SINGULAR MEASURES HAVING ABSOLUTELY CONTINUOUS CONVOLUTION POWERS

BY

SADAHIRO SAEKI

1. Introduction

Throughout this paper, let G be an arbitrary *nondiscrete* locally compact abelian group with dual Γ, m_G Haar measure on G, and $\mathbf{M}(G)$ the convolution measure algebra of G (cf. [4]). We shall denote by $\mathbf{M}_a(G)$ and $\mathbf{M}_s(G)$ the closed ideal of all absolutely continuous measures and the closed subspace of all singular measures (with respect to m_G) in $\mathbf{M}(G)$, respectively. If B is a space of measures or functions, the symbol B^+ (or B_+) will stand for the set of real-valued nonnegative members of B.

In their interesting paper [2] E. Hewitt and H. S. Zuckerman constructed a nonzero measure $\mu \in \mathbf{M}_s^+(G)$ such that $\mu * \mu$ is absolutely continuous and its Radon-Nikodym derivative with respect to m_G is in $L^p(G) = L^p(G, m_G)$ for all real $p \geq 1$ (see also [3]). Later K. Stromberg [7] used their methods to prove that $\mathbf{M}(G)$ contains a "large" independent set comprizing such measures. The construction of such a measure μ by Hewitt and Zuckerman depends upon the so-called Riesz product technique and some structure theorems for locally compact abelian groups, and consequently it seems quite difficult to make any nontrivial constraint on the support of μ if one adheres to their methods. Notice, however, that it is easy to make supp μ sufficiently large by convoluting μ with an appropriate discrete measure (cf. 4.6 of [2]). We are therefore interested in the problem to construct a nonzero singular measure μ so that supp μ is as small as possible while $\mu * \mu$ has some preassigned nice properties. It is the purpose of the present paper to show that such a construction is possible in some sense.

2. The results

Let $\mathbf{D}(2)$ denote the set of all sequences $b = (b_1, b_2, \ldots)$ consisting of 0 and 1 alone, so that the cardinality of $\mathbf{D}(2)$ is equal to the cardinality of the continuum. We write $\mathbf{Z}$ and $\mathbf{Z}_+$ for the set of all integers and the set of all nonnegative integers, respectively. Suppose that $\mu \in \mathbf{M}(G)$, $1 \leq p \leq \infty$ and $f \in L^p(G)$. By writing $\mu \in L^p(G)$ we mean that μ is in $\mathbf{M}_a(G)$ and its Radon-Nikodym derivative with respect to m_G is in $L^p(G)$. If $\mu = g m_G$ for some $g \in L^1 \cap L^p(G)$, we define $\|f - \mu\|_p = \|f - g\|_p$. Our results can be stated as follows.

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2.1. THEOREM. *Suppose that* $f_1, f_2, \ldots, f_r$ *are finitely many nonzero functions in* $L^1_+ \cap L^2(G)$, *and that* $v(p)$ *is a strictly positive continuous function of* $p \in (1, 2)$ *such that* $\lim_{p\to 1} v(p) = \infty$. *Then there exists a set* $\{\mu_{jb}: 1 \le j \le r, b \in \mathbf{D}(2)\}$ *of non-zero measures in* $\mathbf{M}_s^+(G)$ *with the following properties:*

(i) *For each* $j = 1, 2, \ldots, r$, *the supports of the measures* μ_{jb} *with* $b \in \mathbf{D}(2)$ *are disjoint compact subsets of* supp f_j, *and the closure of their union has zero Haar measure.*

(ii) *For each* $(j, b) \in \{1, 2, \ldots, r\} \times \mathbf{D}(2)$, $\mu_{jb} * \mu_{jb}$ *is in* $L^p(G)$ *for all real* $p \ge 1$, $\|f_j * f_j - \mu_{jb} * \mu_{jb}\|_1 < v(2)$ *and*

$$\|(\hat{f}_j)^2 - (\hat{\mu}_{jb})^2\|_p < v(p) \quad \textit{for all } p \in (1, 2].$$

(iii) *If* (j, b) *and* (k, c) *are two different elements of* $\{1, 2, \ldots, r\} \times \mathbf{D}(2)$, *then* $\mu_{jb} * \mu_{kc}$ *is in* $L^p(G)$ *for all* $p \ge 1$,

$$\|f_j * f_k - \mu_{jb} * \mu_{kc}\|_1 < v(2),$$

and $\hat{\mu}_{jb}\hat{\mu}_{kc} \in L^1(\Gamma)$. *If* $j \ne k$, *then* $\|\hat{f}_j\hat{f}_k - \hat{\mu}_{jb}\hat{\mu}_{kc}\|_1 < v(2)$.

2.2. COROLLARY. *Let* U *be a neighborhood of the identity of* Γ, *and* $v(p)$ *a function of* $p \in (1, 2]$ *as in Theorem* 2.1. *Then there exists a probability measure* μ *in* $\mathbf{M}_s(G)$ *such that* supp μ *is compact*, $m_G(\text{supp }\mu) = 0$, *and*

$$\left(\int_\Gamma |\hat{\mu}|^{2p}\, d\gamma\right)^{1/p} < m_\Gamma(U)^{1/p} + v(p) \qquad (1 < p \le 2).$$

2.3. COROLLARY. *Let* F *be a* σ*-compact subset of* G. *Then there exist* $\mu \in \mathbf{M}_s^+(G)$ *and* $f \in \mathbf{C}^+(G)$ *such that* (i) m_G (supp μ) $= 0$, (ii) $f(x) > 0$ *for all* $x \in F$, (iii) $\hat{\mu}^2 \in L^p(\Gamma)$ *for all* $p > 1$, *and* (iv) $g(x) \ge f(x)$ *for almost all* $x \in G$, *where* g *denotes the Radon-Nikodym derivative of* $\mu * \mu$ *with respect to* m_G.

Let r be a natural number, and let E be a subset of $\mathbf{Z}^r_+$ which contains all the unit vectors $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. We say that such a set E is *dominative* if whenever $m = (m_j) \in \mathbf{Z}^r_+$, $n = (n_j) \in E$ and $m_j \le n_j$ for all j, *then* $m \in E$. Notice that if $\mu_1, \ldots, \mu_r$ are nonzero measures in $\mathbf{M}_s^+(G)$, and if E is the set of all $(n_j) \in \mathbf{Z}^r_+$ such that $\mu_1^{n_1} * \cdots * \mu_r^{n_r} \in \mathbf{M}_s(G)$, then E is dominative.

2.4. THEOREM. *Suppose that* $K_1, K_2, \ldots, K_r$ *are compact subsets of* G *with positive Haar measure, that* Φ *is a separable subset of* $\mathbf{M}_s(G)$, *and that* E *is a dominative subset of* $\mathbf{Z}^r_+$. *Then there exists a set* $\{\mu_{jb}: 1 \le j \le r, b \in \mathbf{D}(2)\}$ *of nonzero measures in* $\mathbf{M}_s^+(G)$ *with the following properties:*

(a) *For each* $j = 1, 2, \ldots, r$, *the supports of the measures* μ_{jb} *with* $b \in \mathbf{D}(2)$ *are disjoint compact subsets of* K_j, *and the closure of their union has zero Haar measure.*

(b) *If* $(m_1, \ldots, m_r) \in E$, $b \in \mathbf{D}(2)$, *and* $\nu \in \Phi$, *then*

$$\nu * \mu_{1b}^{m_1} * \cdots * \mu_{rb}^{m_r} \in \mathbf{M}_s(G).$$

(c) *If* $(n_1, \ldots, n_r) \in \mathbf{Z}_+^r \backslash E$ *and* $b \in \mathbf{D}(2)$, *then*

$$\mu_{1b}^{n_1} * \cdots * \mu_{rb}^{n_r} \in \mathbf{M}_a(G).$$

(d) *If* (j, b) *and* (k, c) *are two elements of* $\{1, 2, \ldots, r\} \times \mathbf{D}(2)$ *with* $b \neq c$, *then* $\mu_{jb} * \mu_{kc}$ *is in* $L^p(G)$ *for all real* $p \geq 1$.

