ with $x = 11, 13, 16, 17, 19, 23, 25, 27, 29$. Now we apply Theorem 3.2 a second time with $b = 0$, $r = 2$, hence $q = 5$ and s equal to any of the nine values listed for $5x$; we obtain eight new t-values which go with $r = 2$ ($t = 2 \cdot 5 \cdot 25$ was already counted once). These $10 + 7 + 8 = 25$ t-values are either well known (the 10) or are given by the results of [3].

Next we apply Theorems 3.1 and 3.2 in conjunction with Theorem 1.6, using $r = 2$, $b = 1$ and 2, to obtain the 12 new t-values: $4 \cdot 9$, $4 \cdot 9^2$, $8 \cdot 17$, $8 \cdot 19$ and $16x$ with $x = 23, 25, 27, 29, 31, 37, 41, 43$. Lastly, we make a sequential application of Theorems 3.2 and 3.1. Setting $b = 0$, $r = 9$, and $q = s$ in Theorem 3.2 yields the existence of sets of 10 auxiliary matrices of row lengths $(9x)^2$ with $x = 19, 23, 25$, and 27. Now we apply Theorem 3.1 with $b = 1$, $r = 2$, $q = 9$, and s equal to any of the four values listed for $9x$. This yields our final four values for t; namely, $36x$ with $x = 19, 23, 25, 27$.

We remark that Kleinfeld [4] proved that for $r = 2$, $t = 3$ is impossible; the author [2] has excluded the additional values $t = 5, 7$. To date, these three are the only excluded values under or over 1,000.

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