EMBEDDING SPACES

BY

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0. Introduction

In a recent paper [10], Helen Robinson gives an improvement of a theorem of Dax relating smooth and topological embeddings of manifolds in the metastable range. Their result is also a consequence of the techniques of Morlet [9]. In this paper we extend the results of Robinson using Morlet's idea of relating embeddings and immersions, and recent results of Millett on *PL*-immersions [8]. What we show is that in a range of dimensions above the metastable (Corollary 3 of Theorem A) the obstructions to deforming higher homotopy groups of topological embeddings to smooth embeddings lie in the Haefliger knot groups. We also relate topological and piecewise linear (*PL*) embeddings. In Section 2, we relate *PL* embeddings to the space of maps, extending a result of Lusk [6]. For a range of dimensions, this reduces the computation of the homotopy groups of spaces of topological, piecewise linear and smooth embeddings to a purely homotopy problem.

1. The relationship between smooth and topological embeddings

Let $(M^p, \partial M) \subset (N^n, \partial N)$ be smooth manifolds, M compact (with possibly $\partial M = \emptyset$, $\partial N = \emptyset$). Let $E^t(M, N)$ (resp. $E^d(M, N)$) be the space of locally flat topological (resp. smooth) embeddings rel ∂. These may be treated as spaces with the *C-O* topology (resp. C^∞-topology) or as Δ-sets (see Appendix for a detailed discussion). $\mathrm{Im}^t\,(M, N)$ (resp. $\mathrm{Im}^d\,(M, N)$) will be the corresponding spaces of immersions rel ∂. Also Maps (M, N) will be the space of continuous maps rel ∂. Let T be a closed normal tube of M in N, and $\mathring{T}$ an open normal tube containing T, defined with respect to some metric on N. Then $E^t(T, N)$ (resp. $E^d(T, N)$) will denote the space of locally flat (smooth) embeddings of T in N rel $T \cap \partial N$; and similarly for $E(T, \mathring{T})$. Finally, let $E(T, \mathring{T} \bmod M)$ be the subspace of $E(T, \mathring{T})$ of embeddings fixed on $M \cup (T \cap \partial N)$. We assume $n \geq 5$ throughout this paper.

By the isotopy extension theorem (see [2]) the restriction map $E(T, N) \to E(M, N)$ is a fibration (i.e., $E(T, N)$ is a fibre space over a union of components of $E(M, N)$) with fibre $E(T, N \bmod M)$. In either category, $E(T, \mathring{T} \bmod M)$ is a deformation retract of $E(T, N \bmod M)$. Thus (up to homotopy equivalence) the following are fibrations in both categories:

$$\text{(a)} \qquad \begin{aligned} &E(T, \mathring{T} \bmod M) \to E(T, N) \to E(M, N), \\ &E(T, \mathring{T} \bmod M) \to E(T, \mathring{T}) \to E(M, \mathring{T}). \end{aligned}$$

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Likewise the restriction map Im $(T, N) \to$ Im (M, N) is a fibration with fibre Im $(T, N \bmod M)$. Since M is compact, an immersion of T in N fixed on M is an embedding in a neighborhood of M, and $E(T, \quad \bmod M)$ is a deformation retract of Im $(T, N \bmod M)$. Thus (up to homotopy equivalence) the following are fibrations in both categories:

$$\text{(b)} \qquad \begin{aligned} &E(T, \mathring{T} \bmod M) \to \operatorname{Im}(T, N) \to \operatorname{Im}(M, N), \\ &E(T, \mathring{T} \bmod M) \to \operatorname{Im}(T, \mathring{T}) \to \operatorname{Im}(M, \mathring{T}). \end{aligned}$$

Writing $\pi_j^{t/d}(E(M, N))$ for $\pi_j(E^t(M, N), E^d(M, N))$, etc., we have (cf. [9]):

THEOREM A (t/d).

$$\pi_j^{t/d}(\operatorname{Im}(M, N)) \cong \pi_j^{t/d}(E(M, N)) \simeq \pi_j^{t/d}(E(M, T)), \quad j > 0.$$

Proof. By (a) and (b) we have in both categories:

$$\text{(c)} \qquad \pi_j(E(T, N), E(T, \mathring{T})) \simeq \pi_j(E(M, N), E(M, \mathring{T})), \quad j > 0,$$

$$\text{(d)} \qquad \pi_j(\operatorname{Im}(T, N), \operatorname{Im}(T, \mathring{T})) \simeq \pi_j(\operatorname{Im}(M, N), \operatorname{Im}(M, \mathring{T})), \quad j > 0,$$

$$\text{(e)} \qquad \pi_j(\operatorname{Im}(T, N), E(T, N)) \simeq \pi_j(\operatorname{Im}(M, N), E(M, N)), \quad j > 0.$$

By Theorem 3.1 of [1],

$$\text{(f)} \qquad \pi_j^{t/d}(E(T, N)) \simeq \pi_j^{t/d}(\operatorname{Im}(T, N)) \text{ and } \pi_j^{t/d}(E(T, \mathring{T})) \simeq \pi_j^{t/d}(\operatorname{Im}(T, T)), \quad j > 0.$$

Thus if we can prove

$$\text{(g)} \qquad \pi_j^{t/d}(\operatorname{Im}(T, \mathring{T})) \simeq \pi_j^{t/d}(\operatorname{Im}(T, N)), \quad j > 0,$$

the result will follow. In fact, by (e) and (f),

$$\begin{aligned} \pi_j(\operatorname{Im}^t(M, N), E^t(M, N)) &\simeq \pi_j(\operatorname{Im}^t(T, N), E^t(T, N)) \\ &\simeq \pi_j(\operatorname{Im}^d(T, N), E^d(T, N)) \\ &\simeq \pi_j(\operatorname{Im}^d(M, N), E^d(M, N)), \end{aligned}$$

or

$$\pi_j^{t/d}(\operatorname{Im}(M, N)) \simeq \pi_j^{t/d}(E(M, N)), \quad j > 0,$$

proving the first isomorphism.

By (f) and (g), $\pi_j^{t/d}(E(T, N)) \simeq \pi_j^{t/d}(E(T, \mathring{T}))$, and by (c),

$$\begin{aligned} \pi_j(E^t(M, N), E^t(M, \mathring{T})) &\simeq \pi_j(E^t(T, N), E^t(T, \mathring{T})) \\ &\simeq \pi_j(E^d(T, N), E^d(T, \mathring{T})) \\ &\simeq \pi_j(E^d(M, N), E^d(M, \mathring{T})), \end{aligned}$$

or

$$\pi_j^{t/d}(E(M, N)) \simeq \pi_j^{t/d}(E(M, \mathring{T})), \quad j > 0,$$

proving the second isomorphism.

Proof of (g). If τN denotes the tangent bundle of N, we have by the covering homotopy property a fibration in both categories:

$$\text{(h)} \qquad R_0(\tau T, \tau \mathring{T}) \to R(\tau T, \tau N) \to \text{Maps}\,(M, N),$$

where $R(\tau T, \tau N)$ is the space of bundle monomorphisms and $R_0(\tau T, \tau \mathring{T})$ are bundle monomorphisms over the inclusion. Similarly,

$$\text{(h}') \qquad R_0(\tau T, \tau \mathring{T}) \to R(\tau T, \tau \mathring{T}) \to \text{Maps}\,(M, \mathring{T}).$$

Thus

$$\text{(i)} \qquad \pi_j^{t/d}(R(\tau T, \tau N)) \simeq \pi_j^{t/d}(R_0(\tau T, \tau \mathring{T})) \simeq \pi_j^{t/d}(R(\tau T, \tau \mathring{T})), \quad j > 0.$$

Since by the Immersion Theorem (see [1]), we have in both categories

(j) $\pi_j(R(\tau T, \tau N)) \simeq \pi_j(\text{Im}\,(T, N))$ and $\pi_j(R(\tau T, \tau \mathring{T})) \simeq \pi_j(\text{Im}\,(T, \mathring{T}))$, all j, (g) follows.

Example. Taking $M = D^p$, we have $T = D^n$, and since $E^t(D^p, D^n)$ is contractible by the Alexander trick, we get (up to homotopy equivalence) the fibrations

$$\text{(1)} \qquad E^d(D^p, D^n) \to E^d(D^p, N) \to E^t(D^p, N)$$

$$\text{(2)} \qquad E^d(D^p, D^n) \to \text{Im}^d\,(D^p, D^n) \to \text{Im}^t\,(D^p, D^n).$$

COROLLARY 1 (t/d). $\pi_j^{t/d}(E(M, N)), j > 0,$ *depends only on M and its normal bundle ν in N.*

Let $V^t_{n,p}$ and $V^d_{n,p} \approx O(n)/O(n-p)$ be the topological and smooth Stiefel manifolds of germs of embeddings of R^p in R^n. Let $V^{t/d}_{n,p}$ be the homotopy theoretic fibre of the inclusion $V^d_{n,p} \to V^t_{n,p}$.

COROLLARY 2 (t/d). *The fibre of $E^d(M, N) \to E^t(M, N)$ is the space of sections $\Gamma(K(\nu))$ of a fibre space $K(\nu)$ over M with fibre $V^{t/d}_{n,p}$.*

Proof. Since by Theorem A,

$$\pi_j^{t/d}(\text{Im}\,(M, \mathring{T})) = \pi_j^{t/d}(E(M, \mathring{T})) = \pi_j^{t/d}(E(M, V)),$$

we need to find the fibre of $\text{Im}^d\,(M, \mathring{T}) \to \text{Im}^t\,(M, \mathring{T})$. By [4], $\text{Im}\,(M, \mathring{T})$ is the space of sections of a fibre space over M with fibre $V_{n,p}$, and the result follows.

Remark. Corollary 2 means, for example, that the obstructions to deforming a class in $\pi_j(E^t(M, N))$ into a class in $\pi_j(E^d(M, N))$ lie in

$$H^i\,(\textstyle\sum^{j-1}\,(M/\partial M), \pi_i(V^{t/d}_{n,p})),$$

where if $\partial M = \emptyset$, $M/\partial M = M \cup pt$.

Applying the proof of Theorem A with $E^{pl}(M, N)$ in place of $E^{d}(M, N)$ we obtain for M and N piecewise linear manifolds:

THEOREM A (t/pl).

$$\pi_j^{t/pl}(\mathrm{Im}\,(M, N)) \simeq \pi_j^{t/pl}(E(M, N)) \simeq \pi_j^{t/pl}(E(M, \mathring{T})), \quad j > 0.$$

We also get a similar theorem for pl/d or rather pd/d, $pd =$ piecewise smooth. To use these theorems we need information on $V_{n,p}$.

THEOREM (Haefliger-Millett [8]). *If* $n - p \geq 3$,

$$\pi_i(V_{n,p}^{pl}) \to \pi_i(G_n, G_{n-p})$$

is an epimorphism for $i = 2n - p - 3$ *and an isomorphism for* $i < 2n - p - 3$.

Remark. Millett's proof applies equally well to the topological category, so that their theorem holds for $V_{n,p}^{t}$.

Now taking $M = D^p$, $N = D^n$ in Theorem A (t/pl) and using the fact that $E^t(D^p, D^n)$ and $E^{pl}(D^p, D^n)$ are trivial by the Alexander trick, we have $\pi_j^{t/pl}(\mathrm{Im}\,(D^p, D^n)) = 0$. Since $\mathrm{Im}\,(D^p, D^n) = \Omega^p(V_{n,p})$ we get $\pi_j(V_{n,p}^{pl}) \to \pi_j(V_{n,p}^{t})$ is an isomorphism for $j > p$. Combining this with the Haefliger-Millett Theorem we have:

PROPOSITION (t/pl). *If* $n - p \geq 3$, $V_{n,p}^{pl} \to V_{n,p}^{t}$ *is a homotopy equivalence.*

Remark. If $p = n - 1$, $V_{n,p}^{pl} = PL_n$ and $V_{n,p}^{t} = Top_n$ and by [5], $Top_n/PL_n \approx K(Z_2, 3)$, $n \geq 5$.

By the argument of Corollary 2 with E^{pl} in place of E^d, this gives the theorem of Morlet, Rourke-Sanderson, Kirby-Siebenmann.

COROLLARY 2 (t/pl). *If* $n - p \geq 3$, $E^{pl}(M, N) \to E^t(M, N)$ *is a homotopy equivalence.*

(That the components map surjectively follows from the taming theorem, Theorem 5.51 of [11].)

Remark. If $p = n - 1$, $\pi_j(E^{pl}(M, N)) \to \pi_j(E^t(M, N))$ is a monomorphism if $j \geq 3$ and an epimorphism if $j \geq 4$.

Next for $V_{n,p}^{t/d}$ we have by the Haefliger-Millett Theorem:

PROPOSITION (t/d). *If* $n - p \geq 3$, $\pi_{j-1}(V_{n,p}^{t/d}) = \pi_j(G, O, G_{n-p})$ *for* $j < 2n - p - 3$.

Hence Corollary 2 implies:

THEOREM (Dax, Robinson). $\pi_j^{t/d}(E(M, N)) = 0$ *for* $j \leq 2n - 3p - 3$.

Proof. $\pi_i(G, O, G_{n-p}) = 0$ for $i \leq 2n - 2p - 3$ by [7]. Now $j \leq 2n - 3p - 3$ implies $j + p \leq 2n - 2p - 3 < 2n - p - 3$. Also $3p \leq 2n - 3$ implies the components are surjective.

More generally we have:

COROLLARY 3. *For $0 < j < 2n - 2p - 3$ the obstructions to the existence of a lift of a class in $\pi_j(E^t(M, N))$ to $\pi_j(E^d(M, N))$ and to the uniqueness of the lifts of a class in $\pi_{j-1}(E^t(M, N))$ lie in $\pi_{j+k}(G, O, G_{n-p})$, $0 \leq k \leq p$.*

Now let $CE(M, N)$ (resp. $C \operatorname{Im}(M, N)$) be the space of embeddings (immersions) $f: M \times I \to N \times I$ such that $f \mid M \times 0 \cup \partial M \times 1 =$ inclusion and $f^{-1}(N \times 1) = M \times 1$. Then by the same argument as in Theorem A we have:

THEOREM A (C).

$$\pi_j^{t/d}(C \operatorname{Im}(M, N)) \simeq \pi_j^{t/d}(CE(M, N)) \simeq \pi_j^{t/d}(CE(M, \mathring{T})), \quad j > 0.$$

(Similarly for t/pl and pd/d.)

COROLLARY 1 (C). *$\pi_j^{t/d}(CE(M, N))$, $j > 0$, depends only on M and its normal bundle ν in N.*

Let $P(X, Y)$ be paths in X beginning at the base point and ending in Y.

COROLLARY 2 (C). *The fibre of $CE^d(M, N) \to CE^t(M, N)$ is the space of sections $\Gamma(L(\nu))$ of a fibre space $L(\nu)$ over M with fibre $P(V_{n+1, p+1}^{t/d}, V_{n, p}^{t/d})$.*

COROLLARY 3 (C). *If $n - p \geq 3$, $\pi_j^{t/d}(CE(M, N)) = 0$ for $j < 2n - 2p - 3$.*

Remark. One may of course go from information on $\pi_j^{t/d}(E(M, N))$ or $\pi_j^{t/d}CE(M, N)$ to information on $V_{n, p}^{t/d}$, as we did for $V_{n, p}^{t/pl}$. In fact, one may obtain Millett's improvement of Haefliger's theorem this way. Also using information on stability, i.e., on when $\pi_j CE^d(M, N) \to \pi_j CE^d(M \times I, N \times I)$ is an isomorphism. We get information on when

$$\pi_j(V_{n+1, p+1}^t, V_{n, p}^t) \to \pi_{j+1}(V_{n+2, p+2}^t, V_{n+1, p+1}^t)$$

is an isomorphism. This last will appear in a forthcoming paper on stability.

2. The space of PL embeddings

Let M^p, N^n be compact PL manifolds, $n - p \geq 3$.

THEOREM (Casson, Haefliger, Sullivan [12]). *Let*

$$f: (M, \partial M) \to (N, \partial N)$$

be a map such that $f \mid \partial M$ is a PL embedding and f is $(2p - n + 1)$-connected. Then f is homotopic rel *∂M to a PL embedding.*

Now assume $(M^p, \partial M) \subset (N^n, \partial N)$ and let $\tilde{E}^{pl}(M, N)$ be the Δ-set of block embeddings of M in N [2], with base point the inclusion. This is a Kan Δ-set and $\pi_j(\tilde{E}^{pl}(M, N))$ = concordance classes of PL embeddings $\phi: D^j \times M \to D^j \times N$ such that $\phi \mid \partial(D^j \times M)$ = inclusion. By the above theorem we have:

THEOREM B. *If $\pi_i(N, M) = 0$ for $i \leq t$, then*

$$\pi_j(\text{Maps}\,(M, N), \tilde{E}^{pl}(M, N)) = 0 \quad \text{for } 0 < j \leq n + r - 2p - 1.$$

Write $\pi_j^{\text{rel}}(E^{pl}(M, N)) = \pi_j(\tilde{E}^{pl}(M, N), E^{pl}(M, N))$. The following theorem is essentially due to Millett [8] (see remarks below).

THEOREM C. *Let $M^p = D^p \cup$ handles of dimension greater than l, and let N^n be a k-connected ($k \leq n - 4$ if $\pi_1(\partial N) \neq 0$). Then*

$$\pi_j^{\text{rel}}(E(M, N)) = 0 \quad \text{for } j \leq n + \hat{k} - p - 2,\ \hat{k} = \inf\,(k, n - p + l - 1).$$

Proof. (a) By Theorem 2.8 of [2],

$$\pi_j^{\text{rel}}(E(D^q, N)) = 0 \quad \text{for } j \leq n + k - q - 2 \ (k \leq n - 4 \text{ if } \pi_1(\partial V) \neq 0).$$

(b) By Theorem 3.20 of [8], $\pi_j^{\text{rel}}(E(D^q \times S^{p-q}, D^q \times S^{n-q})) = 0$ for $j \leq 2n - p - q - 3$.

From the fibrations

$$\text{(c)}\quad E(D^q \times D^{p-q}, D^q \times D^{n-q}) \to E(D^q \times S^{p-q}, D^q \times S^{n-q}; \text{mod } D^q \times 0) \to E(D^q \times R^{p-q}, D^q \times R^{n-q}; \text{mod } D^q \times 0),$$

$$E(D^q \times S^{p-q}, D^q \times S^{n-q}; \text{mod } D^q \times 0) \to E(D^q \times S^{p-q}, D^q \times S^{n-q}) \to E(D^q, D^q \times S^{n-q}),$$

and the corresponding fibrations for $\tilde{E}$ we get

$$\begin{aligned}\pi_j^{\text{rel}}(E(D^q \times R^{p-q}, D^q \times R^{n-q}; \text{mod } D^q \times 0)) &\simeq \pi_j^{\text{rel}}(E(D^q \times S^{p-q}, D^q \times S^{n-q} \text{ mod } D^q \times 0)) \\ &\simeq \pi_j^{\text{rel}}(E(D^q \times S^{p-q}, D^q \times S^{n-q})) \quad \text{for } j < 2n - 2q - 3,\end{aligned}$$

since

$$\pi_j^{\text{rel}}(E(D^q \times D^{p-q}, D^q \times D^{n-q})) = \pi_j^{\text{rel}}(D^p, D^n) = 0,$$

and

$$\pi_j^{\text{rel}}(E(D^q, D^q \times S^{n-q})) = 0 \quad \text{for } j \leq 2n - 2q - 3$$

by (a).

(d) By (b) and (c): For $p > q$,

$$\pi_j^{\text{rel}}(E(D^q \times R^{p-q}, D^q \times R^{n-q}; \text{mod } D^q \times 0)) = 0 \quad \text{for } j \leq 2n - p - q - 3.$$

Now $M - \mathring{D}^p = \partial M \cup$ handles of dimension $\leq p - l - 1$. Let q = dim of lowest dim handle in this decomposition ($q = 0$ if $\partial M = \emptyset$). Let $M' =$

M − open normal tube of D^q in M, and $N' = N$ − open normal tube of D^q in N. Then we have the fibrations

(e) $$E(M, N; \bmod D^q) \to E(M, N) \to E(D^q, N),$$

$$E(M', N') \to E(M, N; \bmod D^q) \to E(D^q \times R^{p-q}, D^q \times R^{n-q}; \bmod D^q \times 0)$$

and similarly for $\tilde{E}$.

(f) By (a) and (d),

$$\pi_j^{\mathrm{rel}} E(M', N') \to \pi_j^{\mathrm{rel}}(E(M, N; \bmod D^q) \to \pi_j^{\mathrm{rel}} E(M, N))$$

is surjective for $j \leq n + k' - q - 2$, $k' = \inf(k, n - p - 1)$.

Further $\pi_j(N') = \pi_j(N - D^q) = 0$ for $j \leq \inf(k, n - q - 2)$. Hence it follows by induction on the dimensions of the handles that

(g) $\pi_j(E(D^p, N_0)) \to \pi_j(E(M, N))$ is surjective for

$$j \leq n + k' - p + l - 1 = n + (k' + l + 1) - p - 2,$$

where $N_0 = N$ − open normal tube of $(M - \mathring{D}^p)$ in N.

Further, $\pi_j(N_0) = 0$ for $j \leq k'' = \inf(k, n - p + l - 1)$.

Applying (a) to $E(D^p, N_0)$ we get $\pi_j(E(D^p, N_0)) = 0$ for $j \leq n + k'' - p - 2$, and

(h) $$\pi_j(E(M, N)) = 0 \quad \text{for } j \leq n + \hat{k} - p - 2,$$

$$\begin{aligned} \hat{k} &= \inf(k' + l + 1, k'') \\ &= \inf(k + l + 1, n - p + l, k, n - p + l - 1) \\ &= \inf(k, n - p + l - 1). \end{aligned}$$

Remarks. (i) This differs from Millett's result in [8] in three respects:

(1) He assumes $M = D^p \cup$ handles of dim between $l + 1$ and $p - l - 1$ if $\partial M \neq \emptyset$, and the foregoing union a p-handle if $\partial M = \emptyset$.

(2) He omits the condition $k \leq n - 4$ if $\pi_1(\partial V) \neq 0$. This is because he claims that (a) holds without this condition.

(3) His result states that $\pi_j^{\mathrm{rel}}(E(M, N)) = 0$ for $j \leq n + r - p - 2$, $r = \inf(k, n - p - 1)$.

(ii) If M is an l-connected manifold, then M satisfies the hypothesis of the theorem, provided $l \leq p - 4$ if $\pi_1(\partial M) \neq 0$.

By Theorems B and C we have:

THEOREM D. *If $\pi_i(N) = 0$ for $i \leq k$ ($k \leq n - 4$ if $\pi_1(\partial N) \neq 0$), $\pi_i(M) = 0$ for $i \leq l$, and $\pi_i(N, M) = 0$ for $i \leq r$, then*

$$\pi_j(\text{Maps}(M, N), E(M, N)) = 0$$
$$\text{for } 0 < j \leq n + t - p - 2, \ t = \inf(r - p + 1, k, n + l - p + 1).$$

Theorem B may be improved in the sense that $\pi_j(\tilde{E}^{pl}(M, N))$ may be obtained, up to extensions, from homotopy data for all $j \geq 1$. The result is more complicated (and even less computable) than Maps (M, N). (cf. [8].)

Let $\mathscr{H}(N)$ be the space of homotopy equivalences of N fixed on ∂N. One may also consider the Δ-set $\tilde{\mathscr{H}}(N)$ of block homotopy equivalences. However, the singular complex of $\mathscr{H}(N)$ is a deformation retract of $\tilde{\mathscr{H}}(N)$ and we will suppress the symbol $\sim$. Let $W = \overline{N - T}$. Let S be the frontier of T, i.e., the normal sphere bundle of M in N. Let T_1 be the closure of $\mathring{T}$, $T_1 = T \cup S \times I$.[1] Then as in [2] we have the fibrations:

(a) $\tilde{A}(N \bmod M) \to \tilde{A}(N) \to \tilde{E}(M, N)$.
(b) $\tilde{A}(W) \to \tilde{A}(N \bmod M) \to \tilde{E}(T, N \bmod M) \simeq \tilde{E}(T, \mathring{T} \bmod M)$.
(a′) $\tilde{A}(T_1 \bmod M) \to \tilde{A}(T_1) \to \tilde{E}(M, \mathring{T}) = \tilde{E}(M, T_1)$.
(b′) $\tilde{A}(S \times I) \to \tilde{A}(T_1 \bmod M) \to \tilde{E}(T, \mathring{T} \bmod M)$.

Thus we have since (b′) is a subbundle of (b):

(c) $\pi_i(\tilde{A}(W)/\tilde{A}(S \times I)) \simeq \pi_i(\tilde{A}(N \bmod M)/\tilde{A}(T_1 \bmod M))$, all $i > 0$.

Now using that (a′) is a subbundle of (a), we get, using (c):

(d) $$\to \pi_i(\tilde{A}(W)/\tilde{A}(S \times I)) \to \pi_i(\tilde{A}(N)/\tilde{A}(T_1)) \to \pi_i(\tilde{E}(M, N), \tilde{E}(M, \mathring{T})) \to \pi_{i-1}(\tilde{A}(W)/\tilde{A}(S \times I)) \to$$

is exact, $i > 1$.

Now Theorem 3.5 (5) of [0] gives a (natural) exact sequence:
(e) $\to L^s_{n+i+1}(\pi) \to \pi_i^{\mathscr{H}/PL}\tilde{A}(N) \to [\Sigma^i(N/\partial N), G/PL] \to L^s_{n+i}(\pi)$, $i > 0$, where $\pi = \pi_1(N)$, $L^s_k(\pi)$ is the Wall surgery group, and $N/\partial N = N \cup pt$ if $\partial N = \emptyset$.

Taking the sequence (e) with W in place of N, and using the inclusion $(N, \partial N) \to (N, T \cup \partial N)$ we get a map of the W sequence into the N sequence and hence

(f) $\pi_i^{\mathscr{H}/PL}(\tilde{A}(N)/\tilde{A}(W)) \simeq [\Sigma^i(T \cup \partial N)/\partial N, G/PL]$, $i > 0$.

Applying the same argument to T_1 and $S \times I$, we get

(f′) $\pi_i^{\mathscr{H}/PL}(\tilde{A}(T_1)/\tilde{A}(S \times I)) \simeq [\Sigma^i(T \cup \partial T_1/\partial T_1), G/PL]$, $i > 0$.

Since $T \cup \partial T_1/\partial T_1 = T \cup \partial N/\partial N$, we have

(g) $\pi_i^{\mathscr{H}/PL}(\tilde{A}(T_1)/\tilde{A}(S \times I)) \simeq \pi_i^{\mathscr{H}/PL}(\tilde{A}(N)/\tilde{A}(W))$, or

$$\pi_i^{\mathscr{H}/PL}(\tilde{A}(W)/\tilde{A}(S \times I)) \simeq \pi_i^{\mathscr{H}/PL}(\tilde{A}(N)/\tilde{A}(T_1)).$$

Hence from (d) we get,

(h) $$\to \pi_i(\mathscr{H}(W), \mathscr{H}(S \times I)) \to \pi_i(\mathscr{H}(N), \mathscr{H}(T_1)) \to \pi_i(\tilde{E}(M, N), \tilde{E}(M, \mathring{T})) \to \pi_{i-1}(\mathscr{H}(W), \mathscr{H}(S \times I)) \to$$

is exact, $i > 1$.

[1] That is, $\mathring{T}$ is an open normal tube containing T.

Now by Theorem B,

$$\pi_j(\text{Maps}\,(M, T_1), \tilde{E}(M, T_1)) = 0, \quad j > 0,$$

and

$$\pi_j(\tilde{E}(M, T_1)) = \pi_j(\text{Maps}\,(M, T_1)) = \pi_j\mathscr{H}(M), \quad j > 0.$$

This gives[2]:

THEOREM B′.

$$\to \pi_i(\mathscr{H}(W), \mathscr{H}(S \times I)) \to \pi_i(\mathscr{H}(N), \mathscr{H}(T_1))$$
$$\to \pi_i(\tilde{E}(M, N), \mathscr{H}(M)) \to \pi_{i-1}(\mathscr{H}(W), \mathscr{H}(S \times I)) \to$$

is exact, $i > 1$.

Finally, by Theorem C we have:

THEOREM D′. *Theorem* B′ *holds with* $E(M, N)$ *in place of* $\tilde{E}(M, N)$ *for*

$$1 < i < n + \hat{k} - p - 2, \quad \hat{k} = \inf(k, n - p + l - 1),$$

where N *is* k*-connected, and* M *is* l*-connected.*

We also note (see [8]),

THEOREM D″. *If* N *is* k*-connected and* M *is* l*-connected,*
$\pi_j CE(M, N) = 0$ *for* $j \leq n + \hat{k} - p - 3$, $\hat{k} = \inf(k, n - p + l - 1)$.

Final Remark. Consider the case where N is noncompact, but ∂N is compact (or empty) and N has only a finite number of tame ends. It follows from Siebenmann's thesis (Princeton) that if $X \subset N$ is compact, there exists a compact submanifold K, $X \subset K \subset N$, such that $\pi_i(N, K) = 0$ for $i \leq n - 3$. It follows that Theorem B holds for such N provided $r \leq n - 4$. Likewise Theorem C holds. Thus Theorem D holds under the above assumptions. Similarly, Theorem D″. Finally, Theorem D′ holds under these assumptions provided we interpret $\mathscr{H}(W)$ and $\mathscr{H}(N)$ as homotopy equivalences with compact support.

Appendix. Spaces of topological embeddings

Let $(M, \partial M) \subset (N, \partial N)$ be a compact locally flat submanifold of the compact manifold N with $n - m \geq 3$. Let $\mathscr{E}^t(M, N)$ be the space of topological embeddings $f: M \to N$ with $f^{-1}(\partial N) = \partial M$ and $f \mid \partial M$ = inclusion and with the C-O topology. Let $\mathscr{E}^t_{LF}(M, N)$ be the subspace of locally flat embedding. Finally let $E^t(M, N)$ be the semisimplicial complex for which a p-simplex is an embedding $g: \Delta^p \times M \to \Delta^p \times N$ such that:

(a) g commutes with projection onto Δ^p.

[2] Note that by definition $\tilde{E}(M, T_1) = \tilde{E}(M, \mathring{T})$.

(b) $g^{-1}(\Delta^p \times \partial N) = \Delta^p \times \partial M$.

(c) $g \mid \Delta^p \times \partial M =$ inclusion.

(d) g satisfies the locally isotopy extension condition: For each $(t, x) \in \Delta^p \times M$, there exists a neighborhood U of t in Δ^p, a neighborhood V of x in M, and an embedding $h\colon U \times V \times R^{m-n} \to U \times N$ commuting with projection onto U and with $h \mid U \times V \times 0 = g \mid U \times V$.

Then we have the obvious inclusions $E^t(M, N) \subset S\mathscr{E}^t_{LF}(M, N) \subset S\mathscr{E}^t(M, N)$ where S denotes the singular complex.

THEOREM 1. *If $n - m \geq 3$ and $n \geq 5$, the above inclusions are homotopy equivalences.*

This theorem follows from the results of Cernavskii (Topological embeddings of manifolds, Soviet Math. Dokl., vol 10 (1969), pp. 1037–1041), which in turn depend on work of Homma, Bryant and Seebeck, R. D. Edwards and Richard T. Miller. That the first inclusion is a homotopy equivalence follows immediately from Cernavskii's isotopy extension theorem for the space $\mathscr{E}^t_{LF}(M, N)$; i.e., isotopy extension implies local isotopy extension; so in fact $E^t(M, N) = S\mathscr{E}^t_{LF}(M, N)$ for $n \geq 5, n - m \geq 3$.

For the second inclusion we need the following:

LEMMA. *Let (X, A) be a pair of metric spaces. If A is dense in X and locally p-connected at points of X for $0 \leq p \leq n$, then $\pi_q(X, A) = 0$ for $0 \leq q \leq n - 1$.*

DEFINITION. A is *locally p connected at points of X* if for any $x \in X$ and $\varepsilon > 0$ there is a $\delta > 0$ such that if $g\colon S^p \to A$ with $d(g(s), x) < \delta$ for $s \in S^p$; then g extends to $\bar{g}\colon D^{p+1} \to A$ with $d(\bar{g}(s), x) < \varepsilon$ for $s \in D^{p+1}$.

Thus the fact that $S\mathscr{E}^t_{LF}(M, N) \to S\mathscr{E}^t(M, N)$ is a homotopy equivalence follows from the lemma and the result of Cernavskii et al:

THEOREM 2. *If $n - m \geq 3$ and $n \geq 5$, $\mathscr{E}^t_{LF}(M, N)$ is dense in $\mathscr{E}^t(M, N)$ and is locally p connected at points of $\mathscr{E}^t(M, N)$ for all p.*

(Actually Cernavskii states a slightly weaker result than this in the above reference, but as pointed out by Edwards and Miller, Notices A.M.S., vol 19 (1972), A-467, by using the stronger form of Bryant and Seebeck's engulfing lemma the stronger statement above holds.)

Also we note that Corollary 2 (t/pl) and Theorem 1 above imply:

THEOREM 3. *If M and N are compact PL manifolds with $n \geq 5$ and $n - m \geq 3$ then $E^{pl}(M, N) \to S\mathscr{E}^t(M, N)$ is a homotopy equivalence.*

Theorem 3 is also claimed by R. Stern. The above argument depends crucially on ideas of Richard Miller, and indeed Miller has proved Theorem 3 for M an arbitrary finite PL space of codimension 4 (Fiber preserving equivalence—to appear in Trans. AMS).

Conversely, Theorem 3 and Theorem 1 imply Corollary 2 (t/pl).

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