2.5. COROLLARY. *Let* $n \geq 2$ *be a natural number, and* $L^{1/n}(G)$ *the set of all* $\mu \in \mathbf{M}(G)$ *such that* $\mu^n \in \mathbf{M}_a(G)$. *If K is a compact subset of G with positive Haar measure, then*

$$L^{1/n}(G) \cap \{\mu \in \mathbf{M}(K): m_G\,[\operatorname{supp}(\mu^{n-1})] = 0\}$$

is unseparable, and, for each natural number r, contains probability measures $\mu_1, \ldots, \mu_r$ *such that* $m_G\,[\operatorname{supp}(\mu_1 * \ldots * \mu_r)^{n-1}] = 0$.

A consequence of the last corollary is that $L^{1/n}(G)$ does not form a vector space ($n = 2, 3, \ldots$).

In Section 3 we establish some lemmas which will be used in the proof of Theorem 2.1. Theorem 2.1 and the two corollaries to it are proved in Sections 4 and 5, respectively. Section 6 consists of three lemmas which are needed in the proof of Theorem 2.4. Theorem 2.4 and Corollary 2.5 are proved in Section 7. We give some remarks in Section 8, the final section.

3. Some lemmas

In this section we establish four lemmas. We assume that the Haar measures m_G and m_Γ are so adjusted that the inversion formula holds. Let $A(G) = L^1(\Gamma)^\wedge$ denote the Fourier algebra of G; thus each element f of $A(G)$ is the (inverse) Fourier transform of a unique element of $L^1(\Gamma)$, which will be denoted by $\hat{f}$, and the $A(G)$-norm of f is defined to be $\|f\|_A = \|\hat{f}\|_1$. Notice that $\|f * \mu\|_A \leq \|f\|_A \cdot \|\hat{\mu}\|_\infty$ if $f \in A(G)$ and $\mu \in \mathbf{M}(G)$, and that $\|f * g\|_A \leq \|f\|_2 \cdot \|g\|_2$ if f, $g \in L^2(G)$. If $f \in L^1(G)$, supp f denotes the support of the measure fm_G. For a Borel set F in G and $1 \leq p \leq \infty$, we denote by $L^p(E)$ the set of all $f \in L^p(G)$ such that supp $f \subset E$. The following lemma is, though elementary, very useful in our proofs.

3.1. LEMMA. *Let* $f_1, \ldots, f_m \in A(G)$, *and* $g_1, \ldots, g_m \in L^1 \cap L^\infty(G)$. *Also let* $\varepsilon > 0$, $1 \leq q < \infty$, *and K be a compact subset of G. Then there exist* $\delta > 0$ *and a compact subset Y of* Γ *which have the following properties. If* $h \in L^1 \cap L^2(K)$ *and* $|\hat{h}| < \delta$ *on Y, then*

(i) $\|f_j * h\|_A \leq \varepsilon(1 + \|h\|_1)$,
(ii) $\|g_j * h\|_A \leq \varepsilon(1 + \|h\|_2)$,
(iii) $\|g_j * h\|_p \leq \varepsilon(1 + \|h\|_1)$ $(1 \leq p \leq q)$
for all $j = 1, 2, \ldots, m$.

Proof. Let $\delta > 0$ and a compact subset Y of Γ be given. If $h \in L^1 \cap L^2(G)$ and $|\hat{h}| < \delta$ on Y, then we have, for $f \in A(G)$,

$$\begin{aligned} \|f * h\|_A &= \int_\Gamma |\hat{f}\hat{h}|\, d\gamma \\ &\leq \delta \left(\int_Y |\hat{f}|\, d\gamma\right) + \|\hat{h}\|_\infty \left(\int_{\Gamma\setminus Y} |\hat{f}|\, d\gamma\right) \\ &\leq \delta \|f\|_A + \|h\|_1 \left(\int_{\Gamma\setminus Y} |\hat{f}|\, d\gamma\right). \end{aligned}$$

Similarly we have, for $g \in L^2(G)$,

$$\begin{aligned} \|g * h\|_A &\leq \delta \left(\int_Y |\hat{g}|\, d\gamma\right) + \left(\int_{\Gamma\setminus Y} |\hat{g}|^2\, d\gamma\right)^{1/2} \left(\int_\Gamma |\hat{h}|^2\, d\gamma\right)^{1/2} \\ &\leq \delta m_\Gamma(Y)^{1/2} \|g\|_2 + \left(\int_{\Gamma\setminus Y} |\hat{g}|^2\, d\gamma\right)^{1/2} \|h\|_2 \end{aligned}$$

by Schwarz' inequality and Plancherel's theorem. Thus both (i) and (ii) hold if $Y \subset \Gamma$ is sufficiently large and $\delta > 0$ is sufficiently small.

To confirm (iii), we may assume that $g_j \in A(G)$ and $\operatorname{supp} g_j \subset L$ for all $j = 1, 2, \ldots, m$ and for some compact subset L of G. Then $h \in L^1(K)$ implies $\operatorname{supp}(g_j * h) \subset L + K$ and

$$\|g_j * h\|_p \leq \|g_j * h\|_\infty \cdot m_G(L + K)^{1/p} \leq \|g_j * h\|_A \cdot m_G(L + K)^{1/p}$$

for all $j = 1, 2, \ldots, m$ and all $p \geq 1$. This, combined with (i), shows that (iii) holds for appropriate $Y \subset \Gamma$ and $\delta > 0$, which completes the proof.

3.2. Lemma. *Let $g \in L^1_+ \cap L^\infty(G)$, Y a compact subset of Γ, and $\delta > 0$. Then there exist $h_0, h_1 \in L^1_+ \cap L^\infty(G)$ such that for $i = 0, 1$ we have*

(i) supp h_0 *and* supp h_1 *are disjoint subsets of* supp g,
(ii) $m_G(\operatorname{supp} h_i) \leq 3^{-1} m_G(\operatorname{supp} g)$,
(iii) $\|h_i\|_1 = \|g\|_1$, $\|h_i\|_\infty \leq (3 + \delta)\|g\|_\infty$,
(iv) $|\hat{h}_i(\gamma) - \hat{g}(\gamma)| < \delta$ *for all* $\gamma \in Y$.

Proof. Without loss of generality, we may assume that g is a simple function with $\|g\|_1 = 1$ and that supp g is compact. Since Y is a compact subset of Γ, the set

$$V = \{x \in G: |1 - \gamma(x)| < \delta/2 \text{ for all } \gamma \in Y\}$$

is a neighborhood of $0 \in G$. It follows that g can be written in the form $g = a_1\xi(E_1) + \cdots + a_n\xi(E_n)$, where the a_j are real positive numbers and the E_j are disjoint Borel sets in G such that

$$x, x' \in E_j \quad \text{and} \quad \gamma \in Y \Rightarrow |\gamma(x) - \gamma(x')| < \delta/2$$

for all $j = 1, 2, \ldots, n$ (the symbol $\xi(E)$ denotes the characteristic function of E).

