MOST SYMMETRIC SETS ARE OF SYNTHESIS

BY

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Let $\mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$ be the circle group, and $A(\mathbf{T})$ the Fourier algebra of $\mathbf{T}$, i.e.,

$$A(\mathbf{T}) = \left\{ f \in C(\mathbf{T}) \colon \|f\|_{A(\mathbf{T})} = \sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty \right\}.$$

Any set of the form $\{\sum_{n=1}^{\infty} \varepsilon_n x_n \colon \varepsilon_n = 0 \text{ or } 1 \text{ for all } n\}$, where $(x_n)_1^\infty$ is a summable sequence of real numbers, is called *symmetric*. It is not known whether every symmetric set is of (spectral) synthesis for $A(\mathbf{T})$ (cf. [2]). In this note we prove that "most" symmetric sets are of synthesis. Our methods can be applied to yield a simple proof of Theorem 1 of [5].

We first introduce some notation. Let $q = (q_n)_1^\infty$ be a fixed sequence of natural numbers, $F(m) = \{0, \pm 1, \ldots, \pm m\}$ for all $m \geq 1$, and $E(q) = \prod_{n=1}^{\infty} F(q_n)$. To each sequence $x = (x_n)_1^\infty$ of real numbers satisfying $\sum_{n=1}^{\infty} q_n |x_n| < \infty$, we associate the set

$$E_x = E(q, x) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n x_n \in \mathbf{T} \colon \varepsilon = (\varepsilon_n)_1^\infty \in E(q) \right\}$$

and the continuous map $p_x = p(q, x) \colon E(q) \to E_x$ defined by

$$p_x(\varepsilon) = \sum_{n=1}^{\infty} \varepsilon_n x_n \quad (\varepsilon \in E(q)).$$

Let $(I_n)_1^\infty$ be a sequence of compact intervals, each containing 0, and C a positive real number. We define

$$J = \left\{ x = (x_n) \in \prod_{n=1}^{\infty} I_n \colon \sum_{n=1}^{\infty} q_n |x_n| \leq C \right\},$$

and notice that J is a compact metric space under the product topology. Given a compact set K in $\mathbf{T}$, let $A(K) = A(\mathbf{T})|_K$ denote the Fourier restriction algebra to K with the quotient norm. For the other notation used here without explanation, we refer to [4] and [5].

THEOREM 1. *Suppose C/π is irrational. Then quasi-all $x \in J$ have the following properties:*

(a) *The map $p_x \colon E(q) \to E_x$ is one-to-one, and induces an isometric isomorphism from $A(E_x)$ onto the infinite tensor product $A(q)$ of the algebras $A(\{jx_n \colon j \in F(q_n)\})$, $n = 1, 2, 3, \ldots$.*

(b) *E_x is a Dirichlet set of synthesis.*

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The basic ideas of our proof are found in [3] and [5]. The above theorem is an immediate consequence of the following.

LEMMA. *If C/π is irrational, then quasi-all $x \in J$ have the following property: For each natural number N, each $\eta > 0$, and complex numbers $z_1, \ldots, z_N$ of modulus* 1, *there exists a natural number r such that*

(i) $|z_n - \exp(irx_n)| < \eta \quad (1 \leq n \leq N)$,

(ii) $|1 - \exp(ir \sum_{n=N+1}^{\infty} \varepsilon_n x_n)| < \eta \quad (\varepsilon \in E(q))$.

Proof. Let $\eta > 0$ and $z_1, \ldots, z_N$ be given. Let

$$K = K(\eta; z_1, \ldots, z_N)$$

denote the closure of the set of all $x \in J$ for which there exists no natural number r satisfying (i) and (ii). We claim that K has empty interior if C/π is irrational.

