A SPECTRAL SEQUENCE APPROACH TO EMBEDDING SPACES

BY

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1. Introduction

In 1955 Federer [F] constructed a spectral sequence relating the homotopy groups of the space of maps from a CW complex X into a space Y to the cohomology groups of X with coefficients in the homotopy groups of Y. In this paper we present an analogous spectral sequence for certain spaces of PL embeddings of a simplicial complex K into a PL manifold M. In order to measure only the difference between the mapping spaces and the embedding spaces, we will consider relative homotopy groups.

Let Δ^n denote the standard n-simplex. A map $F: K \times \Delta^n \to M \times \Delta^n$ is *level-preserving* if it commutes with projection on the second factor; it is *block-preserving* if it takes $K \times P$ to $M \times P$ for all faces P of Δ^n. Let $\tilde{C}(K, M)$ denote the Δ-set (see [R-S] for definition) whose n-simplices are block-preserving maps of $K \times \Delta^n$ into $M \times \Delta^n$. Let $C(K, M)$ be the sub-Δ-set whose n-simplices are level-preserving maps and let $\tilde{P}L(K, M)$ be the sub-Δ-set whose n-simplices are block-preserving PL embeddings. Finally let $PL(K, M)$ be $C(K, M) \cap \tilde{P}L(K, M)$; it is the usual Δ-set of PL embeddings of K into M. Let K be a k-complex with skeleta K_q and let M be a PL m-manifold. Let $f: K \to M$ be a fixed PL embedding of K into M, to serve as base point for homotopy groups. We assume throughout that $k \leq m - 3$.

THEOREM 1. *There is a spectral sequence $E^{(r)}_{pq}$ such that for $n \geq 3$ the terms E^{∞}_{pq}, $p + q = n$ form a composition series for $\pi_n(\tilde{C}(K, M), \tilde{P}L(K, M), f)$. If M is $(p + 3q - m + 1)$-connected then there is a map*

$$\psi: E^2_{pq} \to H^{2(p+2q)}((K_q \times I^{p+q})^*, \pi_{2(p+2q)}(S^{m+p+q})),$$

where X^* is the quotient space of $X \times X - \Delta$ by the free $\mathbf{Z}_2$-action which interchanges coordinates and the coefficients are local in a bundle described below, such that $\psi^{-1}(0) = 0$ if $p + 4q \leq 2m - 3$.

In Section 2 we derive some results about $\tilde{P}L(K, M)$, including a covering n-concordance theorem for polyhedra. In Section 3 we define the spectral sequence and obtain convergence and vanishing results. Section 4 contains the definition of the map ψ by which one obtains information about the E^2 term, and some examples are given in Section 5.

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2. Fibrations and exact sequences

Let I^n be the n-fold product of $I = [0, 1]$ with itself. A face P of I^n is a subset of the form

$$\{(x_1, \ldots, x_n) \mid x_i = 0 \text{ or } 1 \text{ for at least one value of } i\}.$$

The face $\{(x_1, \ldots, x_{n-1}, 0)\}$ will be identified with I^{n-1}, and $\mathrm{cl}(\partial I^n - I^{n-1})$ will be called J^{n-1}. A map $F: K \times I^n \to M \times I^n$ is *face preserving* if

$$F^{-1}(M \times P) = K \times P$$

for all faces P of I^n; it is *level-preserving* if it commutes with projection onto I^n. It will be convenient to replace Δ^n by I^n in the Δ-sets described in the introduction. Then an element of $\pi_n PL(K, M; f)$ (respectively $\pi_n \tilde{P}L(K, M; f)$) is represented by a PL face-preserving (respectively level-preserving) embedding F of $K \times I^n$ into $M \times I^n$ such that for $t \in \partial I^n$, $F(x, t) = f(x)$. The n-simplices of $\tilde{P}L(K, M)$ are called *n-concordances*; the n-simplices of $PL(K, M)$ are called *n-isotopies*; and a 1-isotopy is called an *isotopy*. Aut (M) is the sub-Δ-set of $PL(M, M)$ whose n-simplices are level-preserving homeomorphisms. An n-simplex of Aut (M) is called an *ambient n-isotopy*. If $F: X \times I^n \to Y \times I^n$ is an n-isotopy, then $F_t: X \to Y$ is defined by $F(x, t) = (F_t(x), t)$. Some known facts about these Δ-sets are that