Now choose and fix any $b \in (0, 1)$ so that $3/(1 - b) < 3 + \delta$. Since G is nondiscrete, there exist disjoint compact sets K_{j0}, $K_{j1} \subset E_j$ such that $m_G(K_{ji}) = 3^{-1}(1 - b)m_G(E_j)$ ($j = 1, 2, \ldots, n$; $i = 0, 1$). We define h_i by setting $h_i = 3(1 - b)^{-1} \sum_{j=1}^{n} a_j\xi(K_{ji})$ (i = 0, 1). Then it is obvious that (i), (ii), and (iii) hold. To confirm (iv), choose any $x_j \in E_j$ for each $j = 1, 2, \ldots, n$. Then we have, for $i = 0, 1$ and $\gamma \in Y$,

$$
\begin{aligned}
|\hat{g}(\gamma) - \hat{h}_i(\gamma)| &\leq \sum_{j=1}^{n} a_j \left| \int_{E_j} \gamma \, dx - 3(1 - b)^{-1} \int_{K_{ji}} r \, dx \right| \\
&\leq \sum_{j=1}^{n} a_j \left\{ \int_{E_j} |\gamma(x) - \gamma(x_j)| \, dx + 3(1 - b)^{-1} \right. \\
&\qquad\qquad \left. \times \int_{K_{ji}} |\gamma(x) - \gamma(x_j)| \, dx \right\} \\
&\leq \delta \sum_{j=1}^{n} a_j m_G(E_j) \\
&= \delta \| g \|_1 \\
&= \delta,
\end{aligned}
$$

which completes the proof.

3.3. Lemma. *Suppose that* $f_1, f_2, \ldots, f_m \in L^1_+ \cap L^\infty(G)$, *that* Y *is a compact subset of* Γ, *and that* $w(p)$ *is a strictly positive continuous function of* $p \in (1, 2]$ *such that* $w(p) \to \infty$ *as* $p \to 1$. *Then there exist* $f_{10}, f_{11} \in L^1_+ \cap L^\infty(G)$ *such that, for* $i = 0, 1$ *and* $j = 1, 2, \ldots, m$, *we have:*

(i) $\operatorname{supp} f_{10}$ *and* $\operatorname{supp} f_{11}$ *are disjoint subsets of* $\operatorname{supp} f_1$;
(ii) $\|f_{1i}\|_1 = \|f_1\|_1$, $m_G(\operatorname{supp} f_{1i}) \leq 3^{-1} m_G(\operatorname{supp} f_1)$;
(iii) $\|f_1 * f_1 - f_{1i} * f_{1i}\|_1 < w(2)$;
(iv) $\max\{\|f_j * (f_1 - f_{1i})\|_1, \|f_j * (f_1 - f_{1i})\|_A\} < w(2)$;
(v) $|\hat{f}_1(\gamma) - \hat{f}_{1i}(\gamma)| < w(2)$ *for all* $\gamma \in Y$;
(vi) $\|(\hat{f}_1)^2 - (\hat{f}_{1i})^2\|_p < w(p)$ *for all* $p \in (1, 2]$.

Proof. Approximating f_1 by an appropriate simple function, we may assume that f_1 has the form $f_1 = \sum_{k=1}^{N} a_k\xi(E_k) = \sum_{k=1}^{N} g_k$, where the a_k are positive real numbers, the E_k are disjoint compact subsets of G, and $g_k = a_k\xi(E_k)$ for $k = 1, 2, \ldots, N$.

We may also assume that $w(p) \geq w(2)$ for all $p \in (1, 2]$. By induction on $k = 1, 2, \ldots, N$, we shall construct $h_{ki} \in L^\infty_+(E_k)$ for $i = 0, 1$, as follows.

Set $h_{00} = h_{01} = 0$, and assume that $h_{0i}, \ldots, h_{(k-1)i}$ have been defined for $i = 0, 1$ and some natural number $k \leq N$. By Lemmas 3.1 and 3.2, there exist

$h_{k0}, h_{k1} \in L_+^\infty(E_k)$ which satisfy the following conditions for $i = 0, 1, 1 \le j \le m$, and $1 \le n \le N$:

(3.3.1) $\operatorname{supp} h_{k0}$ and $\operatorname{supp} h_{k1}$ are disjoint subsets of E_k;
(3.3.2) $\|h_{ki}\|_1 = \|g_k\|_1$, $m_G(\operatorname{supp} h_{ki}) \le 3^{-1} m_G(E_k)$;
(3.3.3) $\|h_{ki}\|_p \le 4\|g_k\|_p = 4a_k m_G(E_k)^{1/p}$ for all $p \ge 1$,
(3.3.4) $|\hat{g}_k(\gamma) - \hat{h}_{ki}(\gamma)| < w(2)/N$ for all $\gamma \in Y$;
(3.3.5) $\|f_j * (g_k - h_{ki})\|_A + \|f_j * (g_k - h_{ki})\|_1 < w(2)/N$;
(3.3.6) $\|g_n * (g_k - h_{ki})\|_A + \|g_n * (g_k - h_{ki})\|_1 < w(2)/(4N^2)$;
(3.3.7) $\|h_{rl} * (g_k - h_{ki})\|_A + \|h_{rl} * (g_k - h_{ki})\|_1 < w(2)/(4N^2)$

for all $r = 0, 1, \ldots, k-1$ and $l = 0, 1$. This completes our induction.

Now we define $f_{1i} = h_{1i} + h_{2i} + \cdots + h_{Ni}$ $(i = 0,1)$, and claim that f_{10} and f_{11} have all the required properties, provided that all the sets E_k have sufficiently small Haar measure. Indeed, (i), (ii), (iv), and (v) follow from (3.3.1), (3.3.2), (3.3.5), and (3.3.4), respectively.

To confirm (iii), we note that

$$\begin{aligned}
&\left\| f_1 * f_1 - \left(h_{1i} + \sum_{k=2}^N g_k\right) * \left(h_{1i} + \sum_{k=2}^N g_k\right)\right\|_1 \\
&\qquad = \left\| 2\sum_{k=2}^N g_k * (g_1 - h_{1i}) + g_1 * g_1 - h_{1i} * h_{1i}\right\|_1 \\
&\qquad \le 2(N-1)w(2)/(4N^2) + \|g_1\|_1^2 + \|h_{1i}\|_1^2 \quad \text{by (3.3.6)} \\
&\qquad < w(2)/(2N) + 2\|g_1\|_1^2 \quad \text{by (3.3.2).}
\end{aligned}$$

Similarly we have

$$\left\| \left(\sum_{k=1}^{n-1} h_{ki} + \sum_{k=n}^N g_k\right)^{(2)} - \left(\sum_{k=1}^{n} h_{ki} + \sum_{k=n+1}^N g_k\right)^{(2)}\right\|_1 < w(2)/(2N) + 2\|g_n\|_1^2 \quad (n = 1, 2, \ldots, N)$$

by (3.3.6), (3.3.7), and (3.3.2), where $f^{(2)} = f * f$ for $f \in L^1(G)$. Adding the last inequalities for $n = 1, 2, \ldots, N$, we obtain

$$\begin{aligned}
\|f_1 * f_1 - f_{1i} * f_{1i}\|_1 &< w(2)/2 + 2(\|g_1\|_1^2 + \cdots + \|g_N\|_1^2) \\
&\le w(2)/2 + 2\|f_1\|_1 \cdot \max\{\|g_k\|_1 : 1 \le k \le N\} \\
&\le w(2)/2 + 2\|f_1\|_1 \cdot \|f_1\|_\infty \cdot M,
\end{aligned}$$

where

$$M = \max\{m_G(E_k) : 1 \le k \le N\}. \tag{3.3.8}$$

Therefore (iii) holds if $2\|f_1\|_1 \cdot \|f_1\|_\infty \cdot M < w(2)/2$.

