# A DEFINABLE NONSEPARABLE INVARIANT EXTENSION OF LEBESGUE MEASURE

BY

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## Abstract

Although the existence of Lebesgue nonmeasurable sets is provable in ZFC, Solovay has proved that no definition made within set theory can be proved in ZFC to define a Lebesgue nonmeasurable set. In contrast to Solovay's result, we construct a definable countably additive translation invariant extension of Lebesgue measure which has character $2^c$.

Solovay [4] has constructed a model of ZFC in which there are no definable Lebesgue nonmeasurable sets. Kakutani and Oxtoby [2] have constructed a countably additive translation invariant extension of Lebesgue measure on the circle group which has character $2^c$.[2] Further investigations of invariant extensions of Lebesgue measure (and also of Haar measures) have appeared (see Hewitt and Ross [1], and its bibliography). None of these constructions of invariant extensions of Lebesgue measure are given by an explicit definition. Typically, one uses a well ordering of the reals, not only to prove that the extension is a proper extension, but also to describe the $\sigma$-algebra. The results of Solovay show that the first is necessary, and suggest that the second might also be unavoidable. The example presented here shows that this is not the case.

Specifically, we give an explicit definition which, provably in ZFC, defines a countably additive translation invariant extension of Lebesgue measure on the circle group (the real numbers [0, 1) mod 1) of character $2^c$. In fact, the proof that it defines a countably additive translation invariant extension of Lebesgue measure is given within ZF plus the countable axiom of choice. (The same result is shown also for the reals).

The construction makes use of a key measure theoretic lemma, which we prove in Section 1 through forcing and conservative extension results. The lemma likely can be proved from standard techniques; nevertheless, the proof is very easy if these sophisticated techniques are used.[3]

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[2] The separability character of a measure space is the least cardinality of a subset $\mathscr{A}$ of measurable sets such that to every measurable set with $\mu(E) < \infty$, and $\varepsilon > 0$, there is an $A \in \mathscr{A}$ with $(E \Delta A) < \varepsilon$. If the character is less than or equal to $\omega$ then the space is called separable.

[3] The referee has communicated two different proofs to us. The first proof uses hyperarithmetic theory, and the second uses descriptive set theory. Unfortunately neither can be regarded as "logic-free." We wish to thank the referee for these and other important remarks.

## 1. The key measure theoretic lemma

In [3], it is shown how to derive "the union of $\omega_1$ sets of Lebesgue measure 0 is of Lebesgue measure 0" from ZFC + $2^\omega > \omega_1$ + a statement which has since become known as Martin's axiom (MA). In [5], it is shown that every countable model of ZFC can be generically extended to a model of ZFC + $2^\omega > \omega_1$ + MA. A consequence of this is that every statement of a certain form (called $\Pi^1_3$) which is provable from ZFC + $2^\omega > \omega_1$ + MA is provable in ZFC.

Actually, we need only the following.

LEMMA 1. *Every $\Pi^1_3$ sentence provable in* ZFC + "*the union of $\omega_1$ sets of Lebesgue measure* 0 *is of Lebesgue measure* 0" *is provable in* ZFC.

Since the measure theoretic lemma we are interested in is easily seen to be provably equivalent, in ZFC, to a $\Pi^1_3$ sentence, it suffices to give a proof of it using "the union of $\omega_1$ sets of measure 0 is of measure 0."

Let $G$ be an uncountable Borel subset of the circle group $\Omega$, and let $A \subset \Omega$ have positive measure. The measure theoretic lemma asserts that some translate $G + x$ has continuumly many elements in common with $A$. Using the fact that sets of positive measure contain closed subsets of positive measure, and that uncountable Borel sets have perfect subsets, we see that the measure theoretic lemma is provably equivalent (in ZFC) to the statement that every closed set of positive measure meets some translate of $G$ in a set that includes a perfect subset. This latter statement can easily be put into $\Pi^1_3$ form.

To prove the lemma, we can assume, without loss of generality that $A$ is Borel. Let $Q$ be the rationals in $\Omega$. Then $A^* = A + Q$ must have measure 1, by the 0, 1 density law.

It suffices to show that for some translate $G + x$,

$$|(G + x) \cap A^*| = c.$$

For if $|(G + x) \cap A^*| = c$, then $|(G + x) \cap (A + y)| = c$ for some $y \in Q$, and so $|G + (x - y) \cap A| = c$.