$$\pi_n(\tilde{P}L(K, M), PL(K, M); f) = 0$$

if K is a PL k-manifold and $k + n \leq m - 2$ (Morlet [M]) and that if $k \leq m - 3$ the map

$$p: \mathrm{Aut}\,(M) \to PL(K, M; f)$$

given by $p(H) = H(f \times 1)$ is a Kan fibration, where 1 denotes the identity map. This means that given a level-preserving PL embedding

$$G: K \times I^n \to M \times I^n$$

and a level-preserving PL homeomorphism

$$g: M \times J^{n-1} \to M \times J^{n-1}$$

such that $G \mid K \times J^{n-1} = g \circ (f \times 1)$ there is a PL level-preserving homeomorphism $H: M \times I^n \to M \times I^n$ such that

$$H \circ (f \times 1) = G \qquad \text{and} \qquad H \mid M \times J^{n-1} = g.$$

This is a consequence of Hudson's Covering n-Isotopy Theorem [H3]. We wish to establish a similar result for $\tilde{P}L(K, M)$.

THEOREM 2 (Covering n-Concordance Theorem). *Let $F: K \times I^n \to M \times I^n$ be a face-preserving PL embedding such that*

$$F^{-1}(\partial M \times I^n) = K_0 \times I^n$$

for some subcomplex K_0 *of* K *and* $F \mid K_0 \times I^n = (F_0 \mid K_0) \times 1$. *Then if* $k \leq m - 3$ *there is a face-preserving ambient isotopy*

$$H: M \times I^n \times I \to M \times I^n \times I,$$

fixing $\partial Q \times I^n \cup Q \times 0$, *such that* $H_1 F = F_0 \times 1$. *If in addition,* $F \mid K \times J^{n-1} = F_0 \times 1$, *then we may assume* H *fixed on* $Q \times J^{n-1}$.

Note. By induction on m the condition that $F \mid X_0 \times I^n = F_0 \mid K_0 \times 1$ is no restriction if $\dim X_0 \leq m - 4$.

Proof. By induction on n we may assume that after a face-preserving ambient isotopy of $M \times I^n$, fixed on $\partial M \times I^n \cup M \times 0$, we have

$$F \mid K \times J^{n-1} \cup K_0 \times I^n = F_0 \times 1.$$

Then we apply Rourke's Concordance Implies Isotopy Theorem [R] to the 1-concordance

$$F: (K \times I^{n-1}) \times I \to (M \times I^{n-1}) \times I$$

to get an ambient isotopy H of $M \times I^n$, fixed on

$$M \times I^{n-1} \times 0 \cup \partial(M \times I^n) \times I = \partial M \times I^n \cup M \times J^{n-1}$$

such that $H_1 F = F_0 \times 1$. Since H is the identity for all faces of I^n except one, H is face-preserving.

COROLLARY. *The restriction map* $\tilde{P}L(K_q, M; f) \to \tilde{P}L(K_{q-1}, M; f)$ *is a Kan fibration.*

Proof. Suppose $G: K_{q-1} \times I^n \to M \times I^n$ and $g: K_q \times J^{n-1} \to M \times J^{n-1}$ are face-preserving PL embeddings such that

$$G \mid K_{q-1} \times J^{n-1} = g \mid K_{q-1} \times J^{n-1}.$$

By induction and the above theorem we may assume $g \mid K_q \times J^{n-1} = f \times 1$. If H is the ambient isotopy of the above theorem such that $H_1 F = f \times 1$, then $H_1^{-1} \mid K_q \times I^n$ is the desired lift of G.