To prove (vi), notice that $1 \le p < \infty$ and $f \in L^{2p/(2p-1)}(G)$ imply

$$\|\hat{f}^2\|_p = \left(\int_\Gamma |\hat{f}|^{2p}\, d\gamma\right)^{1/p} \le \left(\int_G |f|^{2p/(2p-1)}\, dx\right)^{(2p-1)/p}$$

by the Hausdorff-Young inequality. Therefore we have

$$\begin{aligned}
&\left\| (\hat{f}_1)^2 - \left(\hat{h}_{1i} + \sum_{k=2}^{N} \hat{g}_k\right)^2 \right\|_p \\
&\qquad \le 2 \sum_{k=2}^{N} \|\hat{g}_k(\hat{g}_1 - \hat{h}_{1i})\|_p + \|(\hat{g}_1)^2\|_p + \|(\hat{h}_{1i})^2\|_p \\
&\qquad \le 2 \sum_{k=2}^{N} (\|g_k * (g_1 - h_{1i})\|_A + \|g_k * (g_1 - h_{1i})\|_1) \\
&\qquad\quad + (\|g_1\|_{2p/(2p-1)})^2 + (\|h_{1i}\|_{2p/(2p-1)})^2 \\
&\qquad < w(2)/(2N) + 17a_1^2 m_G(E_1)^{(2p-1)/p} \\
&\qquad \le w(2)/(2N) + 17a_1 m_G(E_1) \cdot \|f_1\|_\infty \cdot m_G(E_1)^{1-1/p}
\end{aligned}$$

by (3.3.6) and (3.3.3). Similarly we have, by (3.3.6), (3.3.7) and (3.3.3),

$$\begin{aligned}
&\left\| \left(\sum_{k=1}^{n-1} \hat{h}_{ki} + \sum_{k=n}^{N} \hat{g}_k\right)^2 - \left(\sum_{k=1}^{n} \hat{h}_{ki} + \sum_{k=n+1}^{N} \hat{g}_k\right)^2 \right\|_p \\
&\qquad < w(2)/(2N) + 17a_n m_G(E_n) \cdot \|f_1\|_\infty \cdot m_G(E_n)^{1-1/p}
\end{aligned}$$

for all $n = 1, 2, \ldots, N$; hence

$$\|(\hat{f}_1)^2 - (\hat{f}_{1i})^2\|_p < w(2)/2 + 17\|f_1\|_1 \cdot \|f_1\|_\infty \cdot M^{1-1/p}$$

for all $p \ge 1$. Thus (iii) and (vi) are satisfied if

$$4\|f_1\|_1 \cdot \|f_1\|_\infty \cdot M < w(2) \quad \text{and} \quad 34\|f_1\|_1 \cdot \|f_1\|_\infty \cdot M^{1-1/p} < w(p)$$

for all $p \in (1, 2]$. Since w is a strictly positive continuous function with $\lim_{p\to 1} w(p) = \infty$, and since we can demand that M is arbitrarily small, this completes the proof.

3.4. LEMMA. *Suppose that $f_1, f_2, \ldots, f_m$, Y and w are as in Lemma* 3.3. *Then there exist $2m$ functions $f_{ji} \in L^1_+ \cap L^\infty(G)$ subject to the following conditions. For all $j, k \in \{1, 2, \ldots, m\}$ and $i, l \in \{0, 1\}$, we have:*

(i) $\operatorname{supp} f_{j0}$ *and* $\operatorname{supp} f_{j1}$ *are disjoint subsets of* $\operatorname{supp} f_j$;
(ii) $\|f_{ji}\|_1 = \|f_j\|_1$, $m_G(\operatorname{supp} f_{ji}) \le 3^{-1} m_G(\operatorname{supp} f_j)$;
(iii) $\|f_j * f_k - f_{ji} * f_{kl}\|_1 < w(2)$;
(iv) $\|f_j * f_k - f_{ji} * f_{kl}\|_A < w(2)$ *if* $j \ne k$;
(v) $|\hat{f}_j(\gamma) - \hat{f}_{ji}(\gamma)| < w(2)$ *for all* $\gamma \in Y$;
(vi) $\|(\hat{f}_j)^2 - (\hat{f}_{ji})^2\|_p < w(p)$ *for all* $p \in (1, 2]$.

Proof. Let f_{10} and f_{11} be the functions given by Lemma 3.3 with w replaced by $2^{-1}w$. When $f_{10}, f_{11}, \ldots, f_{(k-1)0}, f_{(k-1)1}$ are defined, we apply Lemma 3.3 with f_1, $\{f_1, \ldots, f_m\}$ and w replaced by f_k, $\{f_1, \ldots, f_m, f_{10}, f_{11}, \ldots, f_{(k-1)0}, f_{(k-1)1}\}$ and $2^{-1}w$, respectively. By induction on $k = 1, 2, \ldots, m$, we find $2m$ functions with the required properties. This completes the proof.

4. Proof of Theorem 2.1

4.1. Let $f_1, \ldots, f_r \in L^1_+ \cap L^2(G)$ and $v \in \mathbf{C}((1, 2])$ be as in Theorem 2.1. There is no loss of generality in assuming that $\|f_j\|_1 = 1$, $f_j \in L^\infty(G)$, and $K_j = \operatorname{supp} f_j$ is compact ($j = 1, 2, \ldots, r$). For each natural number n and $j \in \{1, 2, \ldots, r\}$, we shall construct 2^n functions $f_{jb} \in L^1_+ \cap L^\infty(G)$ of L^1-norm 1, $b \in \{0, 1\}^n$, as follows.

For $n = 1$, we set $f_{j0} = f_{j1} = f_j$ $(1 \le j \le r)$. Suppose that n is a natural number, and that the functions f_{jb} have been defined for all $1 \le j \le r$, $b \in \{0, 1\}^k$, $1 \le k \le n$. Put

$$Y_n = \bigcup_{k=1}^{n} \{\gamma \in \Gamma : |\hat{f}_{jb}(\gamma)| \ge n^{-1} \text{ for some } 1 \le j \le r \text{ and some } b \in \{0, 1\}^k\},$$

and note that Y_n is a compact subset of Γ. Applying Lemma 3.4 with $Y = Y_n$ and $w = 2^{-n}v$, we can find $2^{n+1}r$ functions $f_{jbi} \in L^1_+ \cap L^\infty(G)$, $b \in \{0, 1\}^n$ and $i \in \{0, 1\}$, which satisfy the following conditions ($j, k \in \{1, 2, \ldots, r\}$; $b, c \in \{0, 1\}^n$; $i, l \in \{0, 1\}$):

(4.1.1) $\operatorname{supp}(f_{jb0})$ and $\operatorname{supp}(f_{jb1})$ are disjoint subsets of $\operatorname{supp} f_{jb}$;
(4.1.2) $\|f_{jbi}\|_1 = 1$, $m_G[\operatorname{supp}(f_{jbi})] \le 3^{-1} m_G[\operatorname{supp}(f_{jb})]$;
(4.1.3) $\|f_{jb} * f_{kc} - f_{jbi} * f_{kcl}\|_1 < 2^{-n}v(2)$;
(4.1.4) $\|f_{jb} * f_{kc} - f_{jbi} * f_{kcl}\|_A < 2^{-n}v(2)$ if $(j, b) \ne (k, c)$;
(4.1.5) $|\hat{f}_{jb}(\gamma) - \hat{f}_{jbi}(\gamma)| < 2^{-n}$ for all $\gamma \in Y_n$;
(4.1.6) $\|(\hat{f}_{jb})^2 - (\hat{f}_{jbi})^2\|_p < 2^{-n}(p)$ for all $p \in (1, 2]$.

