MEASURES WHOSE POISSON INTEGRALS ARE PLURIHARMONIC II

BY

FRANK FORELLI

1. Introduction

Let V be a vector space over $\mathbf{C}$ of complex dimension n with an inner product. If x and y are in V, then we will denote by $\langle x, y\rangle$ the inner product of x and y. We will denote by B the class of all x in V such that $\langle x, x\rangle < 1$, by $\bar{B}$ the class of all x in V such that $\langle x, x\rangle \leq 1$, and by S the class of all x in V such that $\langle x, x\rangle = 1$. We recall that the Poisson kernel of B is the function $\beta\colon \bar{B} \times B \to (0, \infty)$ defined by

$$\beta(x, y) = [(1 - \langle y, y\rangle)/(1 - \langle x, y\rangle)(1 - \langle y, x\rangle)]^n.$$

(We remark that β is the Poisson kernel with respect to the Bergman metric on B and not the Euclidean metric.)

If Y is a locally compact Hausdorff space, then we will denote by $M_+(Y)$ the class of all Radon measures on Y. Thus if $\mu \in M_+(Y)$ and $E \subset Y$, then $\mu(E) \geq 0$. We will denote by $M(Y)$ the complex linear span of those μ in $M_+(Y)$ for which $\mu(Y) < \infty$. (Thus if Y is compact, then $M(Y)$ is the complex linear span of $M_+(Y)$.) We recall that if X and Y are sets, if f is a function defined on the Cartesian product $X \times Y$, and if $(s, t) \in X \times Y$, then f_s and f^t are the functions defined on Y and X respectively by $f_s(y) = f(s, y)$ and $f^t(x) = f(x, t)$.

If $\mu \in M(S)$, then we define $\mu^\#\colon B \to \mathbf{C}$ by $\mu^\#(y) = \int \beta^y \, d\mu$. Thus $\mu^\# \in C^\infty(B)$. We will denote by σ the Radon measure on S which assigns to each open subset of S its Euclidean volume divided by the Euclidean volume of S (for the purpose of defining σ we regard S as the Euclidean sphere of real dimension $2n - 1$). Thus $\sigma(S) = 1$.

There is the following question.

1.1. If $\mu \in M(S)$, if $\mu^\#$ is pluriharmonic, and if $n \geq 2$, then do we have $\mu \ll \sigma$?

The purpose of this paper (which is a sequel to [2]) is to state and prove Theorem 3.15 and Corollary 4.7 which bear on the question 1.1. The results of the paper [2] suggest that the answer to the question 1.1 is yes. Theorem 3.15 and Corollary 4.7 of this paper support this suggestion.

Received January 27, 1975.

2. A class of measures whose Poisson integrals are pluriharmonic (Theorem 2.4)

If Y is a set, then we will denote by $V_+(Y)$ the class of all measures on Y and we will denote by $V(Y)$ the complex linear span of those μ in $V_+(Y)$ for which $\mu(Y) < \infty$. We recall that if $\mu \in V_+(Y)$ and if $E \subset Y$, then $\mu \llcorner E: 2^Y \to [0, \infty]$ is defined by $(\mu \llcorner E)(F) = \mu(E \cap F)$. Thus $\mu \llcorner E \in V_+(Y)$. There is the following fact of measure theory.

2.1. Proposition. *Let Y be a set. If $\mu \in V_+(Y)$, if $f: Y \to [0, \infty]$, if f is μ measurable, and if we define $\lambda: 2^Y \to [0, \infty]$ by*

$$\lambda(E) = \int f \, d(\mu \llcorner E),$$

then $\lambda \in V_+(Y)$. Thus if $g \in L^1(\mu)$ and if we define $\alpha: 2^Y \to \mathbf{C}$ by

$$\alpha(E) = \int g \, d(\mu \llcorner E),$$

then $\alpha \in V(Y)$.

With regard to Proposition 2.1 we write $d\lambda = f\,d\mu$ and $d\alpha = g\,d\mu$. There is the following fact of the theory of Radon measures.

2.2. Proposition. *Let Y be a locally compact Hausdorff space. If $\mu \in M_+(Y)$, if $f: Y \to [0, \infty]$, if f is μ measurable, if $\int f\,d\mu < \infty$, and if $d\lambda = f\,d\mu$, then $\lambda \in M_+(Y)$. Thus if $g \in L^1(\mu)$ and if $d\alpha = g\,d\mu$, then $\alpha \in M(Y)$.*

We recall the following fact of the theory of functions on B.