We first deal with the case $\sum_1^\infty q_n|I_n| = \infty$, where $|I_n|$ denotes the length of I_n. Suppose by way of contradiction that K has nonempty interior. Then there exist finitely many open intervals $V_n \subset I_n$ $(1 \leq n < M, M > N)$ such that

$$\emptyset \neq J \cap \left(V_1 \times \cdots \times V_{M-1} \times \prod_{n=M}^{\infty} I_n\right) \subset K. \tag{1}$$

Since $\sum_1^\infty q_n|I_n| = \infty$, we can assume that there exist $y_n \in V_n$ $(1 \leq n < M)$ such that

$$0 < C - (q_1|y_1| + \cdots + q_{M-1}|y_{M-1}|) < q_M|I_M|/2.$$

Moreover, there is no loss of generality in assuming that $C, \pi, y_1, \ldots, y_{M-1}$ are rationally independent. Choose $y_M \in I_M$ so that

$$q_M|y_M| = C - (q_1|y_1| + \cdots + q_{M-1}|y_{M-1}|). \tag{2}$$

Hence $\pi, y_1, \ldots, y_M$ are rationally independent. By the Kronecker theorem, we can find a natural number r such that

$$|z_n - \exp(iry_n)| < \eta \quad (1 \leq n \leq N), \tag{3}$$

$$|1 - \exp(iry_n)| < \eta/(2Mq_n) \quad (N < n \leq M). \tag{4}$$

Define W to be the set of all $x \in J$ satisfying these conditions:

$$C - (q_1|x_1| + \cdots + q_M|x_M|) < \eta/(2r); \tag{2$'$}$$

$$|z_n - \exp(irx_n)| < \eta \quad (1 \leq n \leq N); \tag{3$'$}$$

$$|1 - \exp(irx_n)| < \eta/(2Mq_n) \quad (N < n \leq M). \tag{4$'$}$$

Then W is open in J and contains the element $y = (y_1, \ldots, y_M, 0, 0, \ldots)$. Hence X is not empty by (1), where

$$X \equiv W \cap \left(V_1 \times \cdots \times V_{M-1} \times \prod_{n=M}^{\infty} I_n\right) \subset K. \tag{1$'$}$$

Choose any $x \in X$; then (i) holds by (3)′. Moreover, $\varepsilon \in E(q)$ implies

$$\begin{aligned}\left|1 - \exp\left(ir \sum_{n=N+1}^{\infty} \varepsilon_n x_n\right)\right| &\leq \sum_{n=N+1}^{\infty} |1 - \exp(ir\varepsilon_n x_n)| \\ &\leq \sum_{n=N+1}^{M} |\varepsilon_n| \cdot |1 - \exp(irx_n)| + r \sum_{n=M+1}^{\infty} |\varepsilon_n x_n| \\ &< (M - N)\eta/(2M) + r\left(C - \sum_{n=1}^{M} q_n|x_n|\right) \\ &< \eta\end{aligned}$$

by (4)′ and (2)′. We have thus proved that every element of X satisfies (i) and (ii). Since X is open, this implies $X \cap K = \emptyset$, which contradicts (1)′.

Now we consider the case $\sum_1^\infty q_n|I_n| < \infty$. In this case, the irrationality of C/π is unnecessary. Suppose that (1), with $M - 1$ replaced by M, holds for some open intervals $V_n \subset I_n$ $(1 \leq n \leq M)$. We choose $y_n \in V_n$ $(1 \leq n \leq M)$ so that $\pi, y_1, \ldots, y_M$ are rationally independent and $q_1|y_1| + \cdots + q_M|y_M| < C$. Take any natural number r satisfying (3) and (4), and also a natural number $L > M$ so that

$$(5) \qquad \sum_{n=L}^{\infty} q_n|I_n| < \eta/(4r).$$

Now define W to be the set of all $x \in J$ satisfying (3)′, (4)′, and

$$(6) \qquad |1 - \exp(irx_n)| < \eta/(4Lq_n) \quad (M < n \leq L).$$

Then we have $(y_1, \ldots, y_M, 0, 0, \ldots) \in W$, and argue similarly as before to obtain a contradiction.

In either case, the closed set $K = K(\eta; z_1, \ldots, z_N)$ has empty interior. Therefore the lemma follows by a routine argument of countability.

Proof of Theorem 1. Choose and fix an arbitrary element x of J which has the property stated in the preceding lemma. In order to prove Theorem 1, it suffices to show that x satisfies (a) and (b).

Part (a) is an immediate consequence of Theorem 3 in [4], and we shall only confirm (b). It is obvious that E_x is a Dirichlet set. Given a natural number N, put

$$E_N = E(x, N) = \left\{\sum_{n=N+1}^{\infty} \varepsilon_n x_n \in \mathbf{T} : \varepsilon \in E(q)\right\}.$$

Since p_x is a one-to-one map, the closed sets $\sum_1^N \varepsilon_n x_n + E_N$, $\varepsilon_n \in F(q_n)$ for $1 \leq n \leq N$, are disjoint. For each pseudomeasure $Q \in PM(E_x)$, we can therefore write

$$(1) \qquad Q = \sum_{\varepsilon} Q_\varepsilon * \delta\left(\sum_{n=1}^{N} \varepsilon_n x_n\right),$$

where $\varepsilon = (\varepsilon_n)_1^N$ ranges over the set $\prod_1^N F(q_n)$, $Q_\varepsilon = Q_{N,\varepsilon}$ is an element of $PM(E_N)$ for each ε, and $\delta(t)$ denotes the unit point mass at $t \in \mathbf{T}$. Define a measure $\mu_N = \mu_N(Q) \in M(E_x)$ by setting

$$(2) \qquad \mu_N = \sum_\varepsilon \hat{Q}_\varepsilon(0)\delta\left(\sum_{n=1}^N \varepsilon_n x_n\right).$$

If we can show that $\mu_N \to Q$ as $N \to \infty$ in the weak* topology of $PM(\mathbf{T})$, the proof will be complete.

Let $j \in \mathbf{Z}$ be given. Setting $z_n = \exp(ijx_n)$ for $1 \leq n \leq N$, we apply the lemma to find a sequence $(r_k)_1^\infty$ of natural numbers such that

$$(3) \qquad \lim_{k\to\infty} \exp(ir_k x_n) = z_n \quad (1 \leq n \leq N),$$

$$(4) \qquad \lim_{k\to\infty} \|\exp(ir_k t) - 1\|_{\mathbf{C}(E_N)} = 0.$$

As is well known, there is an absolute constant M such that

$$\|e^{irt} - e^{ijt}\|_{A(K)} \leq M\|e^{irt} - e^{ijt}\|_{\mathbf{C}(K)}$$

for all compact subsets K of $\mathbf{T}$ (see, for example, [4; Lemma 1]). It follows from (1), (2), (3), and (4) that

$$\begin{aligned}
&|\hat{\mu}_N(-j) - \hat{Q}(-r_k)| \\
&= \left|\sum_\varepsilon \left\{\hat{Q}_\varepsilon(0)\exp\left(ij\sum_{n=1}^N \varepsilon_n x_n\right) - \hat{Q}_\varepsilon(-r_k)\exp\left(ir_k\sum_{n=1}^N \varepsilon_n x_n\right)\right\}\right| \\
&\leq \sum_\varepsilon \left\{|\hat{Q}_\varepsilon(0) - \hat{Q}_\varepsilon(-r_k)| + |\hat{Q}_\varepsilon(-r_k)|\sum_{n=1}^N q_n|z_n - \exp(ir_k x_n)|\right\} \\
&\leq \sum_\varepsilon \|Q_\varepsilon\|_{PM}\left\{M\|1 - \exp(ir_k t)\|_{\mathbf{C}(E_N)} + \sum_{n=1}^N q_n|z_n - \exp(ir_k x_n)|\right\} \\
&\qquad \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Hence we have

$$(5) \qquad \|\mu_N\|_{PM} \leq \|Q\|_{PM} \quad (N = 1,2,3,\ldots).$$

Moreover, we see

$$\begin{aligned}
|\hat{\mu}_N(-j) - \hat{Q}(-j)| &\leq |\hat{\mu}_N(-j) - \hat{Q}(-r_k)| + |\hat{Q}(-r_k) - \hat{Q}(-j)| \\
&\leq o(1) + M\|Q\|_{PM}\|\exp(ir_k t) - \exp(ijt)\|_{C(E_x)} \\
&\leq o(1) + M\|Q\|_{PM}\|1 - e^{ijt}\|_{\mathbf{C}(E_N)} \quad \text{as } k \to \infty.
\end{aligned}$$