This completes our inductive construction of the f_{jb}.

Now let $b = (b_1, b_2, \ldots) \in \mathbf{D}(2)$ and $1 \le j \le r$ be given. We write

$$b(n) = (b_1, b_2, \ldots, b_n) \in \{0, 1\}^n \quad \text{for } n = 1, 2, \ldots,$$

and identify each $f_{jb(n)}$ with the measure $f_{jb(n)}m_G \in \mathbf{M}(G)$. The definitions of the sets Y_n and (4.1.5) show that $\lim_{n\to\infty} \hat{f}_{jb(n)}(\gamma)$ exists for all $\gamma \in \Gamma$. Since $\operatorname{supp} f_{jb(n)} \subset K_j$ for all $n \ge 1$ by (4.1.1), it follows from (4.1.2) that the sequence $(f_{jb(n)})_{n=1}^{\infty}$ converges weak* to a probability measure $\mu_{jb} \in \mathbf{M}(K_j)$. Setting

$$\mathbf{D}_0(2) = \{b \in \mathbf{D}(2) : b_1 = 0\},$$

we claim that the set $\{\mu_{jb} : 1 \le j \le r, b \in \mathbf{D}_0(2)\}$ has the required properties (with $\mathbf{D}(2)$ replaced by $\mathbf{D}_0(2)$).

Indeed, let $1 \le j \le r$ be given. Then (4.1.1) implies that

$$\operatorname{supp}(\mu_{jb}) \subset \bigcap_{n=1}^{\infty} \operatorname{supp}(f_{jb(n)}) \subset K_j \quad (b \in \mathbf{D}_0(2)),$$

that $\operatorname{supp}(\mu_{jb}) \cap \operatorname{supp}(\mu_{jc}) = \emptyset$ whenever b, c are distinct elements of $\mathbf{D}_0(2)$, and that the union of all $\operatorname{supp}(\mu_{jb})$, $b \in \mathbf{D}_0(2)$, is contained in the compact set

$$\bigcap_{n=1}^{\infty} [\bigcup \{\operatorname{supp}(f_{jc}) : c \in \{0, 1\}^n\}] \subset K_j.$$

We therefore conclude from (4.1.1) and (4.1.2) that the closure of the union of all supp (μ_{jb}) has zero Haar measure. This establishes part (i).

To prove part (ii), choose any $(j, b) \in \{1, \ldots, r\} \times \mathbf{D}_0(2)$. We infer from (4.1.3) that the sequence $(f_{jb(n)} * f_{jb(n)})_{n=1}^{\infty}$ converges to some $g_{jb} \in L^1(G)$ and that $\|f_j * f_j - g_{jb}\|_1 < v(2)$. Since

$$\hat{g}_{jb}(\gamma) = \lim_{n\to\infty} \{\hat{f}_{jb(n)}(\gamma)\}^2 = \{\hat{\mu}_{jb}(\gamma)\}^2 \quad (\gamma \in \Gamma),$$

it follows that $\mu_{jb} * \mu_{jb}$ is absolutely continuous and that $\mu_{jb} * \mu_{jb} = g_{jb}m_G$. Moreover, using (4.1.6), we can easily prove that

$$\|(\hat{f}_j)^2 - (\hat{\mu}_{jb})^2\|_p < v(p) \quad \text{for all } p \in (1, 2].$$

Since $\hat{f}_j \in \mathbf{C}_0(\Gamma) \cap L^2(\Gamma)$, we also have $(\mu_{jb} * \mu_{jb})^\wedge \in L^p(\Gamma)$ for all $p \in (1, 2]$, and so $g_{jb} \in L^q(G)$ for all real $q \geq 1$ by the Hausdorff-Young inequality. This establishes part (ii).

The proof of part (iii) is similar to that of part (ii), and the whole proof is complete.

4.2. A weak version of Theorem 2.1 holds for every nondiscrete locally compact (not necessarily abelian) group. Suppose that G is such a group with left Haar measure m_G, that $f_1, f_2, \ldots, f_r$ are nonzero functions in $L_+^\infty(G)$ with compact support, and that w is a strictly positive continuous function on $[1, \infty)$ such that $w(p) \to \infty$ as $p \to \infty$. Then there exist nonzero measures $\mu_{jb} \in \mathbf{M}^+(G)$, $(j, b) \in \{1, 2, \ldots, r\} \times \mathbf{D}(2)$, such that:

(i) For every $1 \leq j \leq r$, the supports of the μ_{jb}, $b \in \mathbf{D}(2)$, are disjoint compact subsets of supp f_j, and the closure of their union has zero Haar measure;

(ii) For each $(j, b) \in \{1, 2, \ldots, r\} \times \mathbf{D}(2)$,

$$\mu_{jb} * \mu_{jb} = g_{jb}m_G \quad \text{for some } g_{jb} \in \bigcap \{L^p(G): 1 \leq p < \infty\}$$

and $\|f_j * f_j - g_{jb}\|_p < w(p)$ for all real $p \geq 1$;

(iii) If (j, b) and (k, c) are different two elements of $\{1, 2, \ldots, r\} \times \mathbf{D}(2)$, then $\mu_{jb} * \mu_{kc} = gm_G$ for some $g = g_{jbkc} \in \mathbf{C}_c(G)$ and $\|f_j * f_k - g\|_1 < w(2)$; and, if, in addition, $j \neq k$, then $\|f_j * f_k - g\|_\infty < w(2)$.

The proof of this fact is slightly more complicated than but similar to the proof of Theorem 2.1. We omit the details.

5. Proofs of Corollaries 2.2 and 2.3

To prove Corollary 2.2, we choose and fix any $f \in L_+^1 \cap A(G)$ such that $\|f\|_1 = 1$ and supp $\hat{f} \subset U$. By Theorem 2.1, there exists a probability measure $\mu \in \mathbf{M}(G)$, with compact support having zero Haar measure, such that $\|\hat{f}^2 - \hat{\mu}^2\|_p < v(p)$ for all $p \in (1, 2]$. Since $|\hat{f}| \leq 1$ and $\hat{f} = 0$ off U, it follows that

$$\|\hat{\mu}^2\|_p < \|\hat{f}^2\|_p + v(p) \leq m_\Gamma(U)^{1/p} + v(p) \quad (1 < p \leq 2),$$

which establishes Corollary 2.2.

To confirm Corollary 2.3, let F be a σ-compact subset of G. By the well-known structure theorem [4], G contains an open subgroup of the form $\mathbf{R}^N \times H$, where N is a nonnegative integer and H is a compact abelian group. Since F is σ-compact, there exist countably many elements $x_0 = 0, x_1, x_2, \ldots$ of G such that the family

$$\{x_n + [-1, 1]^N \times H: n \in \mathbf{Z}_+\}$$

covers F and is locally finite. Let ξ denote the characteristic function of $[-1, 1]^N \times H$, so that $(\xi * \xi)(x) \geq 1$ for all $x \in [-1, 1]^N \times H$ (we normalize m_G so that $m_G([0, 1]^N \times H) = 1$). By Theorem 2.1, we can find $\mu_1, \mu_2 \in \mathbf{M}^+([-1, 1]^N \times H)$ such that m_G (supp $\mu_i) = 0$, $|\hat{\mu}_i|^2 \in L^p(\Gamma)$ for all $p > 1$ and $i = 1, 2$, and $\|\hat{\xi}^2 - \hat{\mu}_1\hat{\mu}_2\|_1 < 1/2$. Define $\mu \in \mathbf{M}(G)$ by setting $\mu = \sum_{n=0}^{\infty} 2^{-n}\delta(x_n) * (\mu_1 + \mu_2)$, where $\delta(x)$ denotes the unit point measure at $x \in G$. We then claim that μ has all the required properties.

Indeed our choice of the sequence (x_n) shows that

$$\operatorname{supp} \mu = \bigcup \{x_n + \operatorname{supp} \mu_i: n \in \mathbf{Z}_+, \text{ and } i = 1, 2\},$$

and so m_G (supp $\mu) = 0$. Since $|\hat{\mu}_1 + \hat{\mu}_2|^2 \in L^p(\Gamma)$ for all $p > 1$ and since

$$\mu * \mu = \left\{\sum_{m,n=0}^{\infty} 2^{-m-n}\, \delta(x_m + x_n)\right\} * (\mu_1 + \mu_2) * (\mu_1 + \mu_2),$$

it follows that $|\hat{\mu}|^2 \in L^p(\Gamma)$ for all $p > 1$. Now let $g \in A(G)$ be the Radon-Nikodym derivative of $\mu_1 * \mu_2$ with respect to m_G, and set

$$f = g * \left(\sum_{n=0}^{\infty} 2^{-n}\, \delta(x_n)\right).$$

Then we have

$$\|\xi * \xi - g\|_\infty \leq \|\xi * \xi - g\|_A = \|\hat{\xi}^2 - \hat{\mu}_1\hat{\mu}_2\|_1 < 1/2,$$

so that $g(x) > (\xi * \xi)(x) - 1/2 \geq 1/2$ for all $x \in [-1, 1]^N \times H$. Therefore $f(x) > 0$ for all $x \in F$ by the definition of f and our choice of (x_n). Finally we have

$$\mu * \mu \geq \left(\sum_{n=0}^{\infty} 2^{-n}\, \delta(x_n)\right) * \mu_1 * \mu_2 = f m_G,$$

which completes the proof.

6. Some lemmas

In this section, we shall establish some lemmas which will be needed in the proof of Theorem 2.4. For $K \subset G$ and $n \in \mathbf{Z}_+$, we shall write $(K)_n = 0$ if $n = 0$ and $(K)_n = [K \cup \{0\} \cup (-K)] + (K)_{n-1}$ if $n \geq 1$. For $f \in L^1(G)$, we shall define $f^{(n)} = \delta(0)$ if $n = 0$ and $f^{(n)} = f * f^{(n-1)}$ if $n \geq 1$.

LEMMA 6.1. *Suppose that $f_1, f_2, \ldots, f_r$ are finitely many functions in $L^1_+(G)$, that D is a compact subset of G having zero Haar measure, that Y is a compact* *subset of* Γ, *and that* $n_1, n_2, \ldots, n_r \in \mathbf{Z}_+$. *Given* $\varepsilon > 0$, *there exist simple functions* $F_1, F_2, \ldots, F_r \in L^1_+(G)$ *such that, for* $j \in \{1, 2, \ldots, r\}$,

(i) $\operatorname{supp} F_j \subset \operatorname{supp} f_j$,
(ii) $\|F_j\|_1 = \|f_j\|_1$,
(iii) $m_G[D + \sum_{j=1}^r (\operatorname{supp} F_j)_{n_j}] < \varepsilon$,
(iv) $|\hat{f}_j(\gamma) - \hat{F}_j(\gamma)| < \varepsilon$ *for all* $\gamma \in Y$.

Proof. There is no loss of generality in assuming that $\|f_j\|_1 = 1$ and $K_j = \operatorname{supp} f_j$ is compact ($j = 1, 2, \ldots, r$).

For each j, let $\{K_{j1}, K_{j2}, \ldots, K_{jN}\}$ be a finite partition of K_j into disjoint Borel sets such that $m_G(K_{jk}) > 0$ and

$$x, x' \in K_{jk} \quad \text{and} \quad \gamma \in Y \Rightarrow |\gamma(x) - \gamma(x')| < \varepsilon/2 \tag{6.1.1}$$

for all $k = 1, 2, \ldots, N$. Then we note that $\mu_{jk} \in \mathbf{M}^+(K_{jk})$ and $\mu_{jk}(K_{jk}) = \int_{K_{jk}} f_j \, dx$ imply

$$\left| \hat{f}_j(\gamma) - \sum_{k=1}^N \hat{\mu}_{jk}(\gamma) \right| < \varepsilon \quad \text{for all } \gamma \in Y. \tag{6.1.2}$$

(See the proof of Lemma 3.2.) We then select compact subsets L_{jk} of K_{jk}, with $m_G(L_{jk}) > 0$, so that

$$m_G \left[D + \sum_{j=1}^r \left(\bigcup_{k=1}^N L_{jk} \right)_{n_j} \right] < \varepsilon. \tag{6.1.3}$$

This is possible, because (6.1.3) is satisfied if the "diameter" of every L_{jk} is sufficiently small (see Lemma 1 of [5]). Setting

$$F_j = \sum_{k=1}^N \left[m_G(L_{jk})^{-1} \int_{K_{jk}} f_j \, dx \right] \xi(L_{jk}) \quad (j = 1, 2, \ldots, r),$$

we can easily prove that the F_j have all the required properties. This completes the proof.

LEMMA 6.2. *Let* $D \subset G$ *and* $Y \subset \Gamma$ *be as in Lemma* 6.1, *and let* $\varepsilon > 0$ *be given. Suppose that* $f_1, f_2, \ldots, f_r$ *are finitely many functions in* $L^1_+ \cap L^\infty(G)$, *and that* $M_1, M_2, \ldots, M_r$, $S \in \mathbf{Z}_+$ *satisfy* $M_j \leq S$ *for all* $j = 1, 2, \ldots, r$. *Then there exist r functions* $g_j \in L^1_+ \cap L^\infty(G)$, $j \in \{1, 2, \ldots, r\}$, *such that:*

(i) $\operatorname{supp} g_j \subset \operatorname{supp} f_j$;
(ii) $\|g_j\|_1 = \|f_j\|_1$;
(iii) $m_G[D + \sum_{j=1}^r (\operatorname{supp} g_j)_{M_j}] < \varepsilon$;
(iv) $|\hat{f}_j(\gamma) - \hat{g}_j(\gamma)| < \varepsilon$ *for all* $\gamma \in Y$;
(v) $\|f_1^{(N_1)} * \cdots * f_r^{(N_r)} - g_1^{(N_1)} * \cdots * g_r^{(N_r)}\|_1 < \varepsilon$ *for all* $(N_1, \ldots, N_r) \in \{0, 1, \ldots, S\}^r$ *such that* $N_j > M_j$ *for some* $j = 1, 2, \ldots, r$.