2.3. Proposition. *If $f: B \to (0, \infty)$ is pluriharmonic, then $f = \mu^{\#}$ where $\mu \in M_+(S)$.*

If Y is a topological space and if $f: Y \to \mathbf{C}$, then we will denote by $spt(f)$ the support of f. We will denote by $C_{00}(Y)$ the class of all continuous functions $g: Y \to \mathbf{C}$ such that $spt(g)$ is compact and we will denote (as is usual) by $C(Y)$ the class of all continuous functions $f: Y \to \mathbf{C}$.

We will denote by $\mathbf{N}$ the class of all positive integers. If $k \in \mathbf{N}$, then we will denote by H_k the class of all members of the polynomial ring $\mathbf{C}[\chi: \chi \in V^*]$ that are homogeneous of degree k. If $f \in \bigcup_{k=1}^{\infty} H_k$, then we let

$$\|f\| = \sup\{|f(x)|: x \in S\}.$$

We will denote (as is usual) by $\mathbf{D}$ the class of all z in $\mathbf{C}$ such that $z\bar{z} < 1$. With regard to the following theorem (Theorem 2.4) we recall that if $z \in \mathbf{D}$, then

$$\operatorname{Re}[(1 + z)/(1 - z)] = (1 - z\bar{z})/(1 - z)(1 - \bar{z}) = \sum_{k=1}^{\infty} \bar{z}^k + 1 + \sum_{k=1}^{\infty} z^k.$$

Furthermore with regard to the proof of Theorem 2.4 we refer to the proof of Proposition 9.5 of [2].

2.4. THEOREM. *Let* $f \in \bigcup_{k=1}^{\infty} H_k$ *and let* $\|f\| \leq 1$. *If*

$$g = \mathrm{Re}\,[(1+f)/(1-f)],$$

then

$$\int g\, d\sigma \leq 1. \tag{2.1}$$

If $d\mu = g\, d\sigma$ *and if* $n \geq 2$, *then* $\mu^{\#} = g \mid B$. (*Thus* $\mu^{\#}$ *is pluriharmonic and* $\mu(S) = 1$.)

Proof. If $z \in \mathbf{D}$, then $g(zx)$ is continuous on $\bar{B}$ and pluriharmonic on B, hence $\int g(zx)\, d\sigma(x) = g(0) = 1$, hence by the Fatou-Lebesgue lemma (2.1) holds.

By Proposition 2.3, $g \mid B = \lambda^{\#}$ where $\lambda \in M_+(S)$. If $h \in C(S)$, then

$$\operatorname*{limit}_{z \in \mathbf{D},\, z \to 1} \int h(x)\lambda^{\#}(zx)\, d\sigma(x) = \int h\, d\lambda.$$

Thus if $F = \{x\colon x \in S, f(x) = 1\}$ and if $h \in C_{00}(S - F)$, then $\int h\, d\mu = \int h\, d\lambda$, hence if $E \subset S - F$, then $\mu(E) = \lambda(E)$. Furthermore by [2, Corollary 1.9] $\lambda(F) = 0$, hence $\mu = \lambda$ which completes the proof of Theorem 2.4.

2.5. COROLLARY. *If* $n \geq 2$, *then with regard to Theorem* 2.4,

$$\operatorname*{limit}_{z \in \mathbf{D},\, z \to 1} \int |g(zx) - g(x)|\, d\sigma(x) = 0.$$

3. An extreme point of $\{\mu\colon \mu \in M_+(S),\ \mu^{\#} \text{ is } ph,\ \mu(S) = 1\}$ (Theorem 3.15)

If $X \subset V^*$, then we will denote by X_+ the class of all x in V such that $\chi(x) \geq 0$ for every χ in X.

3.1. PROPOSITION. *Let* $H_0 = \mathbf{C}$. *If* $f \in \bigcup_{k=0}^{\infty} H_k$, *if* X *is a basis of* V^*, *and if* $f = 0$ *on* X_+, *then* $f = 0$.

Proof. If $f \in H_0$, then Proposition 3.1 holds. We assume that Proposition 3.1 holds if $f \in H_j$.