Observe that $\mu(A^* - y) = 1$ for all $y \in \Omega$. Choose $K \subset G$ of power $\omega_1$. Then $\mu\left(\bigcap_{y \in K} (A^* - y)\right) = 1$. Let $x \in (A^* - y)$, for all $y \in K$. Then $(G + x) \cap A^*$ is uncountable. Since $(G + x) \cap A^*$ is Borel, it is of power $c$.

We have shown the following.

LEMMA 2. *Every set of positive measure has continuumly many elements in common with some translate of every uncountable Borel subset of $\Omega$.*

## 2. Construction of the measure

In this and succeeding sections, by a measure space we will always mean a triple $(\Omega, \mathscr{A}, \mu)$, where $\mathscr{A}$ is a $\sigma$-algebra of subsets of the circle group $\Omega$, and $\mu$ is a countably additive function from $\mathscr{A}$ into $[0, 1]$, $\mu(\Omega) = 1$.

Let us call a sequence $(x_0, \ldots, x_n)$ from $\Omega$ independent just in case $\sum a_i x_i = 0$ implies all $a_i = 0$, for integers $a_i$. Fix $B \subset \Omega$ to be some countable dense set such that any finite sequence of distinct elements is independent.

LEMMA 3. *There is a perfect $C \subset \Omega$ such that any finite sequence of distinct elements is independent.*

*Proof.* Construct a sequence of sets $S_n$ as follows. Each $S_n$ is the finite union of pairwise disjoint closed intervals, all of whose endpoints lie in $B$. $S_{n+1} \subset S_n$ is obtained from $S_n$ by replacing each $[a, b]$ in $S_n$ with some $[a, a + \varepsilon]$, $[b - \delta, b]$, with $a + \varepsilon < b - \delta$. Arrange for any sequence $(x_0, \ldots, x_n)$ from $S_{n+1}$, no two terms of which are from the same interval in $S_{n+1}$, and for any $-n \leq a_0, \ldots, a_n \leq n$, we have $\sum a_i x_i = 0$ implies all $a_i = 0$. Then set $C = \bigcap_n S_n$.

LEMMA 4. *There is a function $F$ whose domain is $\Omega$, such that each $F(x)$ is an uncountable Borel subgroup of $\Omega$, and no $F(x)$ meets the group generated by $\bigcup_{y \neq x} F(y)$ (except at 0).*

*Proof.* Choose $c$ pairwise disjoint perfect subsets of $C$, and index them by $\Omega$. Take the groups generated by each subdivision. Note that the group generated by a compact set is Borel, since it is a countable union of continuous images of compact sets.

Call $\mathcal{N} \subset \mathcal{P}(\Omega)$ *countably independent* just in case any nontrivial countable intersection of elements of $\mathcal{N}$ and complements of elements of $\mathcal{N}$ is nonempty. In [2, p. 585], an explicit construction of a countably independent set $\mathcal{N}$ of power $2^c$ is given, following Tarski.

LEMMA 5. *There is a unique finitely additive measure $v$ on the Boolean algebra $\mathcal{B}(\mathcal{N})$ generated by $\mathcal{N}$, such that $v(\Omega) = 1$, and $v(A_1 \cap \cdots \cap A_n) = 2^{-n}$, for distinct $A_i \in \mathcal{N}$.*

*Proof.* This follows from the finite independence of $\mathcal{N}$. Each finite sequence of distinct $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{N})$ determines a partition of $\Omega$ with $2^n$ subdivisions. The union of any $k$ distinct subdivisions will be assigned measure $k/2^n$. Any element of $\mathcal{B}(\mathcal{N})$ can be so represented. The independence guarantees that the measure is well defined.

LEMMA 6. *The intersection of any decreasing sequence of nonempty elements of $\mathcal{B}(\mathcal{N})$ is nonempty.*

*Proof.* Let $\{B_n\}$ be the sequence. We can represent each $B_n$ by means of a partition $P_n$ (determined as in Lemma 5 by a finite sequence of distinct elements of $\mathcal{N}$), so that each $P_{n+1}$ is a refinement of $P_n$. Choose a subdivision $s_1$ of $P_1$ which intersects each $B_n$. Choose a subdivision $s_2 \subset s_1$ of $P_2$ which intersects each $B_n$. Continue in this way, defining a sequence $\{s_n\}$ of subdivisions, nested under inclusion. Countable independence guarantees that $\bigcap s_n$ is nonempty. We are done, since each $s_n \subset B_n$.

Now let $j\colon \mathscr{P}(\Omega) \to \mathscr{P}(\mathscr{P}(\Omega))$ be given by $j(A) = \{B \subset \Omega\colon (\forall x \in A)\ (-B$ intersects every coset of the group $F(x)$ in a set of power $< c)$ and $(\forall x \notin A)$ $(B$ intersects every coset of the group $F(x)$ in a set of power $< c)\}$. Let

$$\mathscr{M} = \{B \subset \Omega\colon B \in j(A) \text{ for some } A \in \mathscr{B}(\mathscr{N})\} = \bigcup (j[\mathscr{B}(\mathscr{N})]).$$

Define $\rho\colon \bigcup (\mathrm{Rng}\,(j)) \to \mathscr{P}(\Omega)$ by $\rho(B) = \{x\colon -B$ intersects every coset of $F(x)$ in a set of power $< c\}$.

The following uses the fact that the union of countably many sets of power less than $c$ has power less than $c$.

LEMMA 7. *$\mathscr{M}$ is a Boolean algebra, $\bigcup (\mathrm{Rng}\,(j))$ is a Borel algebra, $\rho(-A) = -\rho(A)$, $\rho\,(\bigcap_n \rho(A_n)) = \bigcap_n \rho(A_n)$.*

Define $m_1\colon \mathscr{M} \to \mathbf{R}$ by $m_1(B) = \nu(\rho(B))$.

LEMMA 8. *$m_1$ is a finitely additive translation invariant measure on $\mathscr{M}$. The intersection of any decreasing sequence of sets of positive $m_1$-measure has Lebesgue outer measure* 1.

*Proof.* The first part follows immediately from Lemma 7, and that translations map cosets onto cosets. Now let $\{A_n\}$ be as in the lemma. Note that by Lemma 7, $-\bigcap A_n$ will have less than $c$ elements in common with every coset of some $F(x)$. Hence by the key measure theoretic Lemma 2, $-\bigcap A_n$ can contain no subset of positive Lebesgue measure, and so $\mu^*\,(\bigcap A_n) = 1$.

LEMMA 9. *Let $(\Omega, \mathscr{A}_1, \mu_1)$ be a measure space, and let $\mu_2$ be a finitely additive measure on a Boolean algebra $\mathscr{A}_2$ of subsets of $\Omega$, $\mu_2(\Omega) = 1$. Furthermore suppose that the intersection of any decreasing sequence of sets of positive $\mu_2$-measure has outer $\mu_1$-measure* 1. *Then there is a unique countably additive measure $\lambda$ on the Borel algebra generated by $\mathscr{A}_1 \cup \mathscr{A}_2$, such that*

$$\lambda(A_1 \cap A_2) = \mu_1(A_1) \cdot \mu_2(A_2).$$

*If $\mu_1$, $\mu_2$ are translation invariant, so is $\lambda$.*

*Proof.* We first observe that there is a unique finitely additive measure $\lambda_0$ on the Boolean algebra generated by $\mathscr{A}_1 \cup \mathscr{A}_2$, such that

$$\lambda_0(A_1 \cap A_2) = \mu_1(A_1) \cdot \mu_2(A_2).$$

The elements of this Boolean algebra can be written in the form $(A_1 \cap B_1) \cup \cdots \cup (A_n \cap B_n)$, where $B_1, \ldots, B_n$ is a partition of $\Omega$ by elements of $\mathscr{A}_2$, and $A_1, \ldots, A_n \in \mathscr{A}_1$. Define

$$\lambda_0((A_1 \cap B_1) \cup \cdots \cup (A_n \cap B_n)) = \mu_1(A_1) \cdot \mu_2(B_1) + \cdots + \mu_1(A_n) \cdot \mu_2(B_n).$$

Using $\mu_2(B) > 0 \to \mu_1^*(B) = 1$, one easily shows this to be well defined.

It now suffices to show that any decreasing sequence of elements $\{C_n\}$ in the Boolean algebra, all of whose $\lambda_0$-measures are at least $\varepsilon > 0$, has nonempty intersection. To this end, represent each $C_n$ as

$$(A_{n1} \cap B_{n1}) \cup \cdots \cup (A_{nk_n} \cap B_{nk_n}),$$

as above, in such a way that the partition $B_{n+1,1}, \ldots, B_{n+1,k_{n+1}}$ is a refinement of the partition $B_{n1}, \ldots, B_{nk_n}$. Again using that $\mu_2(B) > 0$ implies $\mu_1^*(B) = 1$, we observe that if $B_{n+1,i} \subset B_{nj}$, $\mu_2(B_{n+1,i}) > 0$, then $\mu_1(A_{n+1,i} - A_{nj}) = 0$.