Since every E_N contains 0 and its diameter is less than or equal to $2\sum_N^\infty q_n|x_n|$, the last inequalities imply

$$(6) \qquad \lim_{N\to\infty} \hat{\mu}_N(-j) = \hat{Q}(-j) \quad (j \in \mathbf{Z}).$$

Finally we infer from (5) and (6) that the sequence $(\mu_N)_1^\infty$ converges to $Q \in PM(E_x)$ in the weak* topology of $PM(\mathbf{T})$, as was required.

THEOREM 2. *Let G be a metrizable LCA I-group, and $(U_n)_1^\infty$ a sequence of compact subsets of G. Suppose that $\sum_1^\infty \varepsilon_n x_n$ converges for each $\varepsilon \in E(q)$ and each $x = (x_n)_1^\infty \in U = \prod_1^\infty U_n$, that every U_n contains $0 \in G$, and that the interior of U_n is dense in U_n. Under these conditions, define the map p_x and the set E_x similarly as before ($x \in U$). Then quasi-all elements of U have the two properties asserted in Theorem* 1.

Proof. We claim without proof that quasi-all $x \in U$ have this property: given a natural number N, $\eta > 0$, and complex numbers $z_1, \ldots, z_n$ of modulus 1, there exists a continuous character γ of G such that

(i) $$|z_n - \gamma(x_n)| < \eta \quad (1 \le n \le N),$$

(ii) $$\left|1 - \gamma\left(\sum_{n=N+1}^{\infty} \varepsilon_n x_n\right)\right| < \eta \quad (\varepsilon \in E(q)).$$

The proof of this fact is similar to that of the lemma for the case $\sum_1^\infty q_n|I_n| < \infty$. A moment's glance at the proof of Theorem 1 shows that all of such $x \in U$ have the required properties.

COROLLARY. *For quasi-all $x \in J$ (or $x \in U$), the symmetric set*

$$K_x = \left\{\sum_1^\infty \varepsilon_n x_n : \varepsilon_n = 0 \text{ or } 1 \text{ for all } n\right\}$$

is of synthesis.

Proof. This is obvious by the proof of part (b) of Theorem 1.

Remarks. (I) The irrationality of C/π is unnecessary in the lemma if we only require that r is a real positive number. Consequently the same is true in Theorem 1 if $\mathbf{T}$ is replaced by $\mathbf{R}$. On the other hand, if C/π is rational and if $\sum_1^\infty q_n|I_n| = \infty$, then quasi-all $x \in J$ satisfy $\sum_1^\infty q_n|x_n| = C$ and none of such x have the property asserted in the lemma.

(II) Let $(\alpha_n)_1^\infty$ and $(\beta_n)_0^\infty$ be two sequences of real positive numbers, and $f(t)$ a strictly positive real function of $t > 0$. If $\alpha_n \neq 1$ for some n, then quasi-all elements x of the space

$$\left\{x \in \prod_1^\infty I_n : \sum_1^\infty \beta_n |x_n|^{\alpha_n} \le \beta_0\right\}$$

have the following property: Given $\eta > 0$ and $|z_1| = \cdots = |z_N| = 1$ there exist two natural numbers r, M such that

(i) $|z_n - \exp(irx_n)| < \eta f(N)$ for $1 \le n \le N$,

(ii) $|1 - \exp(irx_n)| < \eta f(n)$ for $N < n \le M$, and

(iii) $\sum_{M+1}^\infty \beta_n |x_n|^{\alpha_n} < \eta f(rN)$.

This can be proved along the same lines as the lemma. In the case $2^{-1}\alpha_n = \beta_n = 1$ for all $n \geq 1$, this result yields a strong version of both the main theorem of [1] and Theorem 3 of [3].

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