Let $f \in H_{j+1}$. If $x, y \in X_+$ and $t \geq 0$, then $x + ty \in X_+$, hence $f(x + ty) = 0$, hence $f'(x, y) = 0$. Thus by the induction hypothesis if $(x, y) \in V \times X_+$, then $f'(x, y) = 0$. Thus since X is a basis of V^*, $f' = 0$, hence $f \in \mathbf{C}$ which completes the proof of Proposition 3.1.

3.2. COROLLARY. *If* $f \in \bigcup_{k=1}^{\infty} H_k$, *if* X *is a basis of* V^*, *and if* $f = 0$ *on* $S \cap X_+$, *then* $f = 0$.

3.3. COROLLARY OF 3.2. *Let X be an orthonormal basis of V^*, let $w \in \mathbf{C}$, let $k \in \mathbf{N}$, and let $f \in H_{2k}$. If $f = w$ on $S \cap X_+$, then*

$$f = w\left(\sum_{\chi \in X} \chi^2\right)^k. \tag{3.1}$$

Proof. If we denote by g the right side of (3.1), then $g = w$ on $S \cap X_+$, hence $f - g = 0$ on $S \cap X_+$, hence by Corollary 3.2, $f = g$.

3.4. COROLLARY OF 3.3. *Let X be an orthonormal basis of V^*, let $w \in \mathbf{C}$, let $k \in \mathbf{N}$, and let $f \in H_{2k-1}$. If $f = w$ on $S \cap X_+$ and if $n \geq 2$, then $f = 0$.*

Proof. If $g = \sum_{\chi \in X} \chi^2$, then by Corollary 3.3, $f^2 = w^2 g^{2k-1}$. Let

$$G = \{x\colon x \in V, g(x) \neq 0\}.$$

If $w \neq 0$, then we define $h\colon V \to \mathbf{C}$ by $h \mid G = f/wg^{k-1}$, $h \mid G' = 0$. Thus $h^2 = g$. By the Riemann removable singularity theorem [3, p. 19] h is an entire function. Furthermore if $(z, x) \in \mathbf{C} \times V$, then $h(zx) = zh(x)$. Thus $h \in V^*$. Let Y be the basis of V such that X is the dual basis of Y. If $x = \sum_{y \in Y} y$, then $g(x) = n$. Furthermore since $h \in V^*$, $h(x) = n$, hence $n = 1$ which completes the proof of Corollary 3.4.

We will denote (as is usual) by $\mathbf{T}$ the class of all z in $\mathbf{C}$ such that $z\bar{z} = 1$. We will denote by $A(B)$ the class of all functions in $C(\bar{B})$ that are holomorphic on B.

3.5. COROLLARY OF 3.2. *Let X be a basis of V^*, let $Y = \bigcup_{z \in \mathbf{T}} z(S \cap X_+)$, let $f \in A(B)$, and let $g = \operatorname{Re}(f)$. If $g = 0$ on Y, then $f = f(0)$.*

Proof. On B we have

$$f = f(0) + \sum_{j=1}^{\infty} f_j \tag{3.2}$$

where $f_j \in H_j$.

Let $x \in S \cap X_+$. If $z \in \mathbf{T}$, then $zx \in Y$, hence $g(zx) = 0$, hence $f(zx) = f(0)$. Thus if $z \in \mathbf{D}$, then by (3.2), $\sum_{j=1}^{\infty} z^j f_j(x) = 0$, hence $f_j(x) = 0$.

Thus $f_j = 0$ on $S \cap X_+$, hence by Corollary 3.2, $f_j = 0$ which completes the proof of Corollary 3.5.

With regard to Corollary 3.5, we remark that if $\mu \in M(S)$, if $\mu^\#$ is pluriharmonic, and if $n \geq 2$, then by [2, Theorem 1.7], $\mu(Y) = 0$. We will omit the proof of the following corollary (the statement of which we owe to the referee).

3.6. COROLLARY OF 3.1. *Let $G \subset V$ be open and connected. If $f\colon G \to \mathbf{C}$ is holomorphic, if X is a basis of V^*, if $G \cap X_+ \neq \emptyset$, and if $f = 0$ on $G \cap X_+$, then $f = 0$.*

3.7. THEOREM. *If X is an orthonormal basis of V^* and if $k \in \mathbf{N}$, then $(\sum_{\chi \in X} \chi^2)^k$ is an extreme point of $\{f\colon f \in H_{2k}, \|f\| \leq 1\}$.*

Proof. Let $f = (\sum_{\chi \in X} \chi^2)^k$ and let $g \in H_{2k}$. If $\|f + g\| \leq 1$ and if $\|f - g\| \leq 1$, then $g = 0$ on $S \cap X_+$, hence by Corollary 3.2, $g = 0$ which completes the proof of Theorem 3.7.