Note that the $B_{nm}$ form a finitely branching tree when partially ordered under containment, with $B_{nm}$ on the $n$th level. Consider the subtree consisting of all $B_{nm}$ such that $\mu_2(B_{nm}) > 0$ and $\mu_1(A_{nm}) \geq \varepsilon$. This subtree must be infinite, and therefore must have an infinite path $\{B_{nm_n}\}$. By hypothesis, $\mu_1^* (\bigcap_n B_{nm_n}) = 1$. Since $\mu_1 (\bigcap_n A_{nm_n}) \geq \varepsilon$, we see that $\bigcap_n C_n \neq \emptyset$.

Translation invariance and uniqueness are immediate.

LEMMA 10. *There is a unique countably additive measure $m$ on the Borel algebra $\mathscr{B}_\infty(LM \cup \mathscr{M})$ generated by the Lebesgue measurable sets and $\mathscr{M}$, such that $m(A \cap B) = \mu(A) \cdot m_1(B)$, for Lebesgue measurable $A$, and $B \in \mathscr{M}$. Furthermore, $m$ is translation invariant.*

*Proof.* From Lemmas 8 and 9.

## 3. The definability of the measure

The countable axiom of choice ($AC_\omega$) asserts that every countable set of nonempty sets has a choice function. This is a special case of the axiom of dependent choice (DC), which asserts that for nonempty $A$, if $(\forall x \in A)(\exists y \in A)(R(x, y))$, then $(\exists h: \omega \to A)(\forall n \in \omega)(R(h(n), h(n + 1)))$. It is shown in [4] that ZF + DC does not suffice to prove the existence of a Lebesgue nonmeasurable set.

The purpose of this section is to indicate how the $(\Omega. \mathscr{B}_\infty(LM \cup \mathscr{M}), m)$ of the previous section can be defined and proved to be a translation invariant measure space extending the Lebesgue measure space, within ZF + $AC_\omega$. Note that some form of choice is needed even to prove that Lebesgue measure is countably additive.

We first note that the key measure theoretic Lemma is provably equivalent to a $\Pi^1_3$ sentence within ZF + $AC_\omega$. This is because ZF + $AC_\omega$ suffices to prove that sets of positive measure contain closed sets of positive measure, that uncountable Borel sets have perfect subsets, and that uncountable Borel sets are of cardinality $c$. In addition, every $\Pi^1_3$ sentence provable in ZFC is provable in ZF. Hence the measure theoretic lemma is provable in ZF + $AC_\omega$.

A proof in ZF + $AC_\omega$ of the existence of a definable independent countable dense set can be obtained by iterating the following. There is a definable procedure which takes any sequence of reals and interval of reals into a real in that interval which is not present in the sequence.

For Lemma 4, note that the construction of $C$ puts $C$ naturally in one-to-one correspondence $\phi$ with $2^\omega$. Let $g\colon 2^\omega \to \mathscr{P}(2^\omega)$ be given by

$$g(\alpha) = \{\beta \in 2^\omega : (\forall n)(\beta(2n) = \alpha(n))\}.$$

Now take the $\phi^{-1}[g(\alpha)]$. These form a partition of $C$ into perfect subsets indexed by $2^\omega$. It remains only to remark that $2^\omega$ is in definable one-to-one correspondence with $\Omega$. This is a consequence of the fact that the Schröder-Bernstein theorem is explicitly true. In other words there is a definable procedure which takes any one-to-one $\alpha\colon A \to B$, $\beta\colon B \to A$, to a one-to-one onto $\gamma\colon A \to B$.

The construction referred to in [2, p. 585], is definable, and uses only $AC_\omega$ to prove countable independence. The actual domain used there is in definable one-to-one correspondence with $\Omega$.

The proof of Lemma 6 only uses $AC_\omega$. For each $n$, choose an appropriate partition $P'_n$, equipped with an indexing of the subdivisions of $P'_n$. Then refine this partitions by induction, retaining an indexing of subdivisions, to the $P_n$. Then the rest of the argument involves no use of choice.

The definitions of $j$, $\mathscr{M}$, and $\rho$ are explicit. "Power $< c$" means that there is a monomorphism into $\Omega$, but no surjection onto $\Omega$. We use this somewhat non-standard definition to ensure that $AC_\omega$ suffices to prove that the union of a sequence of sets of power less than $c$ is of power less than $c$. Hence Lemmas 7 and 8 need only $AC_\omega$.

In the proof of Lemma 9, if the partitions and subdivisions are handled as in the proof of Lemma 6, only $AC_\omega$ is used. The measure extension theorem used for Lemma 9 needs only $AC_\omega$.

In the proof of Lemma 10, we need that Lebesgue measure is countably additive, which uses only $AC_\omega$.

## 4. The separability character of the measure

In view of [4], we know that ZF + $AC_\omega$, and in fact ZF + DC, does not suffice to prove that $(\Omega, \mathscr{B}_\infty(LM \cup \mathscr{M}), m)$ differs from the Lebesgue measure space. In this section, we will use the full axiom of choice to show that $(\Omega, \mathscr{B}_\infty(LM \cup \mathscr{M}), m)$ has separability character $2^c$.

It clearly suffices to find a subset of $\mathscr{B}_\infty(LM \cup \mathscr{M})$ of power $2^c$ such that the measure of the symmetric difference of any two elements is $\frac{1}{2}$. Hence it suffices to establish that $j(A) \neq \emptyset$ for all $A \in \mathscr{N}$.

LEMMA 11. *All $j(A)$ are nonempty, $A \subset \Omega$.*

*Proof.* We shall use $c$ also for the least ordinal number of power $c$. Let $(x_\alpha, y_\alpha)$, for $\alpha < c$, be chosen so that the cosets $F(x_\alpha) + y_\alpha$ enumerate

$$\{F(x) + y : x, y \in \Omega\}$$

without repetition.

We define a transfinite sequence $B_\alpha \subset \Omega$, $\alpha < c$, in order that

(i) for all $\beta \leq \alpha$, $x_\beta \in A$, $-(B_\alpha) \cap (F(x_\beta) + y_\beta))$ has power less than $c$,

(ii) for all $\beta \leq \alpha$, $x_\beta \notin A$, $B_\alpha \cap (F(x_\beta) + y_\beta)$ has power less than $c$,

(iii) for all $\gamma \leq \beta \leq \alpha$, $B_\alpha \cap (F(x_\gamma) + y_\gamma) = B_\beta \cap (F(x_\gamma) + y_\gamma)$,

(iv) $B_\alpha \subset \bigcup_{\beta \leq \alpha} (F(x_\beta) + y_\beta)$.

Suppose all $B_\beta$, $\beta < \alpha$, have been defined in accordance with (i)–(iv). Note that by Lemma 4, if $(x, y) \neq (z, w)$ then $F(x) + y$ has at most one element in common with $F(z) + w$. It therefore follows that $\bigcup_{\beta < \alpha} (F(x_\beta) + y_\beta)$ has less than $c$ elements in common with $F(x_\alpha) + y_\alpha$.

If $\alpha \in A$, set

$$B_\alpha = \left(\bigcup_{\beta < \alpha} B_\beta\right) \cup \left((F(x_\alpha) + y_\alpha) - \bigcup_{\beta < \alpha} (F(x_\beta) + y_\beta)\right).$$

If $\alpha \notin A$, set $B_\alpha = \bigcup_{\beta < \alpha} B_\beta$.

Now let $B = \bigcup_{\alpha < c} B_\alpha$. Then $B \in j(A)$.

## 5. Conclusion

From Lemmas 10, 11, and the discussion in Section 3, we have now proved the following.

THEOREM. *There is a formula $\phi(x)$ of set theory such that*

(a) ZF + $AC_\omega$ *proves* $(\exists!\, x)(\phi(x))$ *and* $(\exists x)(\phi(x)$ & *$x$ is a translation invariant measure space on the circle group that extends Lebesgue measure) and*

(b) ZFC *proves* $(\exists x)(\phi(x)$ & *$x$ has separability character* $2^c)$.

The theorem carries over for Lebesgue measure on the reals by means of the following observation. Let $(\Omega, \mathscr{E}, \nu)$ be a measure space. Let $\bar{\mathscr{E}}$ be the family of all $A \subset \mathbf{R}$ such that $(A \cap [n, n + 1)) = B_n + n$, for some (unique) $B_n \in \mathscr{E}$, $n \in \mathbf{Z}$. (Here $\Omega$ is identified with $[0, 1)$). Let $\bar{\nu}$ map $\bar{\mathscr{E}}$ into the nonnegative extended reals by $\bar{\nu}(A) = \sum \nu(B_n)$. Then $\bar{\nu}$ is countably additive. If $\nu$ extends Lebesgue measure on $\Omega$, then $\bar{\nu}$ will extend Lebesgue measure on $\mathbf{R}$. If $\nu$ is $\Omega$-translation invariant, $\bar{\nu}$ will be $\mathbf{R}$-translation invariant. If $(\Omega, \mathscr{E}, \nu)$ has separability character $2^c$, so will $(\mathbf{R}, \bar{\mathscr{E}}, \bar{\nu})$.

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