CLASSIFICATION THEOREMS FOR PARAMETERIZED FAMILIES OF SMOOTH OR PIECEWISE LINEAR MANIFOLD STRUCTURES RESPECTING A SUBMANIFOLD

BY

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Let M be a topological manifold with N a locally flat submanifold. In [15], C. Rourke and B. Sanderson relativized the main diagram of R. Kirby and L. Siebenmann [2] to get stable classification theorems for PL structures on M with N a PL submanifold. Kirby and Siebenmann [4], [6] and Burghelea and Lashof [1] exploit the ideas of immersion theory to get unstable parameterized classification theorems for smooth or PL manifold structures on M. In this paper, we observe that this immersion theoretic approach can be relativized to yield unstable parameterized classification theorems for smooth or PL manifold structures on M with N a smooth or PL submanifold.

R. Miller [10] has given a codimension 4 fiber preserving equivalence theorem which implies that if (K, K_0) is a polyhedral pair, $f: K_0 \to N$ a PL embedding, where N is a PL manifold, and $\dim N - \dim (K \backslash K_0) \geq 4$, then the inclusion $\mathrm{Emb}^{\mathrm{PL}}(K, N; f) \to \mathrm{Emb}^{\mathrm{TOP}}(K, N; f)$ is a weak homotopy equivalence, with all homotopies as small as desired. Here $\mathrm{Emb}^{\mathrm{CAT}}(K, N; f)$, CAT = TOP or PL, is the semisimplicial complex of CAT embeddings of K into N extending f. The primary motivation for this paper is to prove in Section 5 a codimension 3 fiber preserving equivalence theorem and then to prove the above result in codimension 3, but with the further requirement that K be a PL manifold and $\dim N \geq 5$. This codimension 3 result then has its subsequent applications to topological embedding spaces (see [8], [11], [17]).

0. Notation and definitions

In this paper we will be concerned with three categories, namely the category TOP of topological manifolds and continuous maps, the category PL of piecewise linear manifolds and piecewise linear maps, and the category DIFF of C^∞ manifolds and C^∞ maps. We denote the boundary of a CAT manifold N by ∂M. For the objects of DIFF we allow C^∞ manifolds with *corners*, namely we allow coordinate charts which are diffeomorphic to open subsets of

$$R^n_q = \{(x_1, \ldots, x_n) \in R^n \mid x_1 \geq 0, x_2 \geq 0, \ldots, x_q \geq 0\}.$$

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We let $\mathbf{0} \in R^n$ denote the origin of R^n, and we let Δ^n denote the standard n-simplex in R^n.

A CAT *manifold pair* (M, N) is a pair of CAT manifolds M and N with N a CAT locally flat submanifold of the CAT manifold M and $\partial M \cap N = \partial N$. We refer to Milnor [14] and Kuiper-Lashof [7] for the theory of CAT microbundles. Let M be a CAT manifold and let $\tau(M)$ denote the CAT tangent microbundle of M. If M is unbounded, then $\tau(M)$ is the microbundle with total space $M \times M$, zero section the diagonal map $\Delta: M \to M \times M$ given by $\Delta(x) = (x, x)$, and projection the projection map on the second factor. If M is bounded, we let

$$\tau(M) = (\tau M \cup \text{open collar of } \partial M) \mid M.$$

A CAT (n, m)*-microbundle pair* is a pair of CAT microbundles (ξ^n, η^m) over a CAT object X such that η^m is a subbundle of ξ^n, i.e., $E(\eta^m) \subset E(\xi^n)$ in some neighborhood of the zero section, and for each $x \in X$, there exists microbundle charts $h: U \times R^m \to E(\eta)$, $g: U \times R^n \to E(\xi)$, where U is a neighborhood of x, such that

$$\begin{array}{ccc} U \times R^m & \xrightarrow{h} & E(\eta) \\ \cap & & \cap \\ U \times R^n & \xrightarrow{g} & E(\xi) \end{array}$$

commutes.

Let (ξ_0, η_0) and (ξ_1, η_1) be CAT microbundle pairs over CAT objects X_0 and X_1, respectively. A CAT *morphism* $\theta: (\xi_0, \eta_0) \to (\xi_0, \eta_1)$ consists of a neighborhood U of the zero section of (ξ_0, η_0) in $E(\xi_0)$, a CAT map of pairs

$$H: (U, U \cap E(\eta_0)) \to (E(\xi_1), E(\eta_1)),$$

and a CAT map $h: X_0 \to X_1$ such that $p_1 H = h p_0$, $H i_0 = i_1 h$, and for each $x \in X$

$$H|_{U \cap p_0^{-1}(x)}: (U \cap p_0^{-1}(x), U \cap p_0^{-1}(x) \cap E(\eta_0)) \to (p_1^{-1}(h(x)), p_1^{-1}(h(x)) \cap E(\eta_1))$$

is a CAT open embedding of pairs. Here p_j and i_j, $j = 0, 1$, are the projections and zero-sections of (ξ_0, η_0) and (ξ_1, η_1).

1. Relative CAT submersions are CAT bundle pairs

In this section we relativize the "CAT submersions are CAT bundles" theorem of Burghelea and Lashof [1] and of Kirby and Siebenmann [4]. A *relative* CAT *submersion* (CAT = TOP, PL, or DIFF) is a CAT map $p: (E, E') \to (B, B)$ of CAT manifold pairs such that $p: E \to B$ is a CAT submersion and for each point $y \in E'$, there is an open neighborhood N of $p(y)$ in B and an open embedding $f: (V, V') \times N \to (E, E')$ of CAT pairs, where V is an open subset of $p^{-1}(p(y))$ containing y and $V' = V \cap E'$, such that the composition pf is the projection $(V, V') \times N \to N$ and $f(u, p(y)) = u$. The map f is called a relative CAT product chart about (V, V'). Note that if $p\colon (E, E') \to (B, B)$ is a relative CAT submersion, then $p \mid E'\colon E' \to B$ is a CAT submersion.

A CAT *bundle pair* is a relative CAT submersion $p\colon (E, E') \to (B, B)$ such that $p\colon E \to B$ is a locally trivial CAT bundle, and $p \mid E'\colon E' \to B$ is a locally trivial CAT subbundle.

Let (M, N) be a TOP manifold pair. By a *relative* CAT *structure on the manifold pair* $\Delta^k \times (N, M)$ *sliced over* Δ^k we mean a CAT structure Γ on the product $\Delta^k \times M$ such that the projection $(\Delta^k \times (M, N))_\Gamma \to (\Delta^k, \Delta^k)$ is a relative CAT submersion. Note that for each $t \in \Delta^k$, $\Gamma \mid t \times (M, N)$ is a CAT structure on M inducing a CAT structure on N making N a CAT (locally flat) submanifold.

Let Σ be a CAT structure on N. By a *relative* CAT *structure on* $\Delta^k \times (M, N)$ rel Σ *sliced over* Δ^k we mean a relative CAT structure Γ on $\Delta^k \times (M, N)$ sliced over Δ^k with $\Gamma \mid \Delta^k \times N = \Delta^k \times \Sigma$.

The first goal of this section is to prove:

THEOREM 1.1. *Let* (M, N) *be a* TOP *manifold pair and let* Σ *be a* CAT *structure on* N. *Suppose* Γ *is a relative* CAT *structure on* $\Delta^k \times (M, N)$ *sliced over* Δ^k. *If either*

(i) $\dim N \neq 4 \neq \dim \partial N$ *and* $\dim M \geq 6$ (≥ 5 *if* $\partial M = \emptyset$), *or*

(ii) $\dim M \geq 6$ (≥ 5 *if* $\partial M = \emptyset$) *and* $\Gamma \mid \Delta^k \times N = \Delta^k \times \Sigma$, *then the projection* $p_1\colon (\Delta^k \times (M, N))_\Gamma \to (\Delta^k, \Delta^k)$ *is a* CAT *bundle pair.*

By local applications of (1.1) we have:

COROLLARY 1.2. *Let* $p\colon (E, E') \to (B, B)$ *be a relative* CAT *submersion of* CAT *manifold pairs such that* p *is a* TOP *bundle pair. If either*

(i) $\dim (p^{-1}(x) \cap E') \neq 4 \neq \dim \partial(p^{-1}(x) \cap E')$ *and* $\dim (p^{-1}(x)) \geq 6$ (≥ 5 *if* $\partial p^{-1}(x) = \emptyset$), *for all* $x \in B$, *or*

(ii) $p \mid E'\colon E' \to B$ *is a locally trivial* CAT *bundle and* $\dim (p^{-1}(x)) \geq 6$ (≥ 5 *if* $\partial p^{-1}(x) = \emptyset$) *for all* $x \in B$,

then $p\colon (E, E') \to (B, B)$ *is a* CAT *bundle pair.*

Remark 1.3. It is an easy exercise, using the CAT isotopy extension theorem for isotopies respecting a submanifold, to show that if $p\colon (E, E') \to (B, B)$ is a relative CAT submersion then there exists a relative CAT product chart for a neighborhood of a compact subset of the fiber of p. (cf. Corollary 6.9 of [16].) Thus, if p is a *proper relative* CAT *submersion*, i.e., for each compactum K in B, $(p^{-1}(K), p^{-1}(K) \cap E')$ is a compact pair, then $p\colon (E, E') \to (B, B)$ is a CAT bundle with no dimension restrictions on $p^{-1}(x)$.

Our proof of (1.1) is just a relativized version of the corresponding submersions are bundles proof of Kirby and Siebenmann (cf. Essay II, Theorem 1.8 of [4]).

To prove (1.1) we require the following two lemmas.

LEMMA 1.4. *Let* (M, N) *be a connected* CAT *manifold pair and let*

$$h: (M, N) \to (P, P') \times R$$

be a homeomorphism, where (P, P') *is a* TOP *manifold pair.*

(i) *If* $\dim N \neq 4 \neq \dim \partial N$ *and* $\dim M \geq 6$, *then* (M, N) *has the following engulfing property:*

$E(M, N)$. *For any pair of integers* $a \leq b$, *there exists a* CAT *isotopy*

$$h_t: (M, N) \to (M, N), \quad 0 \leq t \leq 1,$$

of id $| M$, *with compact support in* $h^{-1}(P \times (a - 1, b + 1))$, *such that*

$$h_1 h^{-1}((P, P') \times (-\infty, a)) \supset h^{-1}((P, P') \times (-\infty, b]).$$

(ii) *If* $\dim N \neq 4$, $\dim M \geq 5$ *and property* $E(\partial M, \partial N)$ *is verified, then property* $E(M, N)$ *holds true.*

(iii) *If* $\dim M \geq 6$ *and property* $E(N, \emptyset)$ *is verified, then property* $E(M, N)$ *holds true.*

(iv) *If* $\dim M \geq 5$ *and properties* $E(N, \emptyset)$ *and* $E(\partial M, \partial N)$ *are verified, then property* $E(M, N)$ *holds true.*

Proof. To verify (i) when $\dim N \leq 3$, note that P' has a CAT structure and hence N is CAT isomorphic to $P' \times R$ by Moise's 3-dimensional Hauptvermutung, so that property $E(N, \emptyset)$ holds true. Then, the CAT isotopy extension theorem and CAT engulfing verify property $E(M, N)$.

To verify the remaining cases of (i), (ii), (iii), and (iv), use CAT engulfing in a relative collar of $(\partial M, \partial N)$ in (M, N), respecting N, to verify property $E(\partial M, \partial N)$. Then engulf with compact support, respecting N, in int M. This respectful engulfing is achieved by first engulfing in ∂N or N and then using the CAT isotopy extension theorem and CAT engulfing in ∂M or M minus the interior of a regular neighborhood of N in ∂M or M. ∎

LEMMA 1.5. *Let* (M, N) *be a compact* TOP *manifold pair and let* Σ *be a* CAT *structure on* $N \times R$. *Suppose* Γ *is a relative* CAT *structure on* $\Delta^k \times (M, N) \times R$ *sliced over* Δ^k. *If either*

(i) $\dim N \neq 3 \neq \dim \partial N$ *and* $\dim M \geq 5$ $(\geq 4$ *if* $\partial M = \emptyset)$, *or*

(ii) $\dim M \geq 5$ $(\geq 4$ *if* $\partial M = \emptyset)$ *and* $\Gamma \mid \Delta^k \times N \times R = \Delta^k \times \Sigma$,

then for any pair of integers $a < b$ *there exists an open set* U_{ab} *in* $\Delta^k \times M \times R$ *containing* $\Delta^k \times M \times [a, b]$ *such that the projection*

$$p: (U_{ab}, U_{ab} \cap (\Delta^k \times N \times R)) \to (\Delta^k, \Delta^k)$$

is a CAT *bundle pair.*

Proof. Let C be a closed subset of Δ^k and consider the following engulfing property:

$E_r(s, C)$. For any pair of integers $a \leq b$, there exists a CAT isotopy

$$h_t: (\Delta^k \times (M, N) \times R)_\Gamma \to (\Delta^k \times (M, N) \times R)_\Gamma$$

of id $| \Delta^k \times M \times R$ which commutes with projection to Δ^k, such that when $[a - s, b + s] \subset [-r + 2, r - 2]$,

$$h_1(\Delta^k \times M \times (-\infty, a)) \supset C \times M \times (-\infty, b).$$

Note that $E_r(s, C)$ implies $E_u(r, D)$ if $r \geq u$, $v \geq s$, and $C \supset D$. Also, one can easily verify the addition property:

(A) $E_r(s, C)$ and $E_r(t, D)$ implies $E_r(s + t, C \cup D)$.

Fix an integer $r < \infty$, and let U_x be an open neighborhood of a point $x \in \Delta$ and consider a relative CAT product chart

$$f: U_x \times ((M, N) \times (-r, r))_{\Gamma | p^{-1}(x)} \to (\Delta^k \times (M, N) \times R)_\Gamma$$

for the relative CAT submersion p. Such a chart exists (for an arbitrary r) by (1.3) and by noting that $(M, N) \times [-r, r]$ is a compact manifold pair.