We will omit the proof of the following theorem.

3.8. THEOREM (cf. Theorem 3.7). *Let $f \in H_2$. If f is an extreme point of*

$$\{g : g \in H_2, \|g\| \leq 1\},$$

then there is an orthonormal basis X of V^* such that $f = \sum_{\chi \in X} \chi^2$.

If $k \in \mathbf{N}$, then we let $\mathbf{T}_k = \{z : z \in \mathbf{T}, z^k = 1\}$.

Let $k \in \mathbf{N}$. We remark that if $\mu \in M(S)$ and if either of the properties 3.9.1 or 3.9.2 which follow hold, then the other holds.

3.9.1. If $E \subset S$ and if $z \in \mathbf{T}_k$, then $\mu(E) = \mu(zE)$.
3.9.2. If $y \in B$ and if $z \in \mathbf{T}_k$, then $\mu^\#(y) = \mu^\#(zy)$.

We will denote by N_k the class of all μ in $M_+(S)$ such that $\mu^\#$ is pluriharmonic, $\mu(S) = 1$, and the property 3.9.1 holds. Thus N_k is convex and compact. (If $k = 1$, then we let $N = N_k$. Thus N is the class of all μ in $M_+(S)$ such that $\mu^\#$ is pluriharmonic and $\mu(S) = 1$.)

We remark that if $f \in H_k$, if $\|f\| \leq 1$, if $g = \mathrm{Re}\,[(1 + f)/(1 - f)]$, if $d\mu = g\, d\sigma$, and if $n \geq 2$, then by Theorem 2.4, $\mu \in N_k$.

3.10. THEOREM. *Let $k \in \mathbf{N}$, let X be an orthonormal basis of V^*, let $f = (\sum_{\chi \in X} \chi^2)^k$, let $g = \mathrm{Re}\,[(1 + f)/(1 - f)]$, and let $d\mu = g\, d\sigma$. If $n \geq 2$, then μ is an extreme point of N_{2k}.*

Proof. By Theorem 2.4,

$$\mu^\# = \sum_{j=1}^{\infty} \bar{f}^j + 1 + \sum_{j=1}^{\infty} f^j. \tag{3.3}$$

If $\alpha \in N_{2k}$, then by 3.9.2,

$$\alpha^\# = \sum_{j=1}^{\infty} \bar{a}_j + 1 + \sum_{j=1}^{\infty} a_j \tag{3.4}$$

where $a_j \in H_{2kj}$. Thus if $v \in S$ and if $z \in \mathbf{D}$, then

$$\alpha^\#(zv) = \sum_{j=1}^{\infty} \bar{z}^{2kj}\bar{a}_j(v) + 1 + \sum_{j=1}^{\infty} z^{2kj}a_j(v).$$

Furthermore $\alpha^\#(zv)$ is a positive harmonic function on $\mathbf{D}$, hence

$$|a_j(v)| \leq 1. \tag{3.5}$$

Let $\gamma \in M(S)$. If $\mu + \gamma \in N_{2k}$, then by (3.3) and (3.4)

$$\gamma^\# = \sum_{j=1}^{\infty} \bar{c}_j + \sum_{j=1}^{\infty} c_j$$

where $c_j \in H_{2kj}$. Furthermore if $j \in \mathbf{N}$, then by (3.5), $\|f^j + c_j\| \le 1$. Thus if $\mu + \gamma \in N_{2k}$ and $\mu - \gamma \in N_{2k}$, then by Theorem 3.7, $c_j = 0$, hence $\gamma = 0$ which completes the proof of Theorem 3.10.

With regard to Theorem 3.10 we recall that if $n = 1$, if $k \in \mathbf{N}$, and if λ is an extreme point of N_k, then $\lambda \perp \sigma$. We will omit the proof of the following theorem.