Proof. For typographical reason, we shall often write in this proof, for instance, $M(j) = M_j$ and $f^N = f^{(N)}$. There is no loss of generality in assuming that $\|f_j\|_1 = 1$ and $\operatorname{supp} f_j$ is compact for all $j = 1, 2, \ldots, r$. Taking Lemma 3.1 into account, we may also assume that $M_j \geq 1$ for all j, if necessary, by setting $g_j = f_j$ for those j with $M_j = 0$. Given $i \in \{1, 2, \ldots, r\}$, let $P(i)$ be the following assertion: there exist $g_1, g_2, \ldots, g_r \in L^1_+ \cap L^\infty(G)$ which satisfy (i)–(iv) and $(\mathrm{v})_i$

$$\|f_1^{N_1} * \cdots * f_i^{N_i} - g_1^{N_1} * \cdots * g_i^{N_i}\|_1 < \varepsilon$$

for all $(N_1, \ldots, N_i) \in \{0, 1, \ldots, S\}^i$ such that $N_j > M_j$ for some $j = 1, 2, \ldots, i$. Notice that $P(r)$ is nothing but the required conclusion. We shall prove the above assertions by induction on i.

Let $P(0)$ denote the conclusion of Lemma 6.1. Suppose that $i \in \{1, 2, \ldots, r\}$ and that $P(i-1)$ is true. We choose and fix a natural number $T > M_i$ so that

$$A \cdot (M_i/T)^{M_i+1} < \varepsilon/8 \quad \text{where } A = \binom{T}{M_i}. \tag{6.2.1}$$

Let $I_1, I_2, \ldots, I_A$ be the distinct subsets of $\{1, 2, \ldots, T\}$ each of which comprises distinct M_i elements. For each $a = 1, 2, \ldots, A$, we shall construct $r - 1 + T$ functions in $L^1_+ \cap L^\infty(G)$,

$$G_{ja}\ (j \in \{1, 2, \ldots, r\}\backslash\{i\}) \quad \text{and} \quad F_{ta}\ (t \in \{1, 2, \ldots, T\}),$$

as follows. First take a real positive number δ less than ε and a compact subset Y_0 of Γ containing Y; they may be arbitrary but will satisfy some requirements which will be made later. Next set $G_{j0} = f_j$ for $j \neq i$, and select any $F_{t0} \in L^1_+ \cap L^\infty(G)$ so that

$$f_i = F_{10} + F_{20} + \cdots + F_{T0}$$

and

$$\|F_{t0}\|_1 = \|f_1\|_1/T = 1/T \quad \text{for all } t = 1, 2, \ldots, T.$$

We may assume that the F_{t0} have disjoint supports.

Suppose that $a \in \{1, 2, \ldots, A\}$, and that the functions $G_{j(a-1)}$ and $F_{t(a-1)}$ have been defined; we demand that $\|G_{j(a-1)}\|_1 = 1$ and $\|F_{t(a-1)}\|_1 = 1/T$. Set

$$F_{ta} = F_{t(a-1)} \quad \text{for all } t \in \{1, 2, \ldots, T\}\backslash I_a. \tag{6.2.2}$$

By $P(i-1)$, we can find $r - 1 + M_i$ functions G_{ja} $(j \neq i)$ and F_{ta} $(t \in I_a)$ in $L^1_+ \cap L^\infty(G)$ which satisfy the following five conditions:

(6.2.3) $\operatorname{supp} F_{ta} \subset \operatorname{supp} F_{t(a-1)}$, $\|F_{ta}\|_1 = 1/T$;
(6.2.4) $\operatorname{supp} G_{ja} \subset \operatorname{supp} G_{j(a-1)}$, $\|G_{ja}\|_1 = 1$;
(6.2.5) $m_G[D + \sum_{j\neq i} (\operatorname{supp} G_{ja})_{M_j} + (\bigcup_{t\in I_a} \operatorname{supp} F_{ta})_{M_i}] < \varepsilon/A$;
(6.2.6) $\sum_{j\neq i} |\hat{G}_{j(a-1)} - \hat{G}_{ja}| + |\sum_{t\in I_a} (\hat{F}_{t(a-1)} - \hat{F}_{ta})| < \delta/A$ on Y_0;
(6.2.7) $\|G_{1(a-1)}^{N(1)} * \cdots * G_{(i-1)(a-1)}^{N(i-1)} - G_{1a}^{N(1)} * \cdots * G_{(i-1)a}^{N(i-1)}\|_1 < \varepsilon/(2A)$

for all $(N_1, \ldots, N_{i-1}) \in \{0, 1, \ldots, S\}^{i-1}$ such that $N_j > M_j$ for some $j = 1, 2, \ldots, i-1$.

This completes the induction on a.

Now we claim that $g_j = G_{jA}$ $(j \neq i)$ and $g_i = F_{1A} + \cdots + F_{TA}$ satisfy (i), (ii), (iii), and (iv). Parts (i) and (ii) are obvious by (6.2.2), (6.2.3), and (6.2.4). To confirm (iii), we first note that $\operatorname{supp} g_j \subset \operatorname{supp} G_{ja}$ and $\operatorname{supp} F_{tA} \subset \operatorname{supp} F_{ta}$ for all $j \neq i$, all $t \in \{1, 2, \ldots, T\}$, and all $a \in \{1, 2, \ldots, A\}$. Let x be an arbitrary element of $(\operatorname{supp} g_i)_{M_i}$. Since $\operatorname{supp} g_i$ is the union of all $\operatorname{supp} F_{tA}$, $t \in \{1, 2, \ldots, T\}$, there are M_i elements $t(1), \ldots, t(M_i)$ of $\{1, 2, \ldots, T\}$ such that

$$x \in \sum_{k=1}^{M(i)} (\operatorname{supp} F_{t(k)A})_1 \subset \left(\bigcup_{k=1}^{M(i)} \operatorname{supp} F_{t(k)A}\right)_{M_i}.$$

Consequently there exists $a \in \{1, 2, \ldots, A\}$ such that

$$x \in \left(\bigcup \{\operatorname{supp} F_{ta} \colon t \in I_a\}\right)_{M_i}.$$

Hence we have

$$D + \sum_{j=1}^{r} (\operatorname{supp} g_j)_{M_j} \subset \bigcup_{a=1}^{A} \left[D + \sum_{j \neq i} (\operatorname{supp} G_{ja})_{M_j} + \left(\bigcup_{t \in I_a} \operatorname{supp} F_{ta}\right)_{M_i}\right].$$

This fact, combined with (6.2.5), yields (iii). Part (iv) is an easy consequence of (6.2.2) and (6.2.6); in fact, we have

(6.2.8) $\quad |\hat{f}_j(\gamma) - \hat{g}_j(\gamma)| < \delta \quad$ for all $\gamma \in Y_0$ and all $j \in \{1, 2, \ldots, r\}$.