For $x \in \Delta^k$, let $\Gamma_{x,r} = \Gamma \mid x \times M \times (-r, r)$. Then for any CAT isotopy

$$g_t^x: ((M, N) \times (-r, r))_{\Gamma_{x,r}} \to ((M, N) \times (-r, r))_{\Gamma_{x,r}}$$

of id $| M \times (-r, r)$ with compact support and for any CAT map $\alpha: \Delta^k \to [0, 1]$ with support in U_x, define a CAT isotopy

$$h_t^x: (\Delta^k \times (M, N) \times R)_\Gamma \to (\Delta^k \times (M, N) \times R)_\Gamma$$

which commutes with projection to Δ^k by letting $h_t^x f(u, s, v) = f(u, g_{\alpha(u)t}(s, v))$ and by letting h_t^x be the identity off of the image of f.

Note that for fixed r, x, and f, there exists an open neighborhood V of x in U_x such that if $\alpha: \Delta^k \to [0, 1]$ is a CAT function with support $(\alpha) \subset U$ then as g_t^x ranges over the isotopies given by (1.4), one for each interval $[a, b] \subset [-r + 2, r - 2]$, the corresponding isotopies h_t establish property $E_r(1, \alpha^{-1}(1))$. By letting x and f vary, with r fixed, form an open covering $\{U_j\}$ of Δ^k by such sets U.

By taking a fine handle decomposition of Δ^k, we can decompose Δ^k into $k + 1$ closed sets $C_0, \ldots, C_k$, where each C_i is the disjoint union of closed sets C_{ij} each contained in some U_k. Now for each C_i let $\alpha_{ij}: \Delta^k \to [0, 1]$ be a CAT function with $\alpha(C_{ij}) = 1$, with support in some U_k, and for a fixed i, the α_{ij}'s have disjoint support. Then, for a fixed i, the isotopies corresponding to the α_{ij}'s compose to establish properties $E_r(1, C_i)$.

By k applications of the addition property (A), we have that property $E_r(k + 1, \Delta^k)$ holds true. But note that in the above argument, r is as large as we please, so that $E_\infty(k + 1, \Delta^k)$ holds true.

Let

$$h_t\colon (\Delta^k \times (M, N) \times R)_\Gamma \to (\Delta^k \times (M, N) \times R)_\Gamma$$

be the CAT isotopy of id | $\Delta^k \times M \times R$ given by property $E_\infty(k + 1, \Delta^k)$ for the integers $a \leq b$, and let

$$Z_{ab} = h_1(\Delta^k \times M \times (-\infty, a]) - (\Delta^k \times M \times (-\infty, a)).$$

Then $U_{ab} = \bigcup \{h_1^k(Z_{ab}) \mid n = 0, \pm 1, \pm 2, \ldots\}$, where h_1^k is the n-fold composition of h_1, is an open subset of $\Delta^k \times M \times R$ and contains $\Delta^k \times M \times [a, b]$.

Let $U'_{ab} = U_{ab} \cap (\Delta^k \times N \times R)$. Form the quotient

$$q\colon U_{ab} \to B = U_{ab}/\{h_1(x) = x \mid x \in \Delta^k \times M \times R\}$$

and observe that q and

$$q' = q \mid U'_{ab}\colon U'_{ab} \to B' = U'_{ab}/\{h_1(x) = x \mid x \in \Delta^k \times N \times R\}$$

are infinite cyclic covering maps with $h_1 \mid U_{ab}$ and $h_1 \mid U'_{ab}$ infinite cyclic covering translations of q and q', respectively. Let $g\colon B \to \Delta^k$ be the unique map such that the composition $gq = p \mid U_{ab}$. We have thus factored

$$p\colon (U_{ab}, U'_{a,b}) \to (\Delta^k, \Delta^k)$$

as

$$(U_{ab}, U'_{ab}) \xrightarrow{q} (B, B') \xrightarrow{g} (\Delta^k, \Delta^k).$$

Now g is a proper relative CAT submersion, hence a CAT bundle pair by (1.3). Using the fact that $q\colon (U_{ab}, U'_{ab}) \to (B, B')$ is an infinite cyclic covering map of pairs, it is an easy exercise to verify that gq, hence $p \mid U_{ab}$, is a CAT bundle pair. ■

Proof of Theorem 1.1. Fix a point $x \in \Delta^k$ and set $\Gamma' = \Gamma \mid x \times (M, N)$ and identify $(M, N) = (M, N)_{\Gamma'}$. Filter (M, N) by compact CAT manifold pairs

$$(M_0, N_0) \subset (M_1, N_1) \subset \cdots, \quad \text{with} \bigcup (M_i, N_i) = (M, N).$$

Choose disjoint open relative CAT bicollarings $(U_i, V_i) \cong (\delta M_i, \delta N_i) \times R$ of the frontiers $(\delta M_i, \delta N_i)$ with the relative collar of $(\delta M_i, \delta N_i)$ in (M_i, N_i) being $(\delta M_i \delta N_i) \times (-\infty, 0]$. By (1.5) there is an open subset E_i of $(\Delta^k \times \delta M_i \times R)_\Gamma$ containing $\Delta^k \times \delta M_i \times 0$, such that

$$(E_i, E_i \cap (\Delta^k \times \delta N_i \times R))_\Gamma \to (\Delta^k, \Delta^k)$$

is a CAT bundle pair. Let $E'_i = E_i \cap (\Delta^k \times \delta N_i \times R)$. Then, by the relative CAT bundle homotopy theorem, there exists a CAT isomorphism of pairs

$$h_i\colon \Delta^k \times (F_i, F'_i) \to (E_i, E'_i)$$

which commutes with projection to Δ^k and is the identity over x. Let

$$(M'_i, N'_i) = (F_i \cap (\delta M_i \times (-\infty, 0]), F'_i \cap (\delta N^i \times (-\infty, 0])).$$

Let the CAT compact manifold pair $(X_i, Y_i) \subset (\Delta^k \times (M, N)_\Gamma$ be given by

$$(X_i, Y_i) = \{\Delta^k \times ((M_i - E_i, N_i - E_i')) \cup h_i(\Delta^k \times (M_i', N_i'))\}_\Gamma.$$

Then $(X_i, Y_i) \subset (X_{i+1}, Y_{i+1})$ and $\bigcup (X_i, Y_i) = (\Delta^k \times (M, N))_\Gamma$. Note that

$$p\colon (X_i, Y_i) \to (\Delta^k, \Delta^k)$$

is a proper relative CAT submersion, hence a trivial CAT bundle pair by (1.3). Let $f_i\colon \Delta^k \times (M, N) \to (X_i, Y_i)$ be CAT isomorphisms which commute with projection to Δ^k and are the identity over x given by the relative CAT bundle homotopy theorem. By the Δ^k parametered CAT isotopy extension theorem respecting a submanifold, we can arrange inductively that

$$f_{i+1} \mid \Delta^k \times (M_i, N_i) = f_i.$$

Then $\lim f_i = f\colon \Delta^k \times (M, N) \to (\Delta^k \times (M, N))_\Gamma$ is a CAT isomorphism respecting projection to Δ^k, so that $(\Delta^k \times (M, N))_\Gamma \to (\Delta^k, \Delta^k)$ is a (trivial) CAT bundle pair. ■

We now wish to refine (1.1), for applications in Section 5, so that the trivialization of the bundle pair in (1.1) can be realized by a small ambient isotopy of $\Delta^k \times (M, N)$ respecting projection to Δ^k.

Let Γ_0 be a relative CAT structure on (M, N) and let Γ be a relative CAT structure on $\Delta^k \times (M, N)$ sliced over Δ^k. Finally, suppose that $C \subset M$ is a closed subset such that $\Gamma = \Delta^k \times \Gamma_0$ near $\Delta^k \times C$, and $\varepsilon\colon \Delta^k \times M \to (0, \infty)$ is continuous map.

THEOREM 1.6. (SLICED CONCORDANCE RESPECTING A SUBMANIFOLD IMPLIES ISOTOPY RESPECTING A SUBMANIFOLD). *If* CAT = PL *let* $\Lambda \subset \Delta^k$ *be any contractible subpolyhedron, and if* CAT = DIFF *let* Λ *be some face of* Δ^k *or* $\partial\Delta^k$ *minus the interior of some principal face. Assume that* $\Gamma \mid \Lambda \times M = \Lambda \times \Gamma_0$. *Then if* $\dim N \neq 4 \neq \dim (\partial N - C)$, $\dim M \geq 5$, *and* $\dim (\partial M - C) \geq 5$ *if* $\partial M - C \neq \emptyset$, *there exists an* ε *isotopy* h_t, $t \in [0, 1]$, *of* $\mathrm{id} \mid \Delta^k \times M$ *sliced over* Δ^k *and respecting* $\Delta^k \times N$, *to a* CAT *isomorphism of pairs*

$$h_1\colon \Delta^k \times (M, N)_{\Gamma_0} \to (\Delta^k \times (M, N))_\Gamma,$$

so that h_t *fixes* $\Lambda \times M$ *and a neighborhood of* $\Delta^k \times C$.

THEOREM 1.7. (SLICED CONCORDANCE REL A SUBMANIFOLD IMPLIES ISOTOPY REL A SUBMANIFOLD). *In addition to the data of* (1.6) *assume that* $\Gamma \mid \Delta^k \times N = \Delta^k \times \Gamma_0 \mid N$. *Then the conclusion of* (1.6) *holds under the weaker hypothesis that* $\dim M \geq 5$ *and if* $\partial M - C \neq \emptyset$, *then* $\dim (\partial M - C) \geq 5$. *Furthermore,* h_t *also fixes* $\Delta^k \times N$.

We list some corollaries of these two theorems.

COROLLARY 1.8. *Let* (M, N) *be a* TOP *manifold pair with* $\dim N \neq 4 \neq \dim \partial N$ *and* $\dim M \geq 6$ (≥ 5 *if* $\partial M = \emptyset$). *Let* Γ *be a relative* CAT *structure on*

$\Delta^k \times (M, N)$ sliced over Δ^k and let Γ' be a relative CAT *structure on $\Delta^k \times (M', M' \cap N)$ sliced over Δ^k, where M' is an open subset of M. Suppose that $\Gamma' = \Gamma$ on $\Lambda \times M'$. Then Γ' extends to a relative* CAT *structure Γ'' on $\Delta^k \times (M, N)$ sliced over Δ^k and is equal to Γ on $\Lambda \times M$.*

Proof. Let $\sigma = \Gamma \mid * \times (M, N)$, where $* \in \Lambda$. Theorem 1.6 yields a sliced CAT isomorphism of pairs $F: \Delta^k \times (M', N \cap M')_\sigma \to (\Delta^k \times (M', M' \cap N))_{\Gamma'}$. Then $F^{-1}(\Gamma)$ equals $\Lambda \times \sigma$ on $\Lambda \times M'$, so Theorem 1.6 gives a sliced CAT ε-isomorphism of pairs $G: \Delta^k \times (M', M' \cap N)_\sigma \to (\Delta^k \times (M', M' \cap N))_{F^{-1}(\Gamma)}$ equal to the identity on $\Lambda \times M'$. For small ε, $FGF^{-1} \mid \Delta^k \times M'$ extends via the identity to a sliced automorphism H of the pair $\Delta^k \times (M, N)$. Then let $\Gamma'' = H(\Gamma)$. ■

Using Theorem 1.7, one can similarly prove:

COROLLARY 1.9. *Let (M, N) be a* TOP *manifold pair with* $\dim M \geq 6$ (≥ 5 *if $\partial M = \emptyset$). Let Σ be a* CAT *structure on N and let Γ be a relative* CAT *structure on $\Delta^k \times (M, N)$* rel Σ *sliced over Δ^k. Suppose Γ' is a relative* CAT *structure on $\Delta^k \times (M', M' \cap N)$* rel Σ *sliced over Δ^k, where M' is an open subset of M. If $\Gamma' = \Gamma$ on $\Lambda \times M'$, then Γ' extends to a relative* CAT *structure on $\Delta^k \times (M, N)$* rel Σ *sliced over Δ^k and is equal to Γ on $\Lambda \times M$.*

By arguments that are now standard (see Essay I of [4]) and the "sliced concordance implies isotopy" theorem of Kirby and Siebenmann (Theorem 2.1, Essay II of [4]) it suffices, in order prove Theorems 1.6 and 1.7, to prove the following handle lemmas.

THEOREM 1.10. *In addition to the data of Theorem* 1.6 *assume that*

$$(M, N) = B^p \times (R^n, R^m), \qquad C = \partial B^p \times (R^n, R^m),$$

and that Γ_0 is the standard structure on $B^p \times R^n$. If $p + m \neq 4$ and $p + n \geq 5$, then there exists a sliced ambient isotopy h_t, $t \in [0, 1]$, of $\Delta^k \times B^p \times R^n$ respecting $\Delta^k \times B^p \times R^m$, fixing $\Lambda \times B^p \times R^n \cup \Delta^k \times \partial B^p \times (R^n, R^m)$, such that

$$h_1: \Delta^k \times B^p \times (R^n, R^m) \to (\Delta^k \times B^p \times (R^n, R^m))_\Gamma$$

is a CAT *embedding on*

$$\Delta^k \times B^p \times (\text{int } B^n, \text{int } B^m)$$

and over

$$(\Delta^k \times B^p \times (\text{int } B^n, \text{int } B^m))_\Gamma.$$

THEOREM 1.11. *In addition to the data of Theorem* 1.10 *assume that*

$$\Gamma \mid \Delta^k \times B^p \times R^m$$

is standard. Then the conclusion of Theorem 1.10 *holds under the weaker hypothesis that $p + n \geq 5$. Furthermore, h_1 fixes $\Delta^k \times B^p \times R^m$.*

Remark. Theorems 1.6, 1.7, 1.8, and 1.9 are respectful versions of Theorems 2.1 and 2.6 in Essay II of [4]. Also, Theorems 1.10 and 1.11 only yield versions of Theorems 1.6 and 1.7, respectively, for relative structures *near* $\Delta^k \times N$. However, the case of Theorem 1.6 with $N = \emptyset$ (which is Theorem 2.1 in Essay II of [4]) allows us to extend these isotopies to all of $\Delta^k \times M$ with the desired properties.