3.11. THEOREM. *With regard to Theorem* 3.10, *if* $p < 3/2$, *then* $g \in L^p(\sigma)$. *If* $n = 2$, *then* $g \notin L^{3/2}(\sigma)$.

3.12. PROPOSITION. *Let* $k \in \mathbf{N}$, *let* X *be an orthonormal basis of* V^*, *let* $f = (\sum_{\chi \in X} \chi^2)^k$, *let* $g = \operatorname{Re}[(1+f)/(1-f)]$, *let* $d\mu = g\,d\sigma$, *let* $\lambda \in M(S)$, *and let* $n \ge 2$ (*thus* $\mu \in N_{2k}$).

If $\mu + \lambda \in N$ *and* $\mu - \lambda \in N$, *then*

$$\lambda^{\#} = 2 \operatorname{Re}[p/(1-f)] \tag{3.6}$$

where $p \in \sum_{j=1}^{2k-1} H_j$. *Furthermore*

$$(1 - f\bar{f}) + \bar{p}(1+f) + p(1-\bar{f}) \ge 0 \quad \text{on } S. \tag{3.7}$$

Proof. We let $G = \mathbf{T}_{2k}$ and we define $s: M(S) \to M(S)$ as follows: if $E \subset X$, then

$$(s(\alpha))(E) = (1/2k) \sum_{z \in G} \alpha(zE). \tag{3.8}$$

Thus if $\alpha \in N$, then $s(\alpha) \in N_{2k}$. Hence by Theorem 3.10, $s(\lambda) = 0$, thus

$$\lambda^{\#} = 2 \operatorname{Re}\left(\sum_{j=1}^{2k-1} \sum_{m=0}^{\infty} g_{jm}\right) \tag{3.9}$$

where $g_{jm} \in H_{j+2km}$. We recall that

$$\mu^{\#} = 1 + 2 \operatorname{Re}\left(\sum_{j=1}^{\infty} f^j\right). \tag{3.10}$$

Let $x \in S \cap X_+$ and define $h: \mathbf{D} \to (0, \infty)$ by $h(z) = (\mu + \lambda)^{\#}(zx)$. If $z \in \mathbf{D}$, then by (3.9) and (3.10),

$$h(z) = 1 + 2 \operatorname{Re}\left(\sum_{j=1}^{\infty} z^{2kj} + \sum_{j=1}^{2k-1} \sum_{m=0}^{\infty} z^{j+2km} g_{jm}(x)\right). \tag{3.11}$$

We recall that if $\alpha \in M(\mathbf{T})$, then $\hat{\alpha}: \mathbf{Z} \to \mathbf{C}$ is defined by $\hat{\alpha}(j) = \int \bar{z}^j \, d\alpha(z)$. Since h is harmonic

$$h(z) = \hat{\gamma}(0) + 2 \operatorname{Re}\left(\sum_{j=1}^{\infty} \hat{\gamma}(j) z^j\right) \tag{3.12}$$

where $\gamma \in M_+(\mathbf{T})$. By (3.11) and (3.12), $\hat{\gamma}(0) = \hat{\gamma}(2k) = 1$, hence $\gamma \in M_+(G)$, hence

$$\hat{\gamma}(j + 2km) = \hat{\gamma}(j). \tag{3.13}$$

We let $p_j = g_{j0}$. If $x \in S \cap X_+$, then by (3.11), (3.12), and (3.13), $g_{jm}(x) = p_j(x)$, hence $g_{jm}(x) = p_j(x)f^m(x)$. Furthermore $g_{jm} - p_jf^m \in H_{j+2km}$. Thus by Corollary 3.2, $g_{jm} = p_jf^m$, hence if $p = \sum_{j=1}^{2k-1} p_j$, then by (3.9), (3.6) holds.

We have

$$(\mu + \lambda)^{\#} = q/(1 - f)(1 - \bar{f}) \tag{3.14}$$

where $q = (1 - f\bar{f}) + \bar{p}(1 - f) + p(1 - \bar{f})$. By (3.14), $q > 0$ on B, hence (3.7) holds.

3.13. LEMMA. *Let $n = 2$. If $\{g, h\}$ is an orthonormal basis of V^* and if $f = g^2 + h^2$, then $|g - \bar{g}f|^2 = h\bar{h}(1 - f\bar{f})$ on S.*

Proof. Lemma 3.13 follows (by direct verification) from the definition of f and the fact that $g\bar{g} + h\bar{h} = 1$ on S.