We shall now state the requirements for δ and Y_0 that assure the validity of $(\mathrm{v})_i$. First we can and do demand by virtue of Lemma 3.1 that (6.2.3) and (6.2.6) imply

$$\sum_{m=1}^{N} \binom{N}{m} \left\| \left\{\sum_{(t)} F_{t(a-1)}\right\}^{(N-m)} * \left\{\sum_{(t)}' F_{t(a-1)}\right\}^{(m)} - \left\{\sum_{(t)} F_{ta}\right\}^{(N-m)} * \left\{\sum_{(t)}' F_{t(a-1)}\right\}^{(m)} \right\|_1 < \varepsilon/(4A)$$

for all $N = 1, 2, \ldots, S$ and all $a = 1, 2, \ldots, A$, where the sums $\sum_{(t)}$ and $\sum'_{(t)}$ are taken over all $t \in I_a$ and over all $t \in \{1, 2, \ldots, T\} \backslash I_a$, respectively. We therefore infer from (6.2.2) and (6.2.3) that $M_i < N \leq S$ imply

$$\begin{aligned}
\left\| \left\{\sum_{t=1}^{T} F_{t(a-1)}\right\}^{(N)} - \left\{\sum_{t=1}^{T} F_{ta}\right\}^{(N)} \right\|_1
&< \varepsilon/(4A) + \left(\left\|\sum_{(t)} F_{t(a-1)}\right\|_1\right)^N + \left(\left\|\sum_{(t)} F_{ta}\right\|_1\right)^N \\
&= \varepsilon/(4A) + 2(M_i/T)^N \\
&\leq \varepsilon/(4A) + 2(M_i/T)^{M_i+1} \quad \text{for all } a = 1, 2, \ldots, A.
\end{aligned}$$

These inequalities, combined with (6.2.1), yield

(6.2.9) $\quad \|f_i^{N_i} - g_i^{N_i}\|_1 < \varepsilon/4 + 2A(M_i/T)^{M_i+1} < \varepsilon/2$

for all $N_i = M_i + 1, \ldots, S$. Next we note that (6.2.7) implies

$$\|f_1^{N(1)} * \cdots * f_{i-1}^{N(i-1)} - g_1^{N(1)} * \cdots * g_{i-1}^{N(i-1)}\|_1 < \varepsilon/2 \tag{6.2.10}$$

for all $(N_1, \ldots, N_{i-1}) \in \{0, 1, \ldots, S\}^{i-1}$ such that $N_j > M_j$ for some $j = 1, 2, \ldots, i-1$. Now take an arbitrary $(N_1, \ldots, N_i)$ of $\{0, 1, \ldots, S\}^i$ such that $N_j > M_j$ for some $j = 1, \ldots, i$. If $N_i > M_i$, we have, by (6.2.9),

$$\begin{aligned}&\|f_1^{N(1)} * \cdots * f_i^{N(i)} - g_1^{N(1)} * \cdots * g_i^{N(i)}\|_1 \\ &\qquad < \|\{f_1^{N(1)} * \cdots * f_{i-1}^{N(i-1)} - g_1^{N(1)} * \cdots * g_{i-1}^{N(i-1)}\} * f_i^{N(i)}\|_1 + \varepsilon/2,\end{aligned}$$

since $\|f_j\|_1 = \|g_j\|_1 = 1$ for all j. If $N_j > M_j$ for some $j < i$, then the left-hand side of the last inequality is less than

$$\|f_1^{N(1)} * \cdots * f_{i-1}^{N(i-1)} * \{f_i^{N(i)} - g_i^{N(i)}\}\|_1 + \varepsilon/2$$

by (6.2.10). Therefore Lemma 3.1 and (6.2.8) show that $(\mathrm{v})_i$ holds for appropriate choices of δ and Y_0. This completes the induction on i and hence the proof.

LEMMA 6.3. *Let $D \subset G$, $Y \subset \Gamma$, and $f_1, f_2, \ldots, f_r \in L_+^1 \cap L^\infty(G)$ be as in Lemma 6.2, and let $\varepsilon > 0$ be given. Suppose that S is a natural number and that E is a dominative subset of $\mathbf{Z}_+^r$ contained in $\{0, 1, \ldots, S\}^r$. Then there exist r functions $g_1, g_2, \ldots, g_r$ in $L_+^1 \cap L^\infty(G)$ such that:*

(i) $\operatorname{supp} g_j \subset \operatorname{supp} f_j$;
(ii) $\|g_j\|_1 = \|f_j\|_1$;
(iii) *If $(m_1, \ldots, m_r)$ is in E, then* $m_G[D + \sum_{j=1}^r (\operatorname{supp} g_j)_{m_j}] < \varepsilon$;
(iv) $|\hat{f}_j(\gamma) - \hat{g}_j(\gamma)| < \varepsilon$ *for all* $\gamma \in Y$;
(v) *If $(n_1, \ldots, n_r)$ is in $\{0, 1, \ldots, S\}^r \backslash E$, then*

$$\|f_1^{(n_1)} * \cdots * f_r^{(n_r)} - g_1^{(n_1)} * \cdots * g_r^{(n_r)}\|_1 < \varepsilon.$$

Moreover, there exist $2r$ functions $f_{1i}, f_{2i}, \ldots, f_{ri} \in L_+^1 \cap L^\infty(G)$, $i \in \{0, 1\}$, such that, for each i, the functions f_{ji} satisfy the above five conditions with $g_j = f_{ji}$, and such that:

(vi) $\operatorname{supp} f_{j0} \cap \operatorname{supp} f_{j1} = \emptyset$,
(vii) $\|f_j * f_k - f_{j0} * f_{k1}\|_p < \varepsilon$ $(p \in [1, S])$ *for all* $j, k = 1, 2, \ldots, r$.

Proof. The first assertion is an easy consequence of Lemma 6.2. In fact, we first select any $M = (m_j) \in E$, and find r functions $g_1, \ldots, g_r \in L_+^1 \cap L^\infty(G)$ satisfying the conclusion of Lemma 6.2. Next select any $M' = (m_j') \in E \backslash \{M\}$ and apply Lemma 6.2 (to $g_1, \ldots, g_r$ and M') to find appropriate $g_1', \ldots, g_r'$. Repeating this process, we obtain r functions which have all the required properties with ε replaced by $(\operatorname{Card} E) \cdot \varepsilon$.

To prove the second assertion, we argue as follows. We may assume that $\|f_j\|_1 = 1$ for all $j = 1, 2, \ldots, r$. By the absolute continuity of indefinite integral, there exists $\delta \in (0, \varepsilon)$ such that if $h_j \in L^1_+(G)$, $h_j \leq f_j$, and $m_G(\{f_j \neq h_j\}) < \delta$, then $\|f_j - h_j\|_p < \varepsilon$ for all $p \in [1, S]$ and all $j = 1, 2, \ldots, r$. Let $f_{10} = g_1, \ldots, f_{r0} = g_r$ be as in the first assertion of the present lemma. We can demand that $m_G\,[\operatorname{supp} f_{j0}] < \delta$, and also (by Lemma 3.1) that $\|f_j * f_k - f_j * f_{k0}\|_p < \varepsilon$ for all $p \in [1, S]$ and all $j, k = 1, 2, \ldots, r$. Define $f_j' \in L^1(G)$ by setting $f_j' = 0$ on $\operatorname{supp} f_{j0}$ and $f_j' = f_j$ on $G\backslash(\operatorname{supp} f_{j0})$ for $j = 1, 2, \ldots, r$. Applying the first assertion to the $f_j'/\|f_j'\|_1$, we can find r functions $f_{11}, f_{21}, \ldots, f_{r1}$ which satisfy (i)–(v) with the f_j replaced by the $f_j'/\|f_j'\|_1$. We may assume that $\operatorname{supp} f_{j0} \cap \operatorname{supp} f_{j1} = \emptyset$ and also (by Lemma 3.1) that

$$\|(f_j'/\|f_j'\|_1) * f_{k0} - f_{j1} * f_{k0}\|_p < \varepsilon$$

for all $p \in [1, S]$ and all $j, k = 1, 2, \ldots, r$. Then the $2r$ functions f_{ji} obtained in this way satisfy the required conditions with ε replaced by $C\varepsilon$, where C is a finite constant depending only on $f_1, \ldots, f_r$ and S and is independent of ε. This completes the proof.

7. Proofs of Theorem 2.4 and Corollary 2.5

7.1. Let $K_1, