CONTINUOUSLY CHOOSING A RETRACTION OF A SEPARABLE METRIC SPACE ONTO EACH OF ITS ARCS

BY

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1. Introduction

R. H. Bing [1] has asked whether there exists a continuous function which selects a point from each arc of the Euclidean plane. The existence of such a choice function is a corollary to the result suggested in the title since we need only choose a point of the plane and take the image of it under each retraction. The proof is an application, in the style of Hamstrom and Dyer [4], of E. Michael's selection Theorem 3.1‴ [7; p. 368], referred to here as Theorem M. It provides a new proof of a theorem of H. Whitney [9; Theorem 14A], c.f., Theorem W in Section 5.

The results are the only application we know of Theorem M, which itself is the only basic selection theorem not requiring each image set to be closed. The key geometrical fact used in our proof is that the set of increasing self-homeomorphisms of the closed unit interval is a $\mathscr{D}$-set (see Section 3).

2. Statement of the problem

Suppose Y is a topological space and 2^Y is defined to be the collection of non-empty subsets of Y. Suppose further that X is a topological space and $\psi: X \to 2^Y$ is a parametrization by X of some of the subsets of Y. A continuous function $f: X \to Y$ such that $f(x) \in \psi(x)$ for all $x \in X$ is a *selection* for ψ, i.e., a continuous choice function for ψ.

If U is a neighborhood of $x_0 \in X$ and $f: U \to Y$ is a selection for $\psi \mid U$ then we call f a *local selection for* ψ *at* x_0.

If for each $y_0 \in \psi(x_0)$ and $x_0 \in X$ there is a local selection for ψ at x_0 such that $f(x_0) = y_0$ then ψ must satisfy the following condition. The set-valued function $\psi: X \to 2^Y$ is *lower semi-continuous* (lsc) if whenever U is open in Y then $\{x \in X: \psi(x) \cap U \neq \emptyset\}$ is open in X. The reader is encouraged to consult the seminal work of E. Michael [7] for the basic facts in the theory of selections.

Let X and Y be topological spaces and let $C(X, Y)$ denote the set of continuous functions from X into Y. If $A \subset X$ and $B \subset Y$, we define

$$(A, B) = \{f \in C(X, Y): f(A) \subset B\}.$$

The *compact-open topology* (*c-topology*) on $C(X, Y)$ is generated by sub-basic open sets (A, U) where A is a compact subset of X and U is an open subset of

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Y [2; p. 257]. In case the topology of Y is generated by a metric d, we define $C(X, Y; d)$ to be the set of continuous functions whose image is bounded in the metric d. The set $C(X, Y; d)$ can be topologized with the metric D defined by

$$D(f, g) = \sup \{d(f(x), g(x)) : x \in X\}$$

[2; p. 269]. The D topology is called the *topology of uniform convergence.*

Remark 1. The relative c-topology on $C(X, Y; d)$ is a subset of (is coarser than) the topology generated by D.

Remark 2. If X is compact then $C(X, Y) = C(X, Y; d)$ and the two topologies agree.

Remark 3. All function spaces will be assumed to have the c-topology unless otherwise specified.

Let d be a metric for Y and let X be compact and metrizable. Let $X(Y)$ be the set of subspaces of Y homeomorphic to X. Given $X_1, X_2 \in X(Y)$ and a homeomorphism g from X_1 onto X_2, we define the *size of* g to be

$$\sup \{d(x, g(x)) : x \in X_1\}.$$

The *completely regular topology* (*c.r.-topology*) on $X(Y)$ is defined by the metric ρ such that $\rho(X_1, X_2) = \inf \{\text{size of } g : g \text{ is a homeomorphism from } X_1 \text{ onto } X_2\}$

Remark 4. Since X is compact, the topology generated by ρ is independent of the metric d generating the given topology of Y.

Recall that a *retraction* of Y onto a subspace A is a continuous function $f: Y \to A$ such that $f(x) = x$ for each $x \in A$.

Again, let X be compact metrizable and Y be metrizable. We define three set-valued functions:

$$\psi : X(Y) \to 2^Y \text{ where } \psi(X_1) = X_1.$$

$$\Phi : X(Y) \to 2^{C(X, Y)} \text{ where } \Phi(X_1) = \{f : f \text{ is a homeomorphism from } X \text{ onto } X_1\}.$$

$$\theta : X(Y) \to 2^{C(X, Y)} \text{ given by } \theta(X_1) = \{f : f \text{ is a retraction of } Y \text{ onto } X_1\}.$$

For which pairs of spaces X and Y can one find selections or local selections for the set-valued functions ψ or Φ or θ? In particular, when X is a compact metrizable absolute retract and Y is a separable metric space, is there a selection for θ (and therefore ψ)? It is known that there is no local selection for Φ at some $A \in X(Y)$ when $Y = E^2$ and X is the absolute retract, Ungar's "ruler" (see [8] for a description).

We know from Whitney [9; Theorem 14A] or from this paper that there are local selections for Φ when Y is a separable metric space and X is an arc (B^1), a simple closed curve (S^1), or a finite graph such that the order of each vertex is different from two. From Whitney [9] and Riemann-Roch [6, Theorem 1], there are local selections for Φ and a selection for ψ in case X is the unit disk and Y is E^2. It is easy to show the impossibility of local selections for ψ and Φ and θ in case X is a Cantor set and Y is E^2.

Although all of these results can be derived from Whitney's theorem, we are now ready to show how these results and Whitney's theorem can be derived as outlined in the introduction. The following simple observation formalizes a remark made in the introduction.

Remark 5. If F is a selection for θ and $x_0 \in Y$, then F' defined by $F'(A) = F(A)(x_0)$ is a selection for ψ [2; Theorem 2.4(1), p. 260].

Remark 6. If X is an AR and Φ admits a selection in a neighborhood of each $A \in X(Y)$ then there is a selection for ψ (c.f., proof of Corollary 2 in Section 5).

3. Theorem M and remarks on the c.r.-topology of $X(Y)$

Let Y be a normed linear space and K be a closed, convex subset of Y. A *supporting set* of K is a closed convex subset S of K such that $S \neq K$ and such that if an interior point of a line segment of K lies in S then the entire line segment lies in S. The set of all elements of K which do not lie in a supporting set of K will be denoted $I(K)$. We define $\mathscr{D}(Y)$ to be the collection of convex subsets B of Y such that $B \supset I(\bar{B})$. Note that $\bar{B}$ is the topological closure of B in Y.

THEOREM M [7; Theorem 3.1‴ (a) implies (c)]. *If X is a perfectly normal T_1-space and Y is a separable Banach space and Φ: $X \to \mathscr{D}(Y)$ is lsc, then there is a selection for Φ.*

Before demonstrating how Theorem M can be used to prove the main result, let us remark on the choice of topology for $X(Y)$.

PROPOSITION 1. *If $C(X, Y)$ has the topology of uniform convergence and T is a topology on $X(Y)$ such that Φ is lsc and there is a selection F for Φ, then T contains (is finer than) the c.r.-topology on $X(Y)$.*

Proof. Let $X_1 \in X(Y)$ and $\varepsilon > 0$ be given and F be a selection for Φ. Then $F(X_1) = f$ is an embedding of X onto $X_1 \subset Y$. Since Φ is lsc, the set

$$U = \{X' \colon \Phi(X') \cap N_\varepsilon(f) \neq \emptyset\}$$

is open in T. But U is the ε-neighborhood of X_1 in the c.r.-topology. For let $X' \in X(Y)$ be such that $\rho(X_1, X') < \varepsilon$. Let h: $X_1 \to X'$ be a homeomorphism of size less than ε. Then $hf \in \Phi(X') \cap N_\varepsilon(f)$ so that $X_2 \in U$. Conversely, if $X' \in U$ then there is a homeomorphism k: $X \to X'$ such that $D(k, f) < \varepsilon$ and kf^{-1} demonstrates that $\rho(X', X_1) < \varepsilon$. The proof is complete.

In order for there to be a selection for ψ, we may not be forced to choose a topology as fine as the c.r.-topology on $X(Y)$. However, if we choose the

Hausdorff metric on $X(Y)$, so that the distance between X_1 and X_2 is less than ε if and only if $X_1 \subset N_\varepsilon(X_2)$ and $X_2 \subset N_\varepsilon(X_1)$, then there is no selection for ψ in case $X = B^1$ and $Y = E^2$, i.e., Bing's problem. For each $t \in (0, 1]$ let $f(t)$ be the graph of $\sin(1/x)$ restricted to the interval $[t/2, t]$ and let $f(0) = \{0\} \times [-1, 1]$. Then f is an embedding of $[0, 1]$ into $X(Y)$ with the Hausdorff topology. A selection g for ψ restricted to any neighborhood of $f(0)$ would violate the fact that $\cup \{f(t): t \in [0, 1]\}$ is not locally connected at any point of $f(0)$.

4. A reduction and the main lemma

We now assume that X is compact metrizable and Y is metrizable. Let Q be the Hilbert cube considered as a compact, convex subset of Hilbert space, l_2.

PROPOSITION 2. *For there to be a selection (local selections) for ψ or Φ or θ, respectively, for $X(Q)$, it is necessary and sufficient for there to be a selection (local selections) for ψ or Φ or θ, respectively, for every $X(Y)$ where Y is a separable metric space.*

Proof. Sufficiency is obvious. For necessity, suppose that Y is any separable metric space, X is compact and metrizable and f is a topological embedding of Y into Q.

The function $F: X(Y) \to X(Q)$, defined by $F(X_1)$ equals the image of X_1 under f, is an embedding onto $X(f(Y))$. The function $H: C(X, Y) \to C(X, Q)$ defined by $H(g) = fg$ is an embedding onto $C(X, f(Y))$. The function

$$F': C(f(Y), f(Y)) \to C(Y, Y)$$

defined by $F'(g) = f^{-1}gf$ is an onto homeomorphism. This fact depends upon the choice of the c-topology for function spaces. Finally, if $A \subset Q$ and

$$B = \{f \in C(Q, Q): f(Q) \subset A\} \subset C(Q, Q)$$

then the restriction function $w: B \to C(A, A)$, defined by $w(f) = f \mid A$ is continuous.

Suppose now that U is an open subset of $X(Q)$. If g is a selection for $\psi \mid U$ or $\Phi \mid U$ or $\theta \mid U$, respectively, then $f^{-1}gF$ or $H^{-1}gf$ or $F'wgF$ is a selection $\psi \mid F^{-1}(U)$ or $\Phi \mid F^{-1}(U)$ or $\theta \mid F^{-1}(U)$, respectively. The proof is complete.

We now consider the special case in which $X = B^1$ and $Y = Q$. For each $A \in B^1(Q)$ define A^* to be the straight line segment in Q whose endpoints are the endpoints of A. Let $\gamma: B^1(Q) \to 2^{C(Q, l_2)}$ be defined by the rule that $\gamma(A)$ is the set of continuous $f: Q \to l_2$ such that (1) $f(Q) = A^*$, (2) f is fixed on the endpoints of A and (3) $f \mid A$ is a homeomorphism onto A^*. We now show that there is a selection for γ.

(A) The space $B^1(Q)$ is a perfectly normal T_1-space.

Proof. $B^1(Q)$ is metric.

(B) The space $C(Q, l_2)$ is a separable Banach space.

Proof. This is because Q is compact and l_2 is a separable Banach space [5; p. 244]. This fact dictates the use of the c-topology for $C(Y, Y)$ in finding a selection for θ. If (Y, d) could be isometrically embedded in a compact, convex subset of a separable Banach space, then we could have used the topology of uniform convergence.

(C) The set-valued function γ is lsc.

Proof. Let d be the standard metric on l_2 and let ρ and D be the induced metrics on $B^1(Q)$ and $C(Q, l_2)$; see Remarks 2 and 4. Suppose that $A \in B^1(Q)$ and $f \in \gamma(A)$ and $\varepsilon > 0$ are given. We must find a neighborhood V of A in $B^1(Q)$ such that if $B \in V$ then $\gamma(B) \cap N_\varepsilon(f) \neq \emptyset$.

Since Q is compact and f is continuous, let $\delta > 0$ illustrate the uniform continuity of f for the number $\varepsilon/2$. Let V be the ρ-metric $\delta/2$ neighborhood of A in $B^1(Q)$. Suppose that $B \in V$; we must construct $g \in \gamma(B)$ such that $D(g, f) < \varepsilon$.

Let $h\colon B \to A$ be an onto homeomorphism moving each point less than $\delta/2$. Let $h\colon Q \to A$ also denote a fixed continuous extension of h. By the compactness of B, let W be an open neighborhood of B (in Q) such that $d(x, h(x)) < \delta$ for each $x \in \overline{W}$ and such that $\overline{W} \subset N_{\delta/2}(A)$. Let $j\colon Q \to [0, 1]$ be a continuous (Urysohn) function such that $j(Q - W) = 0$ and $j(B) = 1$. Let $f'\colon Q \to A^*$ such that

$$f'(z) = j(z)\cdot fh(z) + (1 - j(z))f(z).$$

Clearly, f' is continuous such that $f' \mid Q - W = f \mid Q - W$ and that $f' \mid B = fh \mid B$ and that $f'(Q) \subset A^*$. Finally, we let $h^*\colon A^* \to B^*$ be the linear homeomorphism such that $h^*(h(p)) = p$ for each end-point p of B^*. Clearly

$$d(h^*(x), x) < \varepsilon/2$$

for each $x \in A^*$.

We now show that $g = h^*f'$ is an element of $\gamma(B) \cap N_\varepsilon(f)$. Since

$$B \xrightarrow{h} A \xrightarrow{f} A^* \xrightarrow{h^*} B^*$$

are each homeomorphisms (properly restricted), so is $g \mid B$. The choice of h^* insures that the endpoints of B are fixed under g. Finally, $D(g, f) < \varepsilon$. If $x \in Q - W$ then

$$d(g(x), f(x)) = d(h^*f(x), f(x)) < \varepsilon/2.$$

If $x \in W$, let $z = j(x)(fh(x)) + (1 - j(x))f(x)$, and then

$$\begin{aligned} d(g(x), f(x)) &= d(h^*(z), f(x)) \\ &\leq d(h^*(z), z) + d(z, f(x)) \\ &\leq \varepsilon/2 + j(x)[d(fh(x), f(x))] \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

The last, since $d(h(x), x) < \delta$ and thus $d(fh(x), f(x)) < \varepsilon/2$.

(D) For each $A \in B^1(Q)$, the set $\gamma(A)$ is an element of $\mathscr{D}(C(Q, l_2))$.

Proof. Let a and b denote the endpoints of $A \in B^1(Q)$. Let h be the unique affine homeomorphism from A^* onto $[0, 1]$ such that $h(a) = 0$. Let h' be a topological homeomorphism of $[0, 1]$ onto A such that $h'(0) = a$. A continuous function f from A onto A^*, fixing the endpoints, is a homeomorphism if and only if hfh' is an increasing function from $[0, 1]$ onto itself.

The set $\gamma(A)$ is convex. If $f, g \in \gamma(A)$ and $t \in [0, 1]$ then $(1 - t)f + tg$ is onto A^* and leaves the endpoints of A fixed. Observe that

$$h[(1 - t)f \mid A + t(g \mid A)]h' = (1 - t)h(f \mid A)h' + th(g \mid A)h'. \tag{1}$$

The right-hand side of this equation is a convex combination of increasing self-homeomorphisms of $[0, 1]$ and is therefore itself an increasing self-homeomorphism. So $(1 - t)f + tg \in \gamma(A)$.

If $f \in \overline{\gamma(A)}$ then $h(f \mid A)h'$ is a nondecreasing continuous function from $[0, 1]$ onto $[0, 1]$. If f_i converges to f then hf_ih' converges to hfh'. A limit of increasing continuous functions from $[0, 1]$ onto $[0, 1]$ is nondecreasing. We now claim that $\gamma(A) \supset I(\overline{\gamma(A)})$. Suppose $f \in \overline{\gamma(A)}$ and $f \notin \gamma(A)$ then $h(f \mid A)h'$ is constant on some closed subinterval T of $[0, 1]$. Let $S = \{f \in \overline{\gamma(A)}: hfh'(T)$ is constant$\}$. Clearly S is convex and $S \neq \overline{\gamma(A)}$. We claim that S is a supporting hyperplane of $\overline{\gamma(A)}$. Suppose $t \in (0, 1)$ and $f \in S$ and $k, m \in \overline{\gamma(A)}$ and $f = (1 - t)k + tm$. It suffices to show that $k \in S$. If not, there exist z and w in T such that $hkh'(z) < hkh'(w)$. By (1),

$$\begin{aligned} hfh'(z) &= h((1 - t)k + tm)h'(z) \\ &< h((1 - t)k + tm)h'(w) \\ &= hfh'(w). \end{aligned}$$

This contradicts the assumption that $f \in S$. We have shown that if $f \in \overline{\gamma(A)}$ and $f \notin \gamma(A)$ then $f \notin I(\overline{\gamma(A)})$, i.e., $\gamma(A) \supset I(\overline{\gamma(A)})$.

LEMMA 1. *There is a selection for γ, i.e., there is a continuous function $G: B^1(Q) \to C(Q, Q)$ such that for each $A \in B^1(Q)$, (1) $G(A): Q \to A^*$ and (2) $G(A) \mid A$ is a homeomorphism onto A^*, fixed on the endpoints of A.*

Proof. Statements (A)-(D) show that Theorem M applies to γ.

5. The theorems

THEOREM 1. *Let Y be a separable metric space. There exists a continuous function $F: B^1(Y) \to C(Y, Y)$ such that $F(A)$ is a retraction of Y onto A for each $A \in B^1(Y)$. (Note that $C(Y, Y)$ has the compact-open topology.)*

Proof. By Proposition 2, we need only prove Theorem 1 for $Y = Q$. Let G be the continuous function of Lemma 1. Let F be defined by the formula $F(A) = (G(A) \mid A)^{-1}G(A)$. Clearly each $F(A)$ is a retraction of Q onto A. We now show that F is continuous.

Let d be the usual metric for Q. Suppose that a sequence $\langle A_i \rangle$ converges to A in $B^1(Q)$; but that $\langle F(A_i) \rangle$ does not converge to $F(A)$. Assume without loss of generality that for some fixed $\varepsilon > 0$ and some sequence $\langle x_i \rangle$ in Q we have

$$d(F(A)(x_i), F(A_i)(x_i)) > \varepsilon.$$

By the compactness of A assume that $x, y \in A$ and that

$$\lim F(A)(x_i) = x \in A \quad \text{and} \quad \lim F(A_i)(x_i) = y \in A.$$

Since $d(x, y) \geq \varepsilon$, we have $G(A)(x) \neq G(A)(y)$.

However, $\lim G(A_i) = G(A)$. So $\lim G(A_i)(F(A_i)(x_i)) = G(A)(y)$ and

$$G(A_i)(F(A_i)(x_i)) = G(A_i)(x_i) \quad \text{for each } i.$$

Therefore, $\lim G(A_i)(x_i) = G(A)(y)$. But $\lim G(A_i)(x_i) = \lim G(A)(x_i)$ and

$$G(A)(F(A)(x_i)) = G(A)(x_i) \quad \text{for each } i.$$

So

$$\begin{aligned} G(A)(y) &= \lim G(A_i)(x_i) \\ &= \lim G(A)(x_i) \\ &= \lim G(A)(F(A)(x_i)) \\ &= G(A) \lim (F(A)(x_i)) \\ &= G(A)(x). \end{aligned}$$

This contradicts $G(A)(y) \neq G(A)(x)$ and so F is continuous.

COROLLARY 1. *For each separable metric space Y, there is a continuous function f: $B^1(Y) \to Y$ such that $f(A) \in A$ for each $A \in B^1(Y)$.*

Proof. Theorem 1 and Remark 5.

THEOREM 2. *For each separable metric space Y and each $A \in B^1(Y)$, there exists a neighborhood U of A in $B^1(Y)$ and a continuous function H: $U \to C(B^1, Y)$ such that $F(C)$ is a homeomorphism of B^1 onto C for each $C \in U$.*

Proof. Again, by Proposition 2, it suffices to prove the case $Y = Q$. Given $A \in B^1(Q)$, let a and b be the endpoints of A and let $\eta = d(a, b)$. Let U be the $\eta/2$ neighborhood of A in $B^1(Q)$. For each $C \in U$, let $h(C)$ be the unique affine homeomorphism of B^1 onto C^* such that $d(h(C)(1), a) < \eta/2$. Let G be the continuous function of Lemma 1, and define

$$H: B^1(Q) \to C(B^1, Q)$$

by the formula $H(C) = (G(C) \mid C)^{-1}h(C)$. Each $H(C)$ is clearly a homeomorphism of B^1 onto $C \in U$. The proof that H is continuous is like the proof that F is continuous in Theorem 1.

Remark 7. The fact that the Moebius band can be embedded in Q shows that U cannot be taken to be $B^1(Q)$.

COROLLARY 2. *For each separable metric space Y, there is a continuous function g: $B^1(Y) \to Y$ such that $g(A)$ is a nonendpoint of A for each $A \in B^1(Y)$.*

Proof. Using the paracompactness of $B^1(Y)$ we can find a well-ordered, locally finite, closed cover $\{U_\alpha : \alpha < \gamma\}$ such that each U_α is contained in an open set V_α over which a function H_α of Theorem 2 is defined. We define g by induction. Suppose g is defined on $\cup_{\beta<\alpha} U_\beta$ then $W = U_\alpha \cap \cup_{\beta<\alpha} U_\beta$ is a closed subset of U_α. Define f: $W \to (0, 1)$ by $f(A) = (H_\alpha(A))^{-1}(g(A))$. Let f' be a continuous extension of the continuous function f into (0, 1). Let $g'(A) = H(A)(f'(A))$ be the continuous extension of g over U_α.

A *completely regular mapping* f: $Y \to Z$ (where Y and Z are metric spaces) satisfies the condition that for each $z \in Z$ and $\varepsilon > 0$ there is $\delta > 0$ such that if $d(z, z_1) < \delta$, then there is an onto homeomorphism $g: f^{-1}(z) \to f^{-1}(z_1)$ of size less that ε [3; Hamstrom].

THEOREM W (essentially Whitney [9; Theorem 14A]). *If Y and Z are separable metric spaces and f: $Y \to Z$ is an onto completely regular mapping such that $f^{-1}(z)$ is homeomorphic to B^1 for each $y \in Y$. Then for each $z \in Z$ there is a neighborhood V of Z and an onto homeomorphism h: $V \times B^1 \to f^{-1}(V)$ such that $\pi_1 = fh$.*

Proof (sketch). The function f^{-1} is continuous from Z to $B^1(Y)$. Let $z \in Z$ and let U be a neighborhood of $f^{-1}(z)$ and H: $U \to C(B^1, Y)$ be those asserted in Theorem 2. Let $V = \{z: f^{-1}(z) \in U\}$ and let h: $V \times B^1$ onto $f^{-1}(V)$ be given by the formula $h(z, t) = H(f^{-1}(z))(t)$.

Remark 8. Theorem W easily implies Theorem 2. Consider

$$S \subset B^1(Y) \times Y \quad \text{where} \quad S = \{(A, y): y \in A\}.$$

The map π_1: $S \to B^1(Y)$ is completely regular; now apply Theorem W.

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