We will denote by $U(V)$ the class of all unitary transformations of V.

3.14. LEMMA. *Let X be an orthonormal basis of V^*, let $f = \sum_{\chi \in X} \chi^2$, and let $p \in V^*$. If $n \geq 2$ and if $|p - \bar{p}f| \leq 1 - f\bar{f}$ on S, then $p = 0$.*

Proof. If Lemma 3.14 holds when $n = 2$, then Lemma 3.14 holds when $n \geq 2$. Thus we let $n = 2$.

Let $X = \{g, h\}$. We have $p = ag + bh$ where $a, b \in \mathbf{C}$. If $x \in S \cap X_+$, then $f(x) = 1$, hence $p(x) = \overline{p(x)}$. Thus $a, b \in \mathbf{R}$. Let $q = ag - bh$ and let $t \in U(V)$. If $g \circ t = g$ and if $h \circ t = -h$, then $p \circ t = q$ and $f \circ t = f$, hence $|q - \bar{q}f| \leq 1 - f\bar{f}$ on S. Thus $|(p + q) - (\bar{p} + \bar{q})f| \leq 2(1 - f\bar{f})$ on S, hence

$$|ag - a\bar{g}f| \leq 1 - f\bar{f}$$

on S. Thus by Lemma 3.13,

$$a^2h\bar{h}(1 - f\bar{f}) \leq (1 - f\bar{f})^2 \tag{3.15}$$

on S. If $E = \{x : x \in S, |f(x)| = 1\}$, then since $n > 1$ E is nowhere dense in S, hence by (3.15), $a^2h\bar{h} \leq 1 - f\bar{f}$ on S, hence $a = 0$. Likewise $b = 0$ which completes the proof of Lemma 3.14.

3.15. THEOREM. *Let X be an orthonormal basis of V^*, let $f = \sum_{\chi \in X} \chi^2$, let $g = \operatorname{Re}[(1 + f)/(1 - f)]$, and let $d\mu = g\, d\sigma$. If $n \geq 2$, then μ is an extreme point of N.*

Proof. Let $\lambda \in M(S)$. It is to be proved that if $\mu + \lambda \in N$ and $\mu - \lambda \in N$, then $\lambda = 0$. By Proposition 3.12,

$$\lambda^{\#} = 2 \operatorname{Re}[p/(1 - f)] \tag{3.16}$$

where $p \in V^*$ and

$$(1 - f\bar{f}) + \bar{p}(1 - f) + p(1 - \bar{f}) \geq 0 \tag{3.17}$$

on S. The left side of (3.17) is equal to $(1 - f\bar{f}) + 2 \operatorname{Re}(p - \bar{p}f)$; thus if $(z, x) \in \mathbf{T} \times S$, then

$$[1 - f(x)\bar{f}(x)] + 2 \operatorname{Re}([p(x) - \bar{p}(x)f(x)]z) \geq 0,$$

hence $|p(x) - \bar{p}(x)f(x)| \leq 1 - f(x)\bar{f}(x)$. Hence by Lemma 3.14, $p = 0$, hence by (3.16), $\lambda^{\#} = 0$ which completes the proof of Theorem 3.15.

4. A class of extreme points of $\{\mu\colon \mu \in M_+(S),\ \mu^{\#} \text{ is } ph,\ \mu(S) = 1\}$ (Corollary 4.7)

If Y, Z, and N are sets, if $\phi\colon Y \to Z$, and if $\mu\colon 2^Y \to N$, then we define $\phi^*(\mu)\colon 2^Z \to N$ by

$$\phi^*(\mu)(E) = \mu(\{y\colon y \in Y,\ \phi(y) \in E\}).$$

With regard to this definition we recall the following fact of measure theory [1, p. 72].

4.1. PROPOSITION. *If Y and Z are compact Hausdorff spaces, if $\phi\colon Y \to Z$ is continuous, and if $\mu \in M_+(Y)$, then $\phi^*(\mu) \in M_+(Z)$. Thus if $\mu \in M(Y)$, then $\phi^*(\mu) \in M(Z)$.*

With regard to Proposition 4.1 we remark that if $f \in C(Z)$, then

$$\int f\, d\phi^*(\mu) = \int f \circ \phi\, d\mu.$$

We will denote by $G(B)$ the class of all holomorphic homeomorphisms of B. With regard to $G(B)$ we refer to [2]. We define $T\colon G(B) \times M(S) \to M(S)$ by

$$dT(Z, \mu) = (\beta^{Z(0)} \circ Z)\, dY^*(\mu)$$

where $Y = Z^{-1}$. We remark that if $d\mu = g\, d\sigma$, then by [2, Proposition 2.4], $dT(Z, \mu) = (g \circ Z)\, d\sigma$. We recall the following fact of the theory of B [2, Proposition 8.3].