Proof of Theorems 1.10 *and* 1.11. *Step* 1. Theorem 1.1 yields a CAT isomorphism $\phi\colon \Delta^k \times B^p \times (R^n, R^m) \to (\Delta^k \times B^p \times (R^n, R^m))_\Gamma$, where in the case of (1.11) ϕ is the identity on $\Delta^k \times B^p \times R^m$. As every CAT automorphism of $\Lambda \times (M, N)$ respecting $\Lambda \times N$ (fixed on $\Lambda \times N$) which commutes with projection to Λ, extends to a CAT automorphism of $\Delta^k \times (M, N)$ respecting $\Delta^k \times N$ (fixed on $\Delta^k \times N$) which commutes with projection to Δ^k, we can assume that $\phi \mid \Lambda \times B^p \times (R^n, R^m)$ is the identity.

Step 2. We wish to alter ϕ so that ϕ is actually the identity near

$$\Delta^k \times \partial B^p \times (R^n, R^m).$$

To accomplish this just stretch out the relative sliced collars.

Step 3. We now have a sliced CAT isomorphism of pairs

$$\phi\colon \Delta^k \times (B^p \times (R^n, R^m) \to (\Delta^k \times B^p \times (R^n, R^m))_\Gamma$$

equal the identity on $\Lambda \times B^p \times (R^n, R^m)$ and near $\Delta^k \times \partial B^p \times (R^n, R^m)$, and in the case of (1.11) ϕ is the identity on $\Delta^k \times B^p \times R^m$. Using the TOP isotopy extension theorem respecting or rel a submanifold, we get a TOP isomorphism with compact support

$$\phi'\colon \Delta^k \times B^p \times (R^n, R^m) \to (\Delta^k \times B^p \times (R^n, R^m))_\Gamma$$

sliced over Δ^k, respecting or rel $\Delta^k \times B^p \times R^m$, equal to ϕ on and over

$$\Delta^k \times B^p \times (B^n, B^m),$$

and equal to the identity near $\Delta^k \times \partial B^p \times (R^n, R^m)$ and on $\Lambda \times B^p \times (R^n, R^m)$. Then define the required ambient isotopy h_t, $t \in [0, 1]$, of $\Delta^k \times B^p \times (R^n, R^m)$ by

$$h_t(u, x) = (u, \phi'(r_{1-t}(u), x)) \quad \text{for } (u, x) \in \Delta^k \times B^p \times R^n$$

and $r_t\colon \Delta^k \to \Delta^k$ a deformation retraction of Δ^k onto Λ. ∎

2. Classification of relative CAT structures by CAT structures on microbundle pairs

Let (M, N) be a TOP manifold pair with $\partial M = \partial N = \emptyset$. Our goal in this and the next section is to analyze the s.s. complex CAT (M, N) (and if Σ is a preferred CAT structure on N, the s.s. complex CAT $(M, N; \Sigma)$) of relative CAT structures (respectively of relative CAT structures on (M, N) rel Σ). A typical k-simplex of CAT (M, N) (respectively, CAT $(M, N; \Sigma)$) is a relative CAT structure on $\Delta^k \times (M, N)$ sliced over Δ^k (respectively a relative CAT structure on $\Delta^k \times (M, N)$ rel Σ sliced over Δ^k).

PROPOSITION 2.1. *The s.s. complex* PL (M, N) *and, if* $\dim N \neq 4$ *and* $\dim M \geq 5$, *the s.s. complex* DIFF (M, N) *are Kan complexes.*

Proof. For the complex PL (M, N), the Kan condition is verified by using a PL retraction $\Delta^k \to \Lambda_{k,i}$ ($= \partial\Delta^k$ minus the interior of the face opposite the ith vertex) to pull back a relative PL structure on $\Lambda_{k,i} \times (M, N)$ to one on $\Delta^k \times (M, N)$.

For CAT = DIFF we will need Theorem 1.1. Let Γ be a relative CAT structure on (M, N). Define $\mathrm{Aut}_{\mathrm{CAT}}(M, N)_\Gamma$ to be the s.s. group of CAT automorphisms of $(M, N)_\Gamma$ and let $\mathrm{Aut}_{\mathrm{TOP}}(M, N)$ be the s.s. group of TOP automorphism of (M, N). There is a natural s.s. map $\mathrm{Aut}_{\mathrm{TOP}}(M, N) \to$ CAT (M, N) given by $H \to H(\Delta^k \times \Gamma)$, which induces an injective s.s. map

$$F\colon \mathrm{Aut}_{\mathrm{TOP}}(M, N)/\mathrm{Aut}_{\mathrm{CAT}}(M, N)_\Gamma \to \mathrm{CAT}\,(M, N).$$

The domain of F is a Kan complex as $\mathrm{Aut}_{\mathrm{CAT}}$ and $\mathrm{Aut}_{\mathrm{TOP}}$ are group complexes, hence Kan complexes [see [9], Section 17.1, Section 18.2]. But Theorem 1.1 says that if $\dim N \neq 4$ and $\dim M \geq 5$, the image of F is the component of CAT (M, N) containing Γ. The Kan property of CAT (M, N) is then verified by varying Γ. ∎

Using Theorem 1.3 we similarly prove:

PROPOSITION 2.2. *The s.s. complex* PL $(M, N; \Sigma)$ *and, if* $\dim M \geq 5$, *the s.s. complex* DIFF $(M, N; \Sigma)$ *are Kan complexes.*

To study CAT (M, N) and CAT $(M, N; \Sigma)$ we introduce local versions of these complexes. A *relative* CAT *structure on* $\Delta^k \times (M, N)$ *near* $\Delta^k \times N$ (rel Σ) *sliced over* Δ^k is a relative CAT structure on a neighborhood of $\Delta^k \times N$ in $\Delta^k \times M$ (rel Σ) sliced over Δ^k. Two relative CAT structures Γ and Γ' on $\Delta^k \times (M, N)$ near $\Delta^k \times N$ (rel Σ) sliced over Δ^k have the same *germ* if $\Gamma = \Gamma'$ on a neighborhood of $\Delta^k \times N$ in $\Delta^k \times M$. Let CAT (M near N) (respectively CAT (M near N; Σ)) be the s.s. complex of germs of relative CAT structures on $\Delta^k \times (M, N)$ near $\Delta^k \times N$ (respectively rel Σ).

PROPOSITION 2.3. *With the hypothesis of Proposition* 2.1, CAT (M near N) *is a Kan complex, and with the hypothesis of Proposition* 2.2, CAT (M near N; Σ) *is a Kan complex.*

We now relate CAT (M near N) and CAT (M near N; Σ) to s.s. complexes of CAT structures on microbundle pairs. A CAT *structure on a* TOP *microbundle pair*

$$(\xi, \mathcal{N})\colon (E(\xi), E(\mathcal{N})) \underset{i}{\overset{p}{\rightleftarrows}} (X, X)$$

over a CAT manifold X is a CAT manifold structure Γ on an open neighborhood U of $i(X)$ in $E(\xi)$ such that $p\colon (U, U \cap E(\mathcal{N}))_\Gamma \to (X, X)$ is a relative CAT submersion. If, in addition, $i\colon (X, X) \to (U, U \cap E(\mathcal{N}))$ is a CAT map, we call Γ a CAT *microbundle structure* on the microbundle pair $(\xi, \mathcal{N})$. Two CAT (microbundle) structures Γ and Γ' on the microbundle pair $(\xi, \mathcal{N})$ have the same *germ* if $\Gamma = \Gamma'$ on an open neighborhood of $i(X)$ in $E(\xi)$. We then define a k-simplex of **CAT** $(\xi, \mathcal{N})$ (respectively CAT $(\xi, \mathcal{N})$) to be the germ of a CAT structure (respectively microbundle structure) on the microbundle pair

$$\Delta^k \times (\xi, \mathcal{N})\colon \Delta^k \times (E(\xi), E(\mathcal{N})) \underset{\mathrm{id}\times i}{\overset{\mathrm{id}\times p}{\rightleftarrows}} \Delta^k \times (X, X).$$

If the microbundle $\mathcal{N}$ has a preferred CAT structure (respectively microbundle structure) Γ', then a k-simplex of **CAT** $(\xi, \mathcal{N}; \Gamma')$ (respectively CAT $(\xi, \mathcal{N}; \Gamma')$) is a germ of a CAT structure (respectively microbundle structure) on the bundle pair $\Delta^k \times (\xi, \mathcal{N})$ such that $\Gamma \mid E(\Delta^k \times \mathcal{N}) = \Delta^k \times \Gamma'$.

Let (M, N) be a TOP manifold pair, $\partial M = \partial N = \emptyset$. As N is locally flat in M,

$$(\tau(M) \mid N, \tau(N))\colon (M \times N, N \times N) \underset{\Delta}{\overset{p_2}{\rightleftarrows}} (N, N)$$

is a TOP microbundle pair, where p_2 is projection on the second factor and $\Delta\colon N \to M \times N$ is the diagonal map $\Delta(x) = (x, x)$. Suppose that N has a CAT structure Σ. Then there are s.s. maps

$$d\colon \mathrm{CAT}\ (M \text{ near } N) \to \mathbf{CAT}\ (\tau(M) \mid N, \tau(N))$$

and

$$d_\Sigma\colon \mathrm{CAT}\ (M \text{ near } N; \Sigma) \to \mathbf{CAT}\ (\tau(M) \mid N, \tau(N); \Sigma \times \Sigma)$$

defined as follows. For $\Gamma \in \mathrm{CAT}\ (M \text{ near } N)^{(k)}$ let $d\Gamma \in \mathbf{CAT}\ (\tau(M) \mid N, \tau(N))$ be the CAT structure $(\Gamma \times \Sigma, \Gamma \mid N \times \Sigma)$ on

$$\Delta^k \times (M \times N, N \times N) = (E(\Delta^k \times \tau(M) \mid N), E(\Delta^k \times \tau(N))).$$

The s.s. map d_Σ is similarly defined.

If N does not possess a CAT structure we remedy the situation as follows. Embed N in R^q, $q > n$, and let $r\colon Q \to N$ be a retraction of an open neighborhood of N in R^q onto N. Consider the pull-back pair

$$\begin{aligned}&(\hat\tau(M) \mid Q, \hat\tau(Q))\\ &\quad = (r^*(\tau(M) \mid N), r^*(\tau(N)))\colon (M \times Q, N \times Q) \underset{j}{\overset{p_2}{\rightleftarrows}} (Q, Q)\end{aligned}$$

where $j(y) = (r(y), y)$.

Then, as Q has a CAT structure,

$$\mathbf{CAT}\ (\hat\tau(M) \mid Q, \hat\tau(Q)) \quad \text{and} \quad \mathrm{CAT}\ (\hat\tau(M) \mid Q, \hat\tau(Q))$$

are defined and the rule $(\Gamma, \Gamma \mid N) \to (\Gamma \times Q, \Gamma \mid N \times Q)$ determines a s.s. map

$$d\colon \mathrm{CAT}\ (M \text{ near } N) \to \mathbf{CAT}\ (\hat\tau(M) \mid Q, \hat\tau(Q))$$

and passage to germs determines a s.s. map

$$d\colon \mathrm{CAT}\,(M \text{ near } N) \to \mathbf{CAT}\,(\hat{\tau}(M) \mid N, \hat{\tau}(N)) \equiv$$
$$\text{inj lim}\,\{\mathrm{CAT}\,(\hat{\tau}(M) \mid U, \hat{\tau}(U))\colon N \subset U \text{ open in } Q\}.$$

Our goal in this section is to show that in most instances, d and d_Σ are homotopy equivalences.

PROPOSITION 2.4. *The s.s. complexes* $\mathbf{PL}\,(\hat{\tau}(M) \mid N, \hat{\tau}(N))$, $\mathrm{PL}\,(\hat{\tau}(M) \mid N, \hat{\tau}(N))$ *and, if* $\dim N \neq 4$ *and* $\dim M \geq 5$, *the s.s. complexes* $\mathbf{DIFF}\,(\hat{\tau}(M) \mid N, \hat{\tau}(N))$ *and* $\mathrm{DIFF}\,(\hat{\tau}(M) \mid N, \hat{\tau}(N))$ *are Kan complexes.*

Proof. The two PL complexes are Kan by using a PL retraction $r\colon \Delta^k \to \Lambda_{k,i}$ to pull back the PL (microbundle) structures.

To verify the Kan conditions for the DIFF complexes, it suffices to show that the s.s. complex BDIFF $(\xi, \mathcal{N})$ is a Kan complex for a TOP (R^n, R^m)-bundle pair $(\xi, \mathcal{N})$ over a CAT manifold X. A k-simplex of BDIFF $(\xi, \mathcal{N})$ is a relative DIFF manifold structure Γ on $\Delta^k \times ((E(\xi), E(\mathcal{N}))$ such that the projection

$$\Delta^k \times (E(\xi), E(\mathcal{N})) \to (\Delta^k \times X, \Delta^k \times X)$$

is a relative CAT submersion. This suffices, as every TOP-microbundle pair contains a TOP (R^n, R^m)-bundle pair, by the relative coring theorem [7]. To verify the Kan condition for BDIFF $(\xi, \mathcal{N})$, note that Theorem 1.1(i) implies that if $\Gamma \in \mathrm{BDIFF}\,(\xi, \mathcal{N})^{(k)}$ then there is a CAT isomorphism

$$h\colon (\Delta^k \times (E(\xi), E(\mathcal{N}))_\Gamma \to (\Delta^k \times (E(\xi), E(\mathcal{N}))_\gamma$$

sliced over Δ^k for some relative CAT structure γ. An application of Theorem 1.6 yields that for any 0-simplex $\gamma = \Delta^0 \times \gamma$ of BDIFF $(\xi, \mathcal{N})$ we have a Kan fibration

$$\mathrm{Aut}_{\mathrm{DIFF}}\,(\xi, \mathcal{N})_\gamma \to \mathrm{Aut}_{\mathrm{TOP}}\,(\xi, \mathcal{N}) \to \mathrm{BDIFF}\,(\xi, \mathcal{N})_\gamma$$

where $\mathrm{BDIFF}\,(\xi, \mathcal{N})_\gamma$ is the component of BDIFF $(\xi, \mathcal{N})$ containing γ. Varying γ yields the result. ■

By a similar application of Theorems 1.1(ii) and 1.7 we have:

PROPOSITION 2.5. *The s.s. complex*

$$\mathbf{PL}\,(\tau(M) \mid N, \tau(N); d(\Sigma)) = \mathrm{PL}\,(\tau(M) \mid N, \tau(N); d(\Sigma))$$

and, if $\dim M \geq 5$, *the s.s. complex*

$$\mathbf{DIFF}\,(\tau(M) \mid N, \tau(N); d(\Sigma)) = \mathrm{DIFF}\,(\tau(M) \mid N, \tau(N); d(\Sigma))$$

are Kan complexes.

PROPOSITION 2.6. *With the hypothesis of Proposition* 2.4 *the natural inclusion*

$$\mathrm{CAT}\,(\hat{\tau}(M) \mid N, \hat{\tau}(N)) \to \mathbf{CAT}\,(\hat{\tau}(M) \mid N, \hat{\tau}(N))$$

is a homotopy equivalence of Kan complexes.

Proof. It suffices to show that if $(\xi, \mathcal{N})$ is a TOP microbundle pair over a CAT manifold X and assuming CAT $(\xi, \mathcal{N})$ and **CAT** $(\xi, \mathcal{N})$ are Kan, then

$$\pi_k\,(\mathbf{CAT}\,(\xi, \mathcal{N}), \mathrm{CAT}\,(\xi, \mathcal{N})) = 0 \quad \text{for all } k.$$

For then CAT $(\xi, \mathcal{N}) \to$ **CAT** $(\xi, \mathcal{N})$ is a homotopy equivalence, and then by taking injective limits we obtain our result.

A typical element of π_k (**CAT** $(\xi, \mathcal{N})$, CAT $(\xi, \mathcal{N})$) is a CAT structure Γ on the microbundle

$$\Delta^k \times (E(\xi), E(\mathcal{N})) \underset{i}{\overset{p}{\rightleftarrows}} \Delta^k \times (X, X)$$

such that $i\colon \partial\Delta^k \times X \to (\partial\Delta^k \times E(\mathcal{N}))_{\Gamma | \partial\Delta^k \times E(\mathcal{N})}$ is a CAT embedding. In the microbundle

$$I \times \Delta^k \times E(\mathcal{N}) \underset{\mathrm{id} \times i}{\overset{\mathrm{id} \times p}{\rightleftarrows}} I \times \Delta^k \times X$$

approximate id $\times$ i by a section i' such that

$$i' \mid 0 \times \Delta^k \times X\colon 0 \times \Delta^k \times X \to (0 \times \Delta^k \times E(\mathcal{N}))_{\Gamma | \Delta^k \times E(\mathcal{N})}$$

is a CAT embedding and $i' = i$ on $I \times (\partial\Delta^k) \times X \cup 1 \times \Delta^k \times X$. We then have a new TOP microbundle pair

$$(\xi', \mathcal{N}')\colon I \times \Delta^k \times (E(\xi), E(\mathcal{N})) \underset{i'}{\overset{\mathrm{id} \times p}{\rightleftarrows}} I \times \Delta^k \times (X, X).$$

By the relative microbundle homotopy theorem, there exists a TOP morphism

$$H\colon (\xi', \mathcal{N}') \to I \times \Delta^k \times (\xi, \mathcal{N})$$

over the identity map of the base $I \times \Delta^k \times X$ and with H equaling the identity over

$$I \times (\partial\Delta^k) \times X \cup 1 \times \Delta^k \times X.$$

Then $\Gamma' = H(I \times \Gamma)$ is a CAT structure on $I \times \Delta^k \times (\xi, \mathcal{N})$ with

$$\Gamma' \mid (0 \times \Delta^k \times (\xi, \mathcal{N}))$$

a CAT microbundle structure on $\Delta^k \times (\xi, \mathcal{N})$. ■

We are now in a position to state and prove the main theorem of this section.

THEOREM 2.7. *For every* TOP *manifold pair* (M, N) *with* $\partial M = \partial N = \emptyset$, *the following s.s. maps are homotopy equivalences of Kan complexes:*

(i) d: CAT (M near N) $\to$ **CAT** $(\hat{\tau}(M) \mid N, \hat{\tau}(N))$, *if* dim $N \neq 4$ *and* dim $M \geq 5$;

(ii) d_Σ: CAT (M near N; Σ) $\to$ CAT $(\tau(M) \mid N, \tau(N); d(\Sigma))$ *if* dim $M \geq 5$.

Proof. This theorem follows routinely from the immersion theory machinery given the following six, easily verified facts.

Fact 1. The rules

$$U \to \mathrm{CAT}\,(M \text{ near } U) \qquad U \to \mathrm{CAT}\,(M \text{ near } U,\ \Sigma \mid U)$$

$$U \to \mathbf{CAT}\,(\hat{\tau}(M) \mid U,\ \hat{\tau}(U)) \quad \text{and} \quad U \to \mathrm{CAT}\,(\tau(M) \mid U,\ \tau(U);\ d(\Sigma \mid U))$$

are contravariantly functorial on inclusions of open subsets of N. They convert monotone union to projective limits and finite union to fiber products.

Fact 2. The rules $U \to \mathrm{CAT}\,(M \text{ near } U)$ and $U \to \mathrm{CAT}\,(M \text{ near } U;\ \Sigma \mid U)$ are naturally contravariantly functorial on open TOP embeddings between open subsets of N. Two open embeddings that are isotopic through open embeddings, induce homotopic maps.

For any subset $A \subset N$, let

$$\mathrm{CAT}\,(M \text{ near } A) = \text{inj lim}\,\{\mathrm{CAT}\,(M \text{ near } U) \colon A \subset U \text{ open in } N\},$$

$$\mathrm{CAT}\,(M \text{ near } A;\ \Sigma \mid A) = \text{inj lim}\,\{\mathrm{CAT}\,(M \text{ near } U;\ \Sigma \mid U) \colon A \subset U \text{ open in } N\}$$

$$\mathbf{CAT}\,(\hat{\tau}(M) \mid A,\ \hat{\tau}(A)) = \text{inj lim}\,\{\mathbf{CAT}\,(\hat{\tau}(M) \mid U,\ \hat{\tau}(U)) \colon A \subset U \text{ open in } N\},$$

$$\mathrm{CAT}\,(\tau(M) \mid A,\ \hat{\tau}(A);\ d(\Sigma \mid A))$$
$$= \text{inj lim}\,\{\mathrm{CAT}\,(\tau(M) \mid U,\ \tau(U);\ d(\Sigma \mid U)) \colon A \subset U \text{ open in } N\}.$$

We also have s.s. maps

$$d_A \colon \mathrm{CAT}\,(M \text{ near } A) \to \mathbf{CAT}\,(\hat{\tau}(M) \mid A,\ \hat{\tau}(A))$$

and

$$d_{\Sigma \mid A} \colon \mathrm{CAT}\,(M \text{ near } A;\ \Sigma \mid A) \to \mathrm{CAT}\,(\tau(M) \mid A,\ \tau(A);\ d(\Sigma \mid A)).$$

Fact 3. If $A \subset B$ is a homotopy equivalence of compacta in N, then the restriction maps

(i) $\mathbf{CAT}\,(\tau(M) \mid B,\ \hat{\tau}(B)) \to \mathbf{CAT}\,(\hat{\tau}(M) \mid A,\ \hat{\tau}(A))$
(ii) $\mathrm{CAT}\,(\tau(M) \mid B,\ \tau(B);\ d(\Sigma \mid B)) \to \mathrm{CAT}\,(\tau(M) \mid A,\ \tau(A);\ d(\Sigma \mid A))$
are homotopy equivalences.

Fact 4. If A is a point of N, then the following maps are homotopy equivalences:

(i) $d \colon \mathrm{CAT}\,(M \text{ near } A) \to \mathbf{CAT}\,(\hat{\tau}(M) \mid A,\ \hat{\tau}(A))$;
(ii) $d_{\Sigma \mid A} \colon \mathrm{CAT}\,(M \text{ near } A;\ \Sigma \mid A) \to \mathrm{CAT}\,(\tau(M) \mid A,\ \tau(A);\ d(\Sigma \mid A))$.

Corollaries 1.8 and 1.9 imply:

Fact 5. For any compact pair $A \subset B$ in N, the restriction maps

(i) $\mathrm{CAT}\,(M \text{ near } B) \to \mathrm{CAT}\,(M \text{ near } A)$ and
(ii) $\mathrm{CAT}\,(M \text{ near } B;\ \Sigma \mid B) \to \mathrm{CAT}\,(M \text{ near } A;\ \Sigma \mid A)$
are Kan fibrations.

Fact 6. For any compact pair $A \subset B$ in N, the restriction maps

(i) $\mathbf{CAT}(\hat{\tau}(M) \mid B, \hat{\tau}(B)) \to \mathbf{CAT}(\hat{\tau}(M) \mid A, \hat{\tau}(A))$ and
(ii) $\mathrm{CAT}(\tau(M) \mid B, \tau(B); d(\Sigma \mid B)) \to \mathrm{CAT}(\tau(M) \mid A, \tau(A); d(\Sigma \mid A))$
are Kan fibrations.

Proof of Fact 6. For CAT $=$ PL, (i) and (ii) are verified for open neighborhoods, $U \subset V$ of $A \subset B$ in N by considering a PL map

$$r: \Delta^k \times U \to \Lambda_{k,i} \times V \cup \Delta^k \times U$$

respecting projection to Δ^k and fixing $(\Lambda_{k,i} \times V) \cup \Delta^k \times W$ where W is a neighborhood of A in N with $W \subset U$. Now use r to pull back a CAT structure on a microbundle over $\Lambda_{k,i} \times V \cup \Delta^k \times U$ to a CAT structure on the pull back bundle over $\Delta^k \times V$.

For CAT $=$ DIFF the proof of Proposition 2.4 shows that for any (R^n, R^m) bundle pair $(\xi, \mathcal{N})$ over a CAT manifold X $\mathrm{BCAT}(\xi, \mathcal{N}) \to \mathrm{BCAT}(\xi|_U, \mathcal{N}|_U)$ is a Kan fibration for every U open in X. By taking injective limits and recalling that every microbundle pair contains a (R^n, R^m)-bundle pair, for some (n, m), we observe that (i) and (ii) are Kan fibrations. ■

These six facts imply our theorem as follows. For $A \subset N$ let $S(A)$ be the statement that d_A and $d_{\Sigma \mid A}$ are homotopy equivalences.

Step 1. $S(A)$ holds for any simplex A by Facts 2, 3, and 4, as any such A can be isotopically shrunk into small neighborhoods of a point in the interior of A.

Step 2. If $S(A)$, $S(B)$, and $S(A \cap B)$ are true for compact A and B, then $S(A \cup B)$ is true by Facts 1, 5, and 6.

Step 3. $S(A)$ is true for A any finite simplicial complex by Steps 1 and 2 and induction.

Step 4. $S(A)$ is true for any compactum A is a coordinate chart of N, as A is the intersection of finite simplicial complexes.

Step 5. $S(A)$ is true for any compactum $A \subset N$ by Steps 2 and 4 as A is the finite union of compactum in coordinate charts of N.

Step 6. $S(N)$ is true, as N is the union of compactum $N_1 \subset N_2 \subset \cdots$ and Fact 1 implies that d and d_Σ are the projective limit of equivalences $d_{N_1}, d_{N_2}, \ldots$, and $d_{\Sigma \mid N_1}, d_{\Sigma \mid N_2}, \ldots$, respectively. Thus d and d_Σ are equivalences by Facts 5 and 6. ■

3. Classification theorems

Our goal in this section is to show that the Kan complexes CAT (M, N) and CAT $(M, N; \Sigma)$ are homotopy equivalent to Kan complexes of liftings of appropriate unstable classifying spaces.

Let BCAT_n (CAT = TOP, PL, or DIFF) denote the classifying space for CAT n-microbundles and γ^n_{CAT} the universal CAT n-microbundle over BCAT_n. Also, let BCAT^n_{n+q} denote the classifying space for CAT $(n+q, n)$-microbundle pairs and $(\gamma^{n+q}_{\mathrm{CAT}}, \gamma^n_{\mathrm{CAT}})$ the universal CAT $(n+q, n)$-microbundle pair over BCAT^n_{n+q}. There are natural maps

$$r_{n,q}\colon \mathrm{BCAT}^n_{n+q} \to \mathrm{BCAT}_n$$

which restrict $(\gamma^{n+q}_{\mathrm{CAT}}, \gamma^n_{\mathrm{CAT}})$ to γ^n_{CAT}. Also, there are natural maps

$$j_{n,q}\colon \mathrm{BCAT}^n_{n+q} \to \mathrm{BTOP}^n_{n+q} \quad \text{and} \quad j_n\colon \mathrm{BCAT}_n \to \mathrm{BTOP}_n.$$

We extend these maps to maps

$$\begin{array}{ccc} \bar{\mathrm{B}}\mathrm{CAT}^n_{n+q} & \xrightarrow{j_{n,q}} & \bar{\mathrm{B}}\mathrm{TOP}^n_{n+q} \\ {\scriptstyle r_{n,q}}\downarrow & & \downarrow{\scriptstyle r_{n,q}} \\ \bar{\mathrm{B}}\mathrm{CAT}_n & \xrightarrow{j_n} & \mathrm{BTOP}_n \end{array}$$

so that $j_{n,q}$ and $r_{n,q}$ are Hurewicz fibrations and the above diagram commutes.

Let $(\xi, \mathcal{N})$ be a TOP $(n+q, n)$-microbundle pair over a CAT manifold X and choose a fixed classifying morphism $\phi\colon (\xi, \mathcal{N}) \to (\gamma^{n+q}_{\mathrm{TOP}}, \gamma^n_{\mathrm{TOP}})$ such that ϕ covers a map $f\colon X \to \mathrm{BTOP}^n_{n+q}$ and $\phi \mid \mathcal{N}$ covers $r_{n,q}f$. Also, choose a fixed classifying morphism $\Psi\colon (\gamma^{n+q}_{\mathrm{CAT}}, \gamma^n_{\mathrm{CAT}}) \to (\gamma^{n+q}_{\mathrm{TOP}}, \gamma^n_{\mathrm{TOP}})$ covering $j_{n,q}$. A typical k-simplex σ of $L(f \text{ to } \mathrm{BCAT}^n_{n+q})$ is a map $\sigma\colon \Delta^k \times X \to \bar{\mathrm{B}}\mathrm{CAT}^n_{n+q}$ such that

$$j_{n,q}\sigma = fp_2\colon \Delta^k \times X \to \bar{\mathrm{B}}\mathrm{TOP}^n_{n+q}.$$

In addition, suppose that there exists a map $g\colon X \to \bar{\mathrm{B}}\mathrm{CAT}_n$ such that $j_n g = r_{n,q}f$. Then a typical k-simplex of $L(f \text{ to } \mathrm{BCAT}^n_{n,q}; g)$ is a k-simplex

$$\sigma\colon \Delta^k \times X \to \bar{\mathrm{B}}\mathrm{CAT}^n_{n+q}$$

of $L(f \text{ to } \mathrm{BCAT}^n_{n+q})$ with $r_{n,q}\sigma = gp_2\colon \Delta^k \times X \to \bar{\mathrm{B}}\mathrm{CAT}_n$.

THEOREM 3.1. *If* CAT $(\xi, \mathcal{N})$ *is Kan, then there is a canonical homotopy equivalence*

$$\theta\colon \mathrm{CAT}\,(\xi, \mathcal{N}) \to L(f \text{ to } \mathrm{BCAT}^n_{n+q})$$

of Kan complexes.

THEOREM 3.2. *Let* Σ *be a* CAT *microbundle structure on* $\mathcal{N}$. *If* CAT $(\xi, \mathcal{N}; \Sigma)$ *is Kan, then there is a canonical homotopy equivalence*

$$\theta_\Sigma\colon \mathrm{CAT}\,(\xi, \mathcal{N}; \Sigma) \to L(f \text{ to } \mathrm{BCAT}^n_{n+q}; \theta(\Sigma))$$

of Kan complexes.

Proof of Theorem 3.1. We introduce several new Kan complexes. If X and Y are spaces, a typical k-simplex of $\{X, Y\}$ is a map $\Delta^k \times X \to Y$. If X and Y are CAT objects, then $\{X, Y\}_{\mathrm{CAT}}$ is the subcomplex of $\{X, Y\}$ of CAT maps.

If X is a CAT object a typical k-simplex of $\mathrm{MCAT}^n_{n+q}(X)$ is a CAT $(n+q, n)$-microbundle pair $(\xi, \mathcal{N})$ over $\Delta^k \times X$ together with a CAT morphism

$$g\colon (\xi, \mathcal{N}) \to (\gamma^{n+q}_{\mathrm{CAT}}, \gamma^n_{\mathrm{CAT}}).$$

Two such triples $(\xi, \mathcal{N}, g)$ and $(\xi', \mathcal{N}', g')$ represent the same simplex if they coincide on a neighborhood of their respective zero sections. A typical k-simplex of $\mathrm{BCAT}^n_{n+q}(X)$ is a CAT $(n+q, n)$-microbundle pair $(\xi, \mathcal{N})$ over $\Delta^k \times X$, with two such microbundle pairs $(\xi, \mathcal{N})$ and $(\xi, \mathcal{N}')$ representing the same k-simplex, if they agree on a neighborhood of their respective zero sections. For CAT = PL or DIFF a typical k-simplex of $\mathrm{LCAT}^n_{n+q}(X)$ is a k-simplex $(\xi, \mathcal{N}, g)$ of $\mathrm{MCAT}^n_{n+q}(X)$ together with a morphism $h\colon I \times (\xi, \mathcal{N}) \to (\gamma^{n+q}_{\mathrm{TOP}}, \gamma^n_{\mathrm{TOP}})$ of CAT microbundle pairs such that

$$\Psi g = h|\colon 0 \times (\xi, \mathcal{N}) \to (\gamma^{n+q}_{\mathrm{TOP}}, \gamma^n_{\mathrm{TOP}}).$$

One easily verifies that the map $\mathrm{LCAT}^n_{n+q}(X) \to \mathrm{MTOP}^n_{n+q}(X)$ which selects

$$(\xi, \mathcal{N}) = 1 \times (\xi, \mathcal{N}) \xrightarrow{h} (\gamma^{n+q}_{\mathrm{TOP}}, \gamma^n_{\mathrm{TOP}})$$

is a Kan fibration. Also, using the universality property of $(\gamma^{\mathrm{CAT}}_{n+q}, \gamma^{\mathrm{CAT}}_n)$ one can show that there are canonical maps

$$\mathrm{BCAT}^n_{n+q}(X) \xleftarrow{\alpha} \mathrm{MCAT}^n_{n+q}(X) \xrightarrow{\beta} \{X, \mathrm{BCAT}^n_{n+q}\}_{\mathrm{CAT}}$$

which are homotopy equivalences.

Now consider the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{BCAT}^n_{n+q}(X) & \xleftarrow{\alpha} & \mathrm{LCAT}^n_{n+q}(X) & \xrightarrow{\bar\beta} & \{X, \bar{\mathrm{B}}\mathrm{CAT}^n_{n+q}\}_{\mathrm{CAT}} & \xrightarrow{i} & \{X, \bar{\mathrm{B}}\mathrm{CAT}^n_{n+q}\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{BTOP}^n_{n+q}(X) & \xleftarrow{\alpha} & \mathrm{MTOP}^n_{n+q}(X) & \xrightarrow{\beta} & \{X, \mathrm{BTOP}^n_{n+q}\}_{\mathrm{CAT}} & \xrightarrow{i} & \{X, \mathrm{BTOP}^n_{n+q}\}
\end{array}$$

where the vertical maps are Kan fibrations and the horizontal maps are homotopy equivalences. For the point $(\xi, \mathcal{N}, \phi)$ of $\mathrm{MTOP}^n_{n+q}(X)$ we know that $\alpha(\xi, \mathcal{N}, \phi) = (\xi, \mathcal{N})$ and $i\beta(\xi, \mathcal{N}, \phi) = f$. Passing to fibers, we have canonical homotopy equivalences of fibers.

$$\mathrm{CAT}\,(\xi, \mathcal{N}) \xrightarrow{\theta} L(f \text{ to } \mathrm{BCAT}^n_{n+q}). \quad \blacksquare$$

Proof of Theorem 3.2. Define the Kan complexes $\mathrm{BCAT}_n(X)$, $\mathrm{MCAT}_n(X)$, and $\mathrm{LCAT}_n(X)$ as in the proof of Theorem 3.1 except, rather than considering $(n+q, n)$-microbundle pairs, consider n-microbundles. We similarily obtain homotopy equivalences

$$\mathrm{BCAT}_n(X) \xleftarrow{\gamma} \mathrm{MCAT}_n(X) \xrightarrow{\delta} \{X, \mathrm{BCAT}_n\}$$

and a Kan fibration

$$\mathrm{LCAT}_n\,(X) \to \mathrm{MCAT}_n\,(X).$$

Consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{BCAT}^n_{n+q}\,(X) & \xleftarrow{\bar\alpha} & \mathrm{LCAT}^n_{n+q}\,(X) & \xrightarrow{\bar\beta} & \{X, \bar{\mathrm{B}}\mathrm{CAT}^n_{n+q}\} \\
\downarrow & \searrow^{r_1} & \downarrow \nwarrow^{r_2} & & \downarrow \quad \searrow \\
\mathrm{BTOP}^n_{n+q}\,(X) & \xleftarrow{\alpha} & \mathrm{MTOP}^n_{n+q}\,(X) & \xrightarrow{\beta} & \{X, \bar{\mathrm{B}}\mathrm{TOP}^n_{n+q}\} \\
\searrow^{r_1} & & \searrow^{r_2} & & \searrow^{r_3} \\
\mathrm{BCAT}_n\,(X) & \xleftarrow{\bar\gamma} & \mathrm{LCAT}_n\,(X) & \xrightarrow{\bar\delta} & \{X, \bar{\mathrm{B}}\mathrm{CAT}_n\} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle r_3} \\
\mathrm{BTOP}_n\,(X) & \xleftarrow{\gamma} & \mathrm{MTOP}_n\,(X) & \xrightarrow{\delta} & \{X, \mathrm{BTOP}_n\}
\end{array}$$

where the vertical maps are Kan fibrations, the horizontal maps are homotopy equivalences, and the slanted maps are the natural restriction maps.

Let $(\xi, \mathscr{N})$ be a TOP $(n + q, n)$-microbundle pair with Σ a CAT microbundle structure on $\mathscr{N}$. Let

$$\phi\colon (\xi, \mathscr{N}) \to (\gamma^{n+q}_{\mathrm{TOP}}, \gamma^n_{\mathrm{TOP}}) \quad \text{and} \quad \bar\phi\colon \mathscr{N} \to \gamma_{\mathrm{TOP}}$$

be classifying morphisms such that the point $(\xi, \mathscr{N}, \phi)$ of $\mathrm{MTOP}^n_{n+q}\,(X)$ has the property that $\beta(\xi, \mathscr{N}, \phi) = f$, $\alpha(\xi, \mathscr{N}, \phi) = (\xi, \mathscr{N})$, $r_2(\xi, \mathscr{N}, \phi) = (\mathscr{N}, \bar\phi)$, $\delta(\mathscr{N}, \bar\phi) = r_{n+q,n}f$, and $\delta(\mathscr{N}, \bar\phi) = \mathscr{N}$. By passing to fibers we get a homotopy commutative square

$$\begin{array}{ccc}
\mathrm{CAT}\,(\xi, \mathscr{N}) & \xrightarrow{\theta} & L(f \text{ to } \mathrm{BCAT}^n_{n+q}) \\
{\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle r_3} \\
\mathrm{CAT}\,(\mathscr{N}) & \xrightarrow{\bar\theta} & L(r_{n+q,n}f \text{ to } \mathrm{BCAT}_n)
\end{array}$$

where θ and $\bar\theta$ are homotopy equivalences and r_3 is a Kan fibration. Let

$$\Delta^0 \times \Sigma \in \mathrm{CAT}\,(\mathscr{N})^{(0)}$$

be a 0-simplex, where Σ is the preferred CAT microbundle structure on $\mathscr{N}$. We then have the following commutative diagram

$$\begin{array}{ccccccc}
\begin{array}{c}\mathrm{Aut}_{\mathrm{DIFF}}\\(\xi \text{ fixing } \mathscr{N})\end{array} & \longrightarrow & \begin{array}{c}\mathrm{Aut}_{\mathrm{TOP}}\\(\xi \text{ fixing } \mathscr{N})\end{array} & \xrightarrow{p_3} & \mathrm{CAT}\,(\xi, \mathscr{N}; \Sigma) & \xrightarrow{\theta_\Sigma} & \begin{array}{c}L(f \text{ to}\\ \mathrm{BCAT}^n_{n+q}\colon \bar\theta(\Sigma))\end{array} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Aut}_{\mathrm{DIFF}}(\xi, \mathscr{N}) & \hookrightarrow & \mathrm{Aut}_{\mathrm{TOP}}\,(\xi, \mathscr{N}) & \xrightarrow{p_1} & \mathrm{CAT}\,(\xi, \mathscr{N}) & \xrightarrow{\theta} & L(f \text{ to } \mathrm{BCAT}^n_{n+q}) \\
\downarrow{\scriptstyle r_5} & & \downarrow{\scriptstyle r_4} & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_3} \\
\mathrm{Aut}_{\mathrm{DIFF}}\,(\mathscr{N}) & \hookrightarrow & \mathrm{Aut}_{\mathrm{TOP}}\,(\mathscr{N}) & \xrightarrow{p_2} & \mathrm{CAT}\,(\mathscr{N}) & \xrightarrow{\theta} & \begin{array}{c}L(r_{n+q,n}f \text{ to}\\ \mathrm{BCAT}_n)\end{array}
\end{array}$$

where p_1, p_2, and p_3 are Kan fibrations from the proofs of Propositions 2.4 and 2.5 (actually they are a collection of fibrations over components of their respective bases, and over each component the fiber is the respective DIFF automorphisms with respect to an element of that component). Also, the restriction maps r_4 and r_5 are Kan fibrations by the CAT isotopy extension theorem. Thus, as $\bar{\theta}$ and θ are homotopy equivalences, we have a homotopy equivalence

$$\text{CAT}\ (\xi, \mathcal{N}; \Sigma) \xrightarrow{\theta_\Sigma} L(f \text{ to } \text{BCAT}^n_{n+q}; \bar{\theta}(\Sigma))$$

of Kan complexes. ■

Let (M^{n+q}, N^q) be a TOP manifold pair with $\partial M = \partial N = \emptyset$, and fix a classifying morphism

$$\phi: (\tau(M)|_N, \tau(N)) \to (\gamma^{n+q}_{\text{TOP}}, \gamma^n_{\text{TOP}})$$

covering a map $f: N \to \text{BTOP}^n_{n+q}$. Extend these to

$$\hat{\phi}: (\hat{\tau}(M)|_N, \hat{\tau}(N)) \to (\gamma^{n+q}_{\text{TOP}}, \gamma^n_{\text{TOP}})$$

over a map $\hat{f}: Q \to BTOP^n_{n+q}$. If N has a preferred CAT structure Σ, then $\Sigma \times \Sigma$ is a CAT microbundle structure on $\tau(N)$ and let $g: N \to \bar{\text{B}}\text{CAT}_n$ be the lift corresponding to $\Sigma \times \Sigma$ under $\text{CAT}\ (\tau(N)) \xrightarrow{\theta} L(r_{n+q,q}f \text{ to } \text{BCAT}_n)$. Combining the results of this and the last section we have:

CLASSIFICATION THEOREM 3.3. *If* $\dim N \neq 4$ *and* $\dim M \geq 5$, *there is a natural homotopy equivalence*

$$\theta: \text{CAT}\ (M \text{ near } N) \to L(f \text{ to } \text{BCAT}^n_{n+q})$$

well defined up to homotopy.

If N has a preferred CAT *structure* Σ *and* $\dim M \geq 5$, *then there is a natural homotopy equivalence*

$$\theta_\Sigma: \text{CAT}\ (M \text{ near } N; \Sigma) \to L(f \text{ to } \text{BCAT}^n_{n+q}; g)$$

well defined up to homotopy.

Actually, we have shown that for any closed $A \subset N$, there are natural homotopy equivalences

$$\theta_A: \text{CAT}\ (M \text{ near } A) \to L(f \text{ to } \text{BCAT}^n_{n+q} \text{ near } A)$$

and

$$\theta_{\Sigma|A}: \text{CAT}\ (M \text{ near } A; \Sigma \mid A) \to L(f \text{ to } \text{BCAT}^n_{n+q} \text{ near } A; g \mid A).$$

In the case of θ_A one might worry that θ_A depends on the embedding of $N \subset R^q$ and the retraction $r: Q \to N$. We leave it to the reader to verify that the homotopy class of the homotopy equivalence θ_A does not depend on the choice of embedding $N \subset R^q$ or retraction $r: Q \to N$ (see 2.3.2 in Essay V of [4]).

Let C be a closed subset of N and consider the homotopy commutative squares

$$\begin{array}{ccc}
\text{CAT}\,(M \text{ near } N) & \xrightarrow{\theta} & L(f \text{ to } \text{BCAT}^n_{n+q}) \\
\downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} \\
\text{CAT}\,(M \text{ near } C) & \xrightarrow{\theta_C} & L(f \text{ to } \text{BCAT}^n_{n+q} \text{ bear } C)
\end{array}$$

$$\begin{array}{ccc}
\text{CAT}\,(M \text{ near } N;\, \Sigma) & \xrightarrow{\theta_\Sigma} & L(f \text{ to } \text{BCAT}^n_{n+q};\, g) \\
\downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s_2} \\
\text{CAT}\,(M \text{ near } C;\, \Sigma \mid C & \xrightarrow{\theta_{\Sigma\mid C}} & L(f \text{ to } \text{BCAT}^n_{n+q} \text{ near } C;\, g \mid C)
\end{array}$$

where, with appropriate dimension restrictions, the horizontal maps are homotopy equivalences and the vertical maps are Kan fibrations (for r_1 and s_1 use Fact 5 from the proof of Theorem 2.7). Thus, there is a homotopy equivalence of the fibers of r_1 and r_2 and s_1 and s_2 over components. So let Γ_0 be a relative CAT structure (rel Σ) on an open neighborhood U of C in M and let g_0 be the lift corresponding to Γ_0 under $\theta(\theta_{\Sigma \mid U})$. Then we have

THEOREM 3.4. *With the hypothesis of Theorem* 3.3, *there are natural homotopy equivalences*

(i) $\text{CAT}\,(M \text{ near } N \text{ rel } \Gamma_0) \xrightarrow{\theta} L(f \text{ to } \text{BCAT}^n_{n+q} \text{ rel } g_0)$ *and*

(ii) $\text{CAT}\,(M \text{ near } N \text{ rel } \Gamma_0;\, \Sigma) \xrightarrow{\theta_\Sigma} L(f \text{ to } \text{BCAT}^n_{n+q} \text{ rel } g_0;\, g)$

of Kan complexes.

We are now in a position to study CAT (M, N) and CAT $(M, N;\, \Sigma)$. Consider the fiber product square

(3.5)
$$\begin{array}{ccc}
\text{CAT}\,(M, N) & \to & \text{CAT}\,(M) \\
\downarrow & & \downarrow \\
\text{CAT}\,(M \text{ near } N) & \to & \text{CAT}_M\,(N)
\end{array}$$

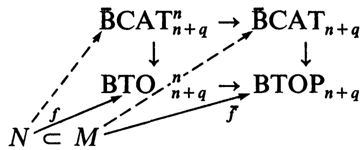
where $\text{CAT}_M\,(N)$ is the Kan complex of germs of CAT structures on a neighborhood of N in M. Let $\bar{f}\colon M \to \text{BTOP}_{n+q}$ classify $\tau(M)$ and $f\colon N \to \text{BTOP}^n_{n+q}$ classify $(\tau(M) \mid N, \tau(N))$. Theorem 3.4 shows that (3.5) is homotopy equivalent to the fiber product square

$$\begin{array}{ccc}
F & \to & L(\bar{f} \text{ to } \text{BCAT}_{n+q}) \\
\downarrow & & \downarrow \\
L(f \text{ to } \text{BCAT}^n_{n+q}) & \to & L(\bar{f} \text{ to } \text{BCAT}_{n+q} \text{ near } N).
\end{array}$$

If $\dim N \neq 4$ and $\dim M \geq 5$, we then have a natural homotopy equivalence

(3.6) $$\text{CAT}\,(M, N) \xrightarrow{\theta} L((\bar{f}, f) \text{ to } (\text{BCAT}_{n+q}, \text{BCAT}^n_{n+q}))$$

where this latter complex is the Kan complex of lifts of $\bar{f}$ and f making the following diagram commute



There is a relative version of (3.6) using arguments similar to (3.4) which says the following. Let D be a closed subset of M such that there is a relative CAT structure Γ_0 on a neighborhood U_0 of D in M. Let

$$(\bar{g}_0, g_0)\colon (U, U \cap N) \to (\bar{B}CAT_{n+q}, \bar{B}CAT^n_{n+q})$$

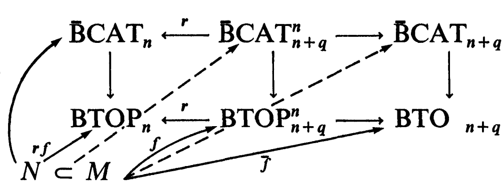
be the lifts associated to this structure. If dim $N \neq 4$ and dim $M \geq 5$, there is a natural homotopy equivalence

(3.7) CAT $(M, N \text{ rel } \Gamma_0) \to L((\bar{f}, f)$ to $(BCAT_{n+q}, BCAT^n_{n+q})$ rel $(\bar{g}_0, g))$.

One can analyze the Kan complex CAT $(M, N; \Sigma)$ in a similar fashion to yield that if dim $M \geq 5$, there is a natural homotopy equivalence

(3.8) CAT $(M, N; \Sigma) \to L((\bar{f}, f)$ to $(BCAT_{n+q}, BCAT^n_{n+q}); g)$

where g is the lift of $r_{n+q,n}f$ to $BCAT_n$ and this latter complex is the complex of lifts of $\bar{f}$ and f making the following diagram commute



Similarly, there is a relative version which states that if D is a closed subset of M, Γ_0 a relative CAT structure rel Σ on a neighborhood U of D in M and $(\bar{g}_0, g_0)\colon (U, U \cap N) \to (\bar{B}CAT_{n+q}, \bar{B}CAT^n_{n+q})$ is the lift associated to this structure with $r_{n+q,n}g_0 = g|_U$, then if dim $M \geq 5$ there is a natural homotopy equivalence

(3.9) CAT $(M, N \text{ rel } \Gamma_0; \Sigma) \to L((\bar{f}, f)$ to $(BCAT_{n+q}, BCAT^n_{n+q})$ rel $(\bar{g}_0, g_0)\colon g)$.

We apply (3.7) by taking $(M, N) = (R^{n+q}, R^n)$ and $C = R^n - \text{int } B^n$, Γ_0 standard, and $\bar{f}$, f, $\bar{g}_0$, and g are constant maps. The left-hand side of (3.7) becomes

$$\text{CAT } (B^n \times R^q, B^n \times 0 \text{ rel } \Gamma_0 \text{ on } \partial B^n \times 0)$$

and the right-hand side is equivalent to $\Omega^n(\mathrm{TOP}^n_{n+q}/\mathrm{CAT}^n_{n+q})$, where $\mathrm{TOP}^n_{n+q}/\mathrm{CAT}^n_{n+q}$ is the fiber of $j\colon \mathrm{BCAT}^n_{n+q} \to \mathrm{BTOP}^n_{n+q}$. By employing the Kan fibration

$$\mathrm{Aut}_{\mathrm{CAT}}(B^n \times R^q, B^n \times 0 \text{ rel } \partial B^n \times 0)$$
$$\to \mathrm{Aut}_{\mathrm{TOP}}(B^n \times R^q, B^n \times 0 \text{ rel } \partial B^n \times 0)$$
$$\to \mathrm{CAT}(B^n \times R^q, B^n \times 0 \text{ rel } \Gamma_0 \text{ on } \partial B^n \times 0)$$

and noting that the total space is contractable by Alexander's device, we have that if $n + q \geq 5$ and $n \neq 4$,

(3.10) $\mathrm{Aut}_{\mathrm{CAT}}(B^n \times R^q, B^n \times 0 \text{ rel } \partial B^n \times 0) \simeq \Omega^{n+1}(\mathrm{TOP}^n_{n+q}/\mathrm{CAT}^n_{n+q})$.

For CAT = PL, the left-hand side is contractible by Alexander's device so that

(3.11) *if* $n + q \geq 5$, $n \neq 4$, $\pi_{n+k}(\mathrm{TOP}^n_{n+q}/\mathrm{PL}^n_{n+q}) = 0$ *for all* $k \geq 1$.

By employing 3.9, a similar argument yields that if $n + q \geq 5$, then

(3.12) $\mathrm{Aut}_{\mathrm{CAT}}(B^n \times R^q \text{ fixed on } B^n \times 0) \simeq \Omega^{n+1}(\mathrm{TOP}_{n+q,n}/\mathrm{CAT}_{n+q,n})$

where $\mathrm{TOP}_{n+q,n}/\mathrm{CAT}_{n+q,n}$ is the fiber of $\mathrm{BCAT}_{n+q,n} \to \mathrm{BTOP}_{n+q,n}$ of classifying spaces for $(n + q)$-microbundles with trivial n-subbundle. To see that this complex arises, note that by the CAT isotopy extension theorems the following is a fibration (up to homotopy):

$$\mathrm{TOP}_{n+q,n}/\mathrm{CAT}_{n+q,n} \subset \mathrm{TOP}^n_{n+q}/\mathrm{CAT}^n_{n+q} \xrightarrow{r} \mathrm{TOP}_n/\mathrm{CAT}_n.$$

For CAT = PL the right-hand side of (3.12) is contractible by Alexander's device, so

(3.13) *if* $n + q \geq 5$, *then* $\pi_{n+k}(\mathrm{TOP}_{n+q,n}/\mathrm{PL}_{n+q,n}) = 0$ *for all* $k \geq 1$.

THEOREM 3.14. *If* $q \leq 2$ *and* $n + q \geq 5$, *then* $\mathrm{TOP}_{n+q,n}/\mathrm{PL}_{n+q,n}$ *is contractible. If* $q \geq 3$ *and* $n + q \geq 5$, *then the natural map*

$$i\colon \mathrm{TOP}_{n+q,n}/\mathrm{PL}_{n+q,n} \to \mathrm{TOP}_{n+q}/\mathrm{PL}_{n+q}$$

is a weak homotopy equivalence.

Proof. By [5], $\pi_k(\mathrm{TOP}_{n+q,n}/\mathrm{PL}_{n+q,n}) = 0$ if $q \leq 2$ and $k \leq n$. Also in [15] it is shown that if $q \geq 3$, then i induces an isomorphism on the kth homotopy groups for $k \leq n$. The result now follows from (3.13) and the fact that $\mathrm{TOP}_{n+q}/\mathrm{PL}_{n+q} = K(Z_2, 3)$. ■

COROLLARY 3.15. *If $q \leq 2$ and $n \geq 5$ then the natural map*

$$j\colon \mathrm{TOP}^n_{n+q}/\mathrm{PL}^n_{n+q} \to \mathrm{TOP}_{n+q}/\mathrm{PL}_{n+q}$$

is a weak homotopy equivalence. If $q \geq 3$ and $n \geq 5$, then $\mathrm{TOP}^n_{n+q}/\mathrm{PL}^n_{n+q}$ is contractible.

Proof. This follows from 3.11, 3.14, the fact that $\mathrm{TOP}_n/\mathrm{PL}_n = K(Z_2, 3)$ for $n \geq 5$ and the fibrations (up to homotopy)

$$\mathrm{TOP}_{n+q,n}/\mathrm{PL}_{n+q,n} \subset \mathrm{TOP}^n_{n+q}/\mathrm{PL}^n_{n+q} \to \mathrm{TOP}_n/\mathrm{PL}_n. \quad \blacksquare$$

By using the techniques of (3.15) and the fact that $\mathrm{TOP}_k/\mathrm{PL}_k$ is contractible for $k \leq 3$ (Essay V of [4]), we have

(3.16) $\mathrm{TOP}^3_5/\mathrm{PL}^3_5$ *is contractible,*

(3.17) $\mathrm{TOP}^3_{3+q}/\mathrm{PL}^3_{3+q} \simeq \mathrm{TOP}_{3+q}/\mathrm{PL}_{3+q}$ *for $q \geq 3$*

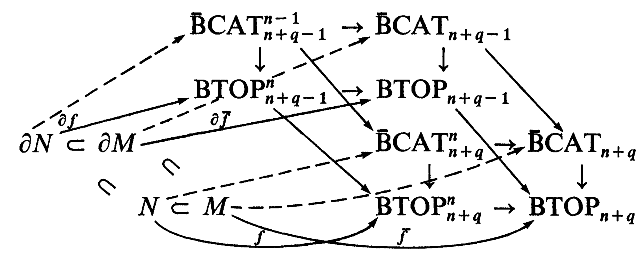
4. Classification theorems for manifold pairs with boundary

Let (M, N) be a TOP manifold pair with $\partial M \neq \emptyset$. Using the techniques of Section 3, we observe that the techniques of Section 4 in Essay V of [4] routinely generalize to yield that if $\dim N \neq 4 \neq \dim \partial N$ and $\dim M \geq 5$, then there is a natural homotopy equivalence

(4.1) $\mathrm{CAT}\,(M, N) \xrightarrow{\theta} L([(\bar{f}, \partial\bar{f}), (f, \partial f)]$

to $[(\mathrm{BCAT}_{n+q}, \mathrm{BCAT}^n_{n+q}), (\mathrm{BCAT}_{n+q-1}, \mathrm{BCAT}^{n-1}_{n+q-1})]$

where $\bar{f}\colon M \to \mathrm{BTOP}_{n+q}$ classifies $\tau(M)$, $\partial\bar{f}\colon \partial M \to \mathrm{BTOP}_{n+q-1}$ classifies $\tau(\partial M)$, $f\colon N \to \mathrm{BTOP}^n_{n+q}$ classifies $(\tau(M)|_N, \tau(N))$, and $\partial f\colon \partial N \to \mathrm{BTOP}^{n-1}_{n+q-1}$ classifies $(\tau(\partial M)|_{\partial N}, \tau(\partial N))$. The right-hand side of (4.1) is the complex of lifts of $\bar{f}$, $\partial\bar{f}$, f, and ∂f such that the following diagram commutes:

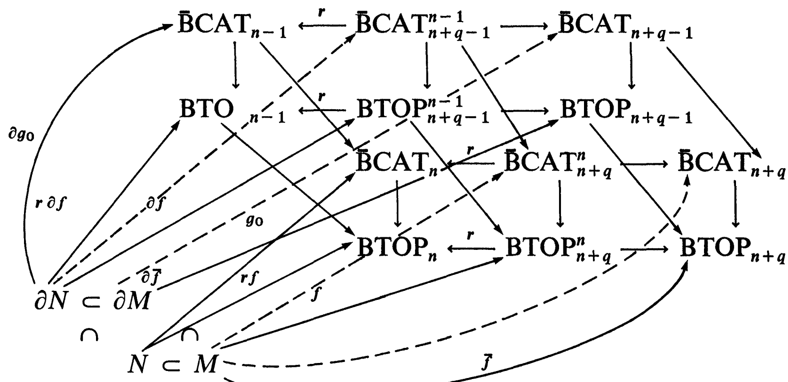


Also, if N has a preferred structure Σ and $(g_0, \partial g_0)\colon (N, \partial N) \to (\mathrm{BCAT}_n, \mathrm{BCAT}_{n-1})$ classifies this structure, then if $\dim \partial M \geq 5$, there is a natural homotopy equivalence

(4.2) $\mathrm{CAT}\,(M, N; \Sigma) \to L([(\bar{f}, \partial\bar{f}), (f, \partial f)]$

to $[(\mathrm{BCAT}_{n+q}, \mathrm{BCAT}^n_{n+q}), (\mathrm{BCAT}_{n+q-1}, \mathrm{BCAT}^{n-1}_{n+q-1})]; (g_0, \partial g_0))$

where this latter complex is the complex of lifts of $\bar{f}$, $\partial\bar{f}$, f, and ∂f such that the following diagram commutes:



(4.1) and (4.2) also have relative versions which the reader can formulate. Using the techniques of (3.10) and (3.12) these relative versions show that

(4.3) *if* $n \neq 4$ *and* $n + q \geq 5$, *then*

$$\mathrm{Aut}_{\mathrm{CAT}}(I \times (B^n \times R^q, B^n \times 0) \text{ rel } X)$$
$$\simeq \Omega^{n+1}(\mathrm{TOP}^n_{n+q}/\mathrm{CAT}^n_{n+q}, \mathrm{TOP}^{n-1}_{n+q-1}/\mathrm{CAT}^{n-1}_{n+q-1})$$

where $X = I \times \partial B^n \times 0 \cup 1 \times B^n \times R^q$,

and

(4.4) *if* $n + q \geq 5$, *then*

$$\mathrm{Aut}_{\mathrm{CAT}}(I \times B^n \times B^q \text{ rel } Y)$$
$$\simeq \Omega^{n+1}(\mathrm{TOP}_{n+q,n}/\mathrm{CAT}_{n+q,n}, \mathrm{TOP}_{n+q-1,n-1}/\mathrm{CAT}_{n+q-1,n-1})$$

where $Y = I \times B^n \times 0 \cup 1 \times B^n \times R^q$.

5. Sliced approximation, equivalence, smoothings, and triangulations

If $F\colon A \times N \to A \times Q$ is an embedding such that F commutes with projection to A, then we say F is an *embedding sliced over* A.

An embedding $F\colon A \times N^n \to A \times Q^{n+q}$ is a CAT *A-locally flat embedding* if F is a CAT embedding sliced over A and for each $(a, x) \in A \times N$, there exists a neighborhood U of a in A, a neighborhood V of x in N, and a CAT embedding

$$H\colon U \times V \times R^q \to U \times Q$$

sliced over U and with $H \mid U \times V \times \mathbf{0} = F \mid U \times V$.

Consider the standard k-simplex Δ^k. Let $0 \in \partial\Delta^k$ and let Λ be a subcomplex of $\partial\Delta^k$ of the following type. If CAT = PL, we allow $\Lambda = \partial\Delta^k$, the empty set, or a retract of $\partial\Delta^k - 0$. If CAT = DIFF, we only allow $\Lambda = \partial\Delta^k$.

THEOREM 5.1. (SLICED APPROXIMATION OF LOCALLY FLAT EMBEDDINGS). *Let N^n and Q^{n+q} be* CAT *manifolds with N compact, $n + q \geq 6$ (≥ 5 if $\partial M = \emptyset$) and let $\Lambda \subset \partial\Delta^k$ be as described above. Suppose $F\colon \Delta^k \times (N, \partial N) \to \Delta^k \times (Q, \partial Q)$ is a* TOP *Δ^k-locally flat embedding such that*

(1) $F^{-1}(\Delta^k \times \partial Q) = \Delta^k \times \partial N$ *and*
(2) $F \mid \Lambda \times (N, \partial N)$ *is a* CAT *Δ^k-locally flat embedding.*

Then, given $\varepsilon\colon Q \to [0, \infty)$ with $\varepsilon \mid p_2F(\Delta^k \times N) > 0$, there exists an ambient isotopy $G_t\colon \Delta^k \times (Q, \partial Q) \to \Delta^k \times (Q, \partial Q)$, $t \in I$, sliced over Δ^k such that

(3) $G_1F\colon \Delta^k \times (N, \partial N) \to \Delta^k \times (Q, \partial Q)$ *is a* CAT *Δ^k-locally flat embedding,*
(4) $G_t \mid \Lambda \times (Q, \partial Q)$ *is the identity, and*
(5) G_t *is within ε of* $(\mathrm{id} \mid \Delta^k) \times \mathrm{id} \mid Q$

if and only if a sequence of obstructions in

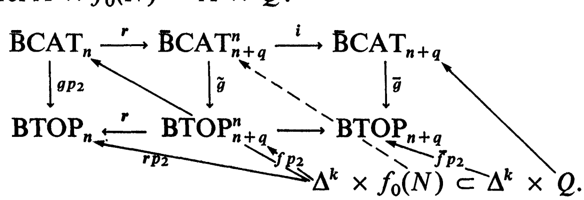
(5) $H^i(\Delta^k \times (N, \partial N)/\Lambda \times (N, \partial N);\ \pi_i\ (\mathrm{TOP}_{n+q}/\mathrm{CAT}_{n+q};\ \mathrm{TOP}_{n+q-1}/\mathrm{CAT}_{n+q-1}, \mathrm{TOP}_{n+q-1,n-1}/\mathrm{CAT}_{n+q-1,n-1};\ \mathrm{TOP}_{n+q,n}/\mathrm{CAT}_{n+q,n}))$ *if* $\partial M \neq \emptyset$; *or*
(6) $H^i(\Delta^k \times N/\Lambda^k \times N;\ \pi_i\ (\mathrm{TOP}_{n+q}/\mathrm{CAT}_{n+q}, \mathrm{TOP}_{n+q,n}/\mathrm{CAT}_{n+q,n}))$ *if* $\partial M = \emptyset$

vanish.

Proof. We assume for simplicity that $\partial M = \emptyset$, as the bounded case follows from a similar argument. By the TOP isotopy extension theorem, there exists a TOP ambient isotopy $H_t\colon \Delta^k \times Q \to \Delta^k \times Q$, $t \in I$, with $H_1\ (\mathrm{id} \mid \Delta^k \times f_0) = F$, where $f_0 = F \mid 0 \times N$. Now $(Q, f_0(N))$ is a TOP manifold pair, with $f_0(N)$ having a preferred CAT manifold structure $f_0(\Sigma)$, where Σ is the given CAT manifold structure on N. Let Γ be the given CAT manifold structure on Q and consider the CAT structure $\Gamma' = H_1^{-1}(\Delta^k \times \Gamma)$ on $\Delta^k \times Q$ sliced over Δ^k. By the classification Theorem 3.8,

$$\mathrm{CAT}\ (Q, f_0(N); f_0(\Sigma)) \simeq L((\bar{f}, f) \text{ to } (\mathrm{BCAT}_{n+q}, \mathrm{BCAT}^n_{n+q}); g)$$

where $\bar{f}\colon M \to \mathrm{BTOP}_{n+q}$ classifies $\tau(M)$, $f\colon f_0(N) \to \mathrm{BTOP}^n_{n+q}$ classifies $(\tau(M) \mid f_0(N), \tau(f_0(N)))$, and $g\colon f_0(N) \to \mathrm{BCAT}_n$ classifies $f_0(\Sigma)$. However, Γ' gives a preferred k-simplex $\bar{g}$ of lifts of $\bar{f}$ to BCAT_{n+q}, and by our hypotheses, $\Gamma' \mid \Lambda \times Q$ is a relative CAT structure on $\Lambda \times (Q, f_0(N))$ rel $f_0(\Sigma)$ sliced over Λ. Thus the obstruction to sliced concording rel $\Lambda \times Q$ the CAT structure Γ' on $\Delta^k \times Q$ to a relative CAT structure Γ'' on $\Delta^k \times (Q, f_0(N))$ rel $f_0(\Sigma)$ sliced over Δ^k is the obstruction to the existence of a map $\tilde{g}\colon \Delta^k \times f_0(N) \to \mathrm{\bar{B}CAT}^n_{n+q}$ with $i\tilde{g} \mid \Lambda \times f_0(N) = \bar{g} \mid \Lambda \times f_0(N)$, making the following diagram commute up to homotopy rel $\Lambda \times f_0(N) \subset \Lambda \times Q$:



These obstructions lie in the cohomology groups given in (5.1). If these obstructions vanish, then Γ' is sliced concordant rel $\Lambda \times Q$ to a relative CAT structure Γ'' on $\Delta^k \times (Q, f_0(N))$ rel $f_0(\Sigma)$ sliced over Δ^k. But by Theorem 1.6, applied to the case where $(M, N) = (Q, \emptyset)$ and

$$(\Delta^k, \Lambda) = (\Delta^k \times I, \Lambda \times I \cup \Delta^k \times 1),$$

there exists an ε-isotopy $R_t\colon \Delta^k \times Q \to \Delta^k \times Q$, $t \in I$, with R_t the identity for $t \in \partial\Delta^k \times I \cup \Delta^k \times 0$, and with $R_1\colon (\Delta^k \times Q)_{\Gamma''} \to (\Delta^k \times Q)_{\Gamma'}$ a CAT isomorphism. The required G_t is given by $G_t = H_t R_t H_t^{-1}$. ∎

Remark. Let $C \subset N$ be a closed subset of N and suppose F is CAT near $\Delta^k \times C$. Our proof of (5.1) shows that the resulting G_t has the property that G_t is the identity near $\Delta^k \times C$ for all $t \in I$.

COROLLARY 5.2. *With the hypothesis of* (5.1), *if* CAT = PL *and* $q \geq 3$, *the resulting* G_t *exists. If* $q \leq 2$, *then* G_t *exists if and only if a well defined obstruction in* $H^3(\Delta^k \times (N, \partial N)/\Lambda \times (N, \partial N); Z_2)$ *vanishes.*

Proof. This follows from (5.1) and (3.14). ∎

COROLLARY 5.3. *With the hypothesis of* (5.1), *if* CAT = DIFF, $q \geq 3$ *and* $k \leq 2q - n - 3$, *the resulting* H *exists.*

Proof. This follows from (5.1), (3.14), and the result of K. Millett that

$$\pi_i\,(\mathrm{PL}_{n+q}/\mathrm{PL}_{n+q,n}, 0_{n+q}/0_n) = 0 \quad \text{for } i < 2q - n - 3 \text{ (cf. [13]).}$$ ∎

THEOREM 5.4. *Let* N^n *and* Q^{n+q} *be* PL *manifolds with* $n + q \geq 6$ (≥ 5 *if* $\partial N = \emptyset$) *and* $q \geq 3$. *Let* k *be a nonnegative integer. Then given* $\varepsilon\colon Q \to [0, \infty)$ *there exists a* $\delta\colon Q \to [0, \infty)$ *such that if* $F = \{f_s \mid s \in \Delta^k\}\colon \Delta^k \times (N, \partial N) \to \Delta^k \times (Q, \partial Q)$ *is a* PL *embedding sliced over* Δ^k *with*

(1) $F^{-1}(\Delta^k \times \partial Q) = \Delta^k \times \partial N$,
(2) $F \mid \partial\Delta^k \times N = (\mathrm{id} \mid \partial\Delta^k) \times f_0$,
(3) $\varepsilon \mid F(\Delta^k \times N) > 0$, *and*
(4) f_s *is within* δ *of* f_0 *for all* $s \in \Delta^k$

then there is a PL *ambient isotopy* G_t, $t \in I$, *of* $\Delta^k \times Q$ *sliced over* Δ^k *with*

(5) $G_1 F = (\mathrm{id} \mid \Delta^k) \times f_0$,
(6) $G_t \mid \partial\Delta^k \times Q$ *is the identity for all* $t \in I$,
(7) $G_t^{-1}(\Delta^k \times \partial Q) = \Delta^k \times \partial N$, *and*
(8) G_t *is within* ε *of the identity.*

Proof. We first show that there exists a TOP ambient isotopy $\bar{G}_t$, $t \in I$, of $\Delta^k \times Q$ sliced over Δ^k satisfying (5)–(8).[2] First assume that F is the product

[2] The author is grateful to the referee for suggesting a simpler proof of the topological version of (5.4).

embedding over a product neighborhood $\partial\Delta^k \times [0, \varepsilon]$ of $\partial\Delta^k$ in Δ^k, i.e., we write $\Delta = \Delta^k$ as $\Delta = \Delta' \cup \partial\Delta' \times [0, \varepsilon]$. By the isotopy extension theorem there is a homeomorphism $H: \Delta' \times Q \to \Delta' \times Q$ sliced over Δ' such that

$$H(\mathrm{id} \mid \Delta' \times f_0) = F \mid \Delta' \times (N, \partial N)$$

with H close to the identity if F is close to id $\mid \Delta' \times f_0$. Then $H \mid \partial\Delta' \times Q$ is close to the identity and $H \mid \partial\Delta' \times f_0N = \mathrm{id}$. Hence $H \mid \partial\Delta' \times Q$ is isotopic to the identity relative to $\partial\Delta' \times f_0N$ by a small isotopy. In other words, we can extend H to $\Delta \times Q$ so that $H(\mathrm{id} \times f_0) = F$, $H \mid \partial\Delta \times Q = \mathrm{id}$ and H is close to the identity. Hence, there is a sliced isotopy H_t of $H = H_0$ to the identity relative to $\partial\Delta^k \times Q$. Then $\bar{G}_t = H_t \circ H_0^{-1}$ satisfies $\bar{G}_0 = $ identity and $\bar{G}_1F = H_1H_0^{-1}F = H_0^{-1}F = \mathrm{id} \times f_0$. Furthermore, $\bar{G}_t$ is close to the identity and $\bar{G}_t \mid \partial\Delta^k \times Q = $ identity.

Now, since F is a TOP Δ^k-locally flat embedding, it may be deformed by a small sliced isotopy K_t fixed over $\partial\Delta^k$ so that $K_1 \circ F$ is of the assumed form. Taking K_{2t} for $0 \le t \le 1/2$ and $\bar{G}_{2t-1}K_1$ for $1/2 \le t \le 1$ gives a new $\bar{G}_t$ satisfying (5)–(8).

We now modify $\bar{G}_t$ so that $\bar{G}_1$ is a PL automorphism of $\Delta^k \times Q$. Let Γ be the given PL structure on Q and Σ the given PL structure on $f_0(N)$. The structures $\Delta^k \times \Gamma$ and $\bar{G}_1^{-1}$ are sliced concordant rel $\partial\Delta^k \times Q$, hence by the proof of (5.1) they are sliced concordant rel $\partial\Delta^k \times Q$ as relative PL structures on $\Delta^k \times (Q, f_0(N))$ rel Σ sliced over Δ^k. Then, by (1.7), there is a small ambient isotopy

$$H_t: \Delta^k \times (Q, f_0(N)) \to \Delta^k \times (Q, f_0(N)), \quad t \in I,$$

sliced over Δ^k with $H_1: (\Delta^k \times (Q, f_0(N))_{\bar{G}_1^{-1}(\Delta^k \times \Gamma)} \to \Delta^k \times (Q, f_0(N))_\Gamma$ a PL isomorphism. We then let $\bar{\bar{G}}_t: \Delta^k \times Q \to \Delta^k \times Q$ be given by

$$\bar{\bar{G}}_t(s, q) = \begin{cases} \bar{G}_{2t}(s, q) & \text{for } 0 \le t \le 1/2 \\ H_{2t-1}(s, q) & \text{for } 1/2 \le t \le 1. \end{cases}$$

Then $\bar{\bar{G}}_t$ is a small TOP ambient isotopy of $\Delta^k \times Q$ sliced over Δ^k, satisfying (5)–(8), and with $\bar{\bar{G}}_1$ a PL automorphism.

Let $\bar{\bar{G}}: \Delta^k \times I \times Q \to \Delta^k \times I \times Q$ be given by $\bar{\bar{G}}(s, t, q) = \bar{\bar{G}}_t(s, q)$. By (5.2), we can assume that $\bar{\bar{G}} \mid \Delta^k \times I \times N$ is a PL embedding of $\Delta^k \times I \times N$ into $\Delta^k \times I \times Q$ sliced over $\Delta^k \times I$.

By the construction of the third paragraph of this proof applied to $\bar{\bar{G}} \mid \Delta^k \times I \times N$, there exists a small ambient isotopy J_t of $\Delta^k \times I \times Q$ sliced over $\Delta^k \times I$ such that J_1 is a PL automorphism of $\Delta^k \times I \times Q$, $J_1 \mid \Delta^k \times 1 \times Q$ is the identity, and

$$J_1\bar{\bar{G}} \mid \Delta^k \times I \times N = (\mathrm{id} \mid \Delta^k \times I) \times f_0.$$

Let $G_s: \Delta^k \times Q \to \Delta^k \times Q$ be the small PL ambient isotopy of $\Delta^k \times Q$ sliced over Δ^k given by $G_s(r, q) = J_1(r, 1 - s, q)$. Then $G_0(r, q) = J_1(r, 1, q)$ is the identity and

$$G_1F(r, q) = G_1(r, f_r(q)) = J_1(r, 0, f_r(q)) = J_1\bar{\bar{G}}(r, 0, f_r(q)) = (r, 0, f_0(q)),$$

so that $G_1F = (\mathrm{id} \mid \Delta^k) \times f_0$. Thus G_t is the required PL ambient isotopy of $\Delta^k \times Q$. ∎

COROLLARY 5.5. (SLICED EQUIVALENCE). *Let N and Q be as in* (5.4). *Suppose $q \geq 3$ and $h\colon N \to Q$ is a* TOP *embedding* (*not necessarily locally flat*) *with $h^{-1}(\partial Q) = \partial N$, Let k be a nonnegative integer. Then given $\varepsilon\colon Q \to [0, \infty)$, with $\varepsilon \mid h(N) > 0$, there exists $\delta\colon Q \to [0, \infty)$ such that if $g\colon N \to Q$ is a* PL *embedding within δ of h with $g^{-1}(\partial Q) = \partial N$ and if $G\colon \Delta^k \times N \to \Delta^k \times Q$ is a* PL *embedding sliced over Δ^k such that*

(1) $G^{-1}(\Delta^k \times \partial Q) = \Delta^k \times \partial N$,
(2) *G is within δ of* $(\mathrm{id} \mid \Delta^k) \times h$ *and*
(3) $G \mid \partial\Delta^k \times N = (\mathrm{id} \mid \Delta^k) \times g$

then there is a PL *ambient isotopy G_t, $t \in I$, of $\Delta^k \times Q$ sliced over Δ^k with*

(4) $G_1G = (\mathrm{id} \mid \Delta^k) \times g$,
(5) *$G_t \mid \partial\Delta^k \times Q$ is the identity, and*
(6) *G_t is within ε of the identity for all $t \in I$.*

Furthermore, the above δ works for all TOP *embeddings sufficiently near h.*

Proof. This follows from (5.4), the fibered general position theorem of Millett [12], and sliced engulfing (cf. [10, p. 248]). ∎

COROLLARY 5.6. *Let* $\mathrm{Emb}^{\mathrm{CAT}}(N^n, Q^{n+q})$ *be the s.s. complex of proper* CAT *embeddings of N^n in Q^{n+q}. If $n + q \geq 6$ (≥ 5 if $\partial N = \emptyset$) and $q \geq 3$ then* $\mathrm{Emb}^{\mathrm{PL}}(N, Q)$ *is locally p-connected at points of* $\mathrm{Emb}^{\mathrm{TOP}}(N, Q)$ *for all p.*

COROLLARY 5.7. (SLICED APPROXIMATION). *Let N^n be a* PL *submanifold of the* PL *manifold Q^{n+q}, with $q \geq 3$ and $n + q \geq 6$ (≥ 5 if $\partial N = \emptyset$). Suppose A is a simplicial complex and B is a subcomplex of A. Let $h\colon A \times N \to A \times Q$ be a proper* TOP *embedding sliced over A with $h \mid B \times N$ a* PL *embedding. Then for any $\varepsilon\colon Q \to [0, \infty)$ there is a proper* PL *embedding $g\colon A \times N \to A \times Q$ sliced over A such that*

(1) *$g \mid B \times N = h \mid B \times N$ and*
(2) *g is within ε of* $(\mathrm{id} \mid \Delta^k) \times h$.

Proof. This is a homotopy result which follows from (5.6) and the fact that a component of $\mathrm{Emb}^{\mathrm{PL}}(N, Q)$ is dense in a component of $\mathrm{Emb}^{\mathrm{TOP}}(N, Q)$ (cf. [10, p. 245]). ∎

Remark 5.8. By local application of (5.5) and (5.7), (5.5) and (5.7) have topological analogues. Thus, in (5.5) and (5.7) we can replace N and Q by TOP manifolds and replace PL embeddings by TOP locally flat embeddings. A stronger topological analogue of (5.4) is proven in [3].

Let $\mathrm{Emb}^{\mathrm{LF}}(N, Q)$ denote the s.s. complex of TOP locally flat embeddings of N into Q. Then (5.6), (5.7), and (5.8) immediately imply:

COROLLARY 5.9. *Let N and Q be as in* (5.7). *Then the natural inclusions*

$$\mathrm{Emb}^{\mathrm{PL}}(N, Q) \to \mathrm{Emb}^{\mathrm{LF}}(N, Q) \to \mathrm{Emb}^{\mathrm{TOP}}(N, Q)$$

are homotopy equivalences, with all homotopies as small as desired.

THEOREM 5.10. (SLICED SMOOTHINGS AND TRIANGULATIONS). *Let Q^{n+q} be a* CAT *manifold with $n + q \geq 6$ (≥ 5 if $\partial Q = \emptyset$), and let N^n be a* TOP *manifold with $n \neq 4, 5$ ($\neq 4$ if $\partial N = \emptyset$). Let $\Lambda \subset \partial\Delta^k$ be as in the beginning of this section. Suppose F: $\Delta^k \times (N, \partial N) \to \Delta^k \times (Q, \partial Q)$ is a* TOP *locally flat embedding sliced over Δ^k with $F^{-1}(\Delta^k \times \partial Q) = \Delta^k \times \partial N$ such that $F(\Lambda \times N)$ is a proper* CAT *submanifold of $\Lambda \times Q$. Then given ε: $Q \to [0, \infty)$ with $\varepsilon \mid p_2F(\Delta^k \times N) > 0$, there exists an ambient isotopy H_t, $t \in I$, of $\Delta^k \times Q$ sliced over Δ^k such that*

(1) *$H_1F(\Delta^k \times N)$ is a proper* CAT *submanifold of $\Delta^k \times Q$,*
(2) *$H_t \mid (\Lambda \times N)$ is the identity for all $t \in I$, and*
(3) *H_t is within ε of* $(\mathrm{id}|_{\Delta^k} \times \mathrm{id}|_Q$

if and only if a sequence of obstructions in

(4) $H^i(\Delta^k \times (N, \partial N)/\Lambda \times (N, \partial N);\ \pi\,(\mathrm{TOP}_{n+q}/\mathrm{CAT}_{n+q};\ \mathrm{TOP}_{n+q-1}/\mathrm{CAT}_{n+q-1}, \mathrm{TOP}^{n-1}_{n+q-1}/\mathrm{CAT}^{n-1}_{n+q-1};\ \mathrm{TOP}^n_{n+q}/\mathrm{CAT}^n_{n+q}))$ *if $\partial N \neq \emptyset$; or*
(5) $H^i(\Delta^k \times \mathrm{N}/\Lambda \times N;\ \pi\,(\mathrm{TOP}_{n+q}/\mathrm{CAT}_{n+q}, \mathrm{TOP}^n_{n+q}/\mathrm{CAT}^n_{n+q}))$ *if $\partial N = \emptyset$ vanishes.*

Proof. This follows in exactly the same manner as (5.1), using (3.6) in place of (3.8). ■

COROLLARY 5.11. *With the hypothesis of* (5.10), *if* CAT = PL, $q \leq 2$, *and $n \geq 5$, the resulting H exists. If $q \geq 3$ and $n \geq 5$, the resulting H exists if and only if a well-defined obstruction in $H^3(\Delta^k \times (N, \partial N)/\Lambda \times (N, \partial N); Z_2)$ vanishes.*

Proof. This follows from (5.10) and (3.15). ■

Remark 5.13. There are obvious relative versions of (5.1)–(5.11).

Remark 5.14. Note that by (5.10), (3.16), and (3.17), if N is a 3-manifold without boundary and Q^{n+q} is a PL manifold without boundary, $q \geq 3$, then the resulting H of (5.10) always exist. If $q \leq 2$, there is an obstruction in $H\ (\Delta^k \times N \mid \Lambda \times N; Z_2)$ which vanishes if and only if H exists.

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