THEOREM 3. *Let*

$$F \longrightarrow E \xrightarrow{p} B \quad \text{and} \quad F' \longrightarrow E' \xrightarrow{p'} B'$$

be a pair of Kan fibrations of Δ*-sets, in which* F', E', *and* B' *are sub-*Δ*-sets of* F, E, *and* B *respectively, with* $p \mid E' = p'$ *and* $F' = E' \cap F$. *Then there is an exact sequence*

$$\cdots \to \pi_n(F, F') \to \pi_n(E, E') \to \pi_n(B, B') \to \pi_{n-1}(F, F') \to \cdots$$
$$\to \pi_3(B, B') \to \pi_2(F, F') \to \pi_2(E, E').$$

Proof. Consider the following diagram, in which the first line is the first homotopy sequence of the triad (E, E', F), the bottom line is the homotopy sequence of the pair (B, B'), and all the vertical maps are induced by p:

$$\begin{array}{ccccccccccccc}
\cdots \to & \pi_n(E', F') & \to & \pi_n(E, F) & \to & \pi_n(E, E', F) & \to & \pi_{n-1}(E', F') & \to & \pi_{n-1}(E, F) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \pi_n(B') & \to & \pi_n(B) & \to & \pi_n(B, B') & \to & \pi_{n-1}(B') & \to & \pi_{n-1}(B) & \to \cdots
\end{array}$$

By the 5-Lemma $p_*\colon \pi_n(E, E', F) \to \pi_n(B, B')$ is an isomorphism for $n \geq 3$. Therefore the second exact sequence of (E, E', F) becomes the sequence of the theorem.

3. The spectral sequence

Let $\{K_q\}$ denote the q-skeleta of K and let $f\colon K \to M$ be a PL embedding. We also let f stand for the restriction of f to any K_q. Let

$$\begin{aligned}
E_q &= \tilde{C}(K_q, M; f), & E'_q &= \tilde{P}L(K_q, M; f),\\
B_q &= \tilde{C}(K_{q-1}, M, f), & B'_q &= \tilde{P}L(K_{q-1}, M; f),\\
F_q &= \tilde{C}(K_q, M \bmod K_{q-1}; f), & F'_q &= \tilde{P}L(K_q, M \bmod K_{q-1}; f)
\end{aligned}$$

where an n-simplex of $C(K_q, M \bmod K_{q-1})$ is a face-preserving map $K_q \times I^n \to M \times I^n$ which agrees with $f \times 1$ on $K_{q-1} \times I^n$. We note that:

(1) $B_q = E_{q-1}$, $E'_q \cap F_q = F'_q$.
(2) $F_q \to E_q \to B_q$ is a Kan fibration and by the corollary to Theorem 2, $F'_q \to E'_q \to B'_q$ is a Kan fibration.
(3) Therefore, by Theorem 3 there is for each q an exact sequence

$$\begin{array}{c}
\cdots \longrightarrow \pi_n(E_q, \ {}'_q) \xrightarrow{\ i\ } \pi_n(B_q, B'_q) \xrightarrow{\ j\ } \pi_{n-1}(F_q, F'_q) \xrightarrow{\ k\ } \pi_{n-1}(E_q, E'_q) \longrightarrow \cdots \\
\| \\
\pi_n(E_{q-1}, \ {}'_{q-1})
\end{array}$$

where i, j, and k come from the proof of Theorem 3.

We now define an exact couple as follows:

Let

$$D_{pq} = \begin{cases} \pi_{p+q}(E_q, E'_q) & \text{if } q \geq 0 \text{ and } p + q \geq 3 \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$E_{pq} = \begin{cases} \pi_{p+q}(F_q, F'_q) & \text{if } q \geq 0 \text{ and } p + q \geq 3 \\ j(\pi_{p+q+1}(B_q, B'_q)) & \text{if } q \geq 0 \text{ and } p + q = 2 \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that if $D = \oplus_{p,q} D_{pq}$ and $E = \oplus_{p,q} E_{pq}$, then the triangle

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ & {}_{k}\nwarrow \quad \swarrow_{j} & \\ & E & \end{array}$$

is exact, where i, j, k are induced from the corresponding homomorphisms in the various exact sequences. Note that the bidegrees of i, j, and k are $(1, -1)$ $(-2, 1)$, and $(0, 0)$ respectively. In the standard way (see [H2], among others) the exact couple (D, E, i, j, k) gives rise to an E^2 spectral sequence (i.e., $E^2 = E$) with differentials $\{d^r\colon E^r_{pq} \to E^r_{p-r,q+r-1}\}$.

Next we show that the above spectral sequence converges strongly. The argument here is like that of Federer [F]. The restriction map induces for each p a homomorphism

$$r_q\colon \pi_{p+q}(\bar{E}, \bar{E}') \to \pi_{p+q}(E_q, E_q'),$$

where $\bar{E} = \tilde{C}(K, M; f)$ and $\bar{E}' = \tilde{P}L(K, M; f)$. Let G_{pq} be the kernel of r_q.

THEOREM 4.

$$\pi_{p+q}(\bar{E}, \bar{E}') = G_{p+q+1,-1} \supset G_{p+q,0} \supset \cdots \supset G_{p+q-k,k} = 0$$

and for $p + q \geq 3$ *and* $r \geq \max\{q + 2, k - q\}$, $E^r_{pq} = G_{p+1,q-1}/G_{pq}$.

Proof. The first statement is easy to check. For the second, we recall that if

$$\begin{array}{ccc} D^{(r)} & \xrightarrow{i^{(r)}} & D^{(r)} \\ & {}_{k^{(r)}}\nwarrow \quad \swarrow_{j^{(r)}} & \\ & E^{(r)} & \end{array}$$

denotes the couple obtained from

$$\begin{array}{ccc} D & \to & D \\ & \nwarrow \quad \swarrow & \\ & E & \end{array}$$

by deriving $r - 2$ times; then $D^{(r)}_{pq} = \operatorname{im} i^{(r-1)}$ and $i^{(r)} = i \mid D^{(r)}$, so

$$i^{(r)} r_q = r_{q-1}\colon \pi_{p+q}(\bar{E}, \bar{E}') \to D^{(r)}_{p+1,q-1}$$

for $p + q \geq 3$. Then

$$\ker i^{(r)} \mid D^{(r)}_{pq} = r_q(G_{p+1,q-1}) \quad \text{for } r \geq k - q,$$

and $\ker r_q \mid G_{p+1,q-1} = G_{pq}$, so $\ker i^{(r)} \mid D^{(r)}_{pq} = G_{p+1,q-1}/G_{pq}$.

Finally, the $(r - 2)$-nd derived couple contains the exact sequence

$$D^{(r)}_{p+r,q-r+1} \xrightarrow{j^{(r)}} E^{(r)}_{pq} \xrightarrow{k^{(r)}} D^{(r)}_{pq} \xrightarrow{i^{(r)}} D^{(r)}_{p+1,q-1}$$

in which the left-hand term is 0 for $r \geq q + 2$. Therefore, $k^{(n)}$ is an isomorphism of $E_{pq}^{(r)}$ onto ker $i^{(r)}$, and the theorem is proved.

COROLLARY. *$E_{pq}^{\infty} = 0$ for $q > k, p + q \geq 3$.*

4. The E^2 term

For any *PL m*-manifold M, let N be a regular neighborhood of the diagonal Δ_M in $M \times M$ which is invariant under the Z_2-action on $M \times M$ of the map $\alpha(x, y) = (y, x)$. Let $T(M)$ be the space obtained by collapsing to a point ω the closure of $M \times M - N$. Let ζ be a point in Δ_M; then $T(M) - (\Delta_M - \zeta)$ can be deformed onto a sphere S^m with a Z_2-action α' fixing the poles ζ and ω, and the deformation is equivariant with respect to the action induced on $T(M)$ by α. The action α' is also induced from α, and is the suspension of the antipodal map on the equator of S^m. Let $g\colon K \to M$ be a general position map which is an embedding on K_0. Then g induces an equivariant map

$$\bar{g}\colon K^2 \xrightarrow{g \times g} M^2 \longrightarrow T(M)$$

such that $\bar{g}(K^2 - \Delta_k) \cap \Delta_M$ has dimension $2m - k$. If M is $(2m - k)$-connected there is a homotopy taking this set to ζ, and by covering the homotopy we may assume that $\bar{g}(K^2 - \Delta_k) \cap \Delta_M \subset \{\zeta\}$. By picking the neighborhood N small enough we may assume that $\bar{g}(K_0 \times K_0 - \Delta_{K_0}) \subset \{\omega\}$. Therefore $\bar{g}$ induces a map $\tilde{g}\colon K^2 - \Delta_k \to S^m$, equivariant with respect to the actions of α on $K^2 - \Delta_k$ and α' on S^m, and if M is $(2k - m + 1)$-connected, the equivariant homotopy class of g is uniquely determined. Let $K^* = (K^2 - \Delta_k)/\alpha$ and consider the bundle $\mathscr{E} = (K^2 - \Delta_k) \times_{Z_2} S^m \to K^*$, where α' is the Z_2 action on S^m. Then equivariant homotopy classes of maps of $K^2 - \Delta_k$ into S^m correspond in a one-to-one fashion to homotopy classes rel K_0^* of sections of this bundle which agree with the constant (at ω) section s_ω on K_0^*, and therefore [E-S] with elements of $H^{2k}(K^*, K_0^*; \pi_{2k}(S^m))$ where the coefficients are local coefficients in the bundle $\mathscr{E}$. Let s_f denote the section corresponding to $f\colon K^2 - \Delta_k \to S^m$. Relative versions of theorems of Harris [H1] show that if $S_{\tilde{g}}$ is homotopic to s_ω rel K_0^*, then g is homotopic to an embedding rel K_0, provided $3k \leq 2m - 3$.

We now return to the spectral sequence. Let $\beta \in E_{pq}^2 = \pi_{p+q}(F_q, F_q')$ for $p + q \geq 3$, $q \geq 0$. Then β is represented by a face-preserving map

$$G\colon K_q \times I^{p+q} \to M \times I^{p+q}$$

such that

$$F \mid K_{q-1} \times I^{p+q} \cup K_q \times \partial I^{p+q} = f \times 1.$$

Then as above G defines an equivariant map of $(K_q \times I^{p+q})^2$ into S^{m+p+q} if M is $(p + 3q - m + 1)$-connected, and $G(K_{q-1} \times I^{p+q} \cup K_q \times \partial I^{p+q}) = \omega$. Therefore, G determines a section of $\mathscr{E}$ which agrees with s_ω on

$$(K_{q-1} \times I^{p+q} \cup K^q \times \partial I^{p+q})^*$$

and so determines an element γ of

$$H^{2(p+2q)}((K_q \times I^{p+q})^*, (K_{q-1} \times I^{p+q} \cup K_q \times \partial I^{p+q})^*; \pi_{2(p+2q)}(S^{m+p+q})),$$

where the coefficients are local in $\mathscr{E}$. Since

$$\dim (K_{q-1} \times I^{p+q} \cup K_q \times \partial I^{p+q})^* \leq 2(p + 2q - 1),$$

the exact cohomology sequence of a pair shows that the inclusion map

$$j: (K_q \times I^{p+q})^* \to ((K_q \times I^{p+q})^*, (K_{a-1} \times I^{p+q} \cup K_q \times \partial I^{p+q})^*)$$

induces an isomorphism j^* on the cohomology groups in dimension $2(p + 2q)$. We define $\psi(\beta) = j^*(\gamma)$. Then if $\psi(\beta) = 0$ and $p + 4q \leq 2m - 3$, G is homotopic, keeping

$$K_{q-1} \times I^{p+q} \cup K_q \times \partial I^{p+q}$$

fixed to an embedding, so $\beta = 0$. This completes the proof of Theorem 1.

5. Examples

Throughout this section, we assume that M is $(4k - m + 1)$-connected.

Example 1. Since $\pi_{2(p+2q)}(S^{m+p+q}) = 0$ for $p + 3q < m$, we have $E^\infty_{pq} = E^2_{pq} = 0$ for $p + q < m - 2k$. (Theorem 1 applies since $k \leq m - 3$ implies $p + 4q \leq 2m - 3$ here.) Since $E^\infty_{pq} = 0$ for $q > k$, we conclude that for $3 \leq n < m - 2k$, $p + q = n$ implies $E^\infty_{pq} = 0$. Therefore

$$\pi_n(\tilde{C}(K, Q; f), \tilde{P}L(K, Q; f)) = 0 \quad \text{for } 3 \leq n < m - 2k,$$

and we recover the result obtainable by general position.

Example 2. Suppose K is a k-manifold, and $n = m - 2k$. Then the only possible nonzero term in the composition series for $\pi_n(\tilde{C}(K, m; f), \tilde{P}L(K, M; f)$ is $E^\infty_{m-3k,k}$. But

$$\psi: E^2_{m-3k,k} \to H^{2(m-k)}((K \times I^{m-2k})^*; Z) = 0$$

since $(K \times I^{m-2k})^*$ is an open $2(m - k)$-manifold. Therefore

$$\pi_n(\tilde{C}(K, M; f), \tilde{P}L(K, M; f)) = 0 \quad \text{for } 3 \leq n \leq m - 2k.$$

Furthermore, since K is a manifold we can apply Morlet's Theorem to get

$$\pi_n(C(K, M; f), PL(K, M; f)) = 0 \quad \text{if } n \leq m - k + 2.$$

We can put this together with some of Federer's results [F] to get information on the homotopy groups of embedding spaces, for example if $\pi_j(M) = G$ and $\pi_i(M) = 0$ for $i \neq j$, then for $3 \leq n \leq m - 2k - 1$ we have

$$\pi_n(PL(K, M; f)) = H^{j-n}(K; G).$$

Example 3. Let T be a triod and let $K = T \times I^2$, $M = E^9$. Repeated use of the Mayer-Vietoris sequence shows that $H^{12}((K \times I^3)^*, \mathbf{Z}) = 0$. This implies that $E^2_{03} = 0$ in the spectral sequence, so we get

$$\pi_3(\tilde{C}(K, E^9), \tilde{P}L(K, E^9)) = 0.$$

Example 4. A number of examples can be concocted from the following general theorem, which will be proved elsewhere. Following Akin [A] we define the *intrinsic i-skeleton* $I^i(K)$ of a complex K to be $\{x \in K \mid$ there is a triangulation of K such that x is contained in the interior of a j-simplex, $j \leq i\}$.

THEOREM 4. *Let K be a finite simply connected k-complex with $H^k(K; \mathbf{Z}) = 0$ such that $I^i(K) = \emptyset$ for $i \leq 3$. Then $H^{2k}(K^*; \mathbf{Z}) = 0$.*

The idea of the proof is to show that under these conditions $K \times K - \Delta$ is simply connected, and then use the spectral sequence of a covering to show that $H^1(K \times K - \Delta; \mathbf{Z}) = 0$ implies $H^{2k}(K^*; \mathbf{Z}) = 0$.

REFERENCES

[A] E. AKIN, *Manifold phenomena in the theory of polyhedra*, Trans. Amer. Math. Soc., vol. 143 (1969), pp. 413–473.

[E-S] S. EILENBERG AND N. STEENROD, *Foundations of algebraic topology*, Princeton University Press, Princeton, New Jersey, 1952.

[F] H. FEDERER, *A study of function spaces by spectral sequences*, Trans. Amer. Math. Soc., vol. 82 (1956), pp. 340–361.

[H1] L. S. HARRIS, *Intersections and embeddings of polyhedra*, Topology, vol. 8 (1969), pp. 1–26.

[H2] S. T. HU, *Homotopy theory*, Academic Press, New York, 1959.

[H3] J. F. P. HUDSON, *Extending piecewise linear isotopies*, Proc. Lond. Math. Soc. (3), vol. 16 (1966), pp. 651–688.

[M] C. MORLET, *Isotopie et pseudo-isotopie*, C. R. Acad. Sci. Paris, t. 266, serie A (1966), pp. 559–650.

[R] C. ROURKE, "Embedded handle theory, concordance, and isotopy," in *Topology of manifolds* (edited by J. C. Cantrell and C. H. Edwards, Jr.), Markham, Chicago, 1970.

[R-S] C. ROURKE AND B. J. SANDERSON, *Δ-sets I, II*, Quart. J. Math. Oxford (2), vol. 22 (1971), pp. 321–338, 465–485.

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