4.2. PROPOSITION. *If $(Z, \mu) \in G(B) \times M(S)$ and if $y \in B$, then*

$$T(Z, \mu)^{\#}(y) = \mu^{\#}(Z(y)). \tag{4.1}$$

Thus if $\mu^{\#}$ is pluriharmonic, then $T(Z, \mu)^{\#}$ is pluriharmonic.

4.3. PROPOSITION. *If $(X, Y) \in G(B) \times G(B)$ and if $\mu \in M(S)$, then*

$$T(XY, \mu) = T(Y, T(X, \mu)).$$

Proof. Proposition 4.3 can be proved by means of the identity (4.1) or (by direct verification) by means of the definition of T.

We define $t\colon G(B) \times N \to N$ by $t(Z, \mu) = T(Z, \mu)/\mu^{\#}(Z(0))$.

4.4. PROPOSITION. *If* $(X, Y) \in G(B) \times G(B)$ *and if* $\mu \in N$, *then*

$$t(XY, \mu) = t(Y, t(X, \mu)).$$

Proof. Proposition 4.4 (like Proposition 4.3) can be proved by means of the identity (4.1).

4.5. PROPOSITION. *If* $p \geq 0$, $q \geq 0$, $p + q = 1$, $\lambda \in N$, $\mu \in N$, *and* $Z \in G(B)$, *then*

$$t(Z, p\lambda + q\mu) = p't(Z, \lambda) + q't(Z, \mu)$$

where

$$p' = p\lambda^{\#}(Z(0))/(p\lambda + q\mu)^{\#}(Z(0)) \quad \text{and} \quad q' = q\mu^{\#}(Z(0))/(p\lambda + q\mu)^{\#}(Z(0)).$$

Proof. If $r = (p\lambda + q\mu)^{\#}(Z(0))$, then

$$\begin{aligned} t(Z, p\lambda + q\mu) &= T(Z, p\lambda + q\mu)/r \\ &= [pT(Z, \lambda) + qT(Z, \mu)]/r \\ &= p't(Z, \lambda) + q't(Z, \mu). \end{aligned}$$

4.6. PROPOSITION. *Let* $(Z, \mu) \in G(B) \times N$. *If* μ *is an extreme point of* N, *then* $t(Z, \mu)$ *is an extreme point of* N.

Proof. If $t(Z, \mu) = p\alpha + q\gamma$ where $p > 0$, $q > 0$, $p + q = 1$, $\alpha \in N$, $\gamma \in N$, and if $Y = Z^{-1}$, then by Proposition 4.4 and Proposition 4.5,

$$\mu = t(Y, p\alpha + q\gamma) = p't(Y, \alpha) + q't(Y, \gamma)$$

where $p' > 0$, $q' > 0$, $p' + q' = 1$, hence $\mu = t(Y, \alpha) = t(Y, \gamma)$, hence by Proposition 4.4, $t(Z, \mu) = \alpha = \gamma$ which completes the proof of Proposition 4.6.

4.7. COROLLARY OF THEOREM 3.15 AND PROPOSITION 4.6. *Let* X *be an orthonormal basis of* V^*, *let* $f = \sum_{\chi \in X} \chi^2$, *let* $g = \operatorname{Re}[(1 + f)/(1 - f)]$, *let* $d\mu = g\, d\sigma$, *and let* $Z \in G(B)$. *If* $n \geq 2$, *then* $t(Z, \mu)$ *is an extreme point of* N.

REFERENCES

1. H. FEDERER, *Geometric measure theory*, Springer-Verlag, New York, 1969.
2. F. FORELLI, *Measures whose Poisson integrals are pluriharmonic.* Illinois J. Math., vol. 18 (1974), pp. 373–388.
3. R. GUNNING AND H. ROSSI, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, 1965.

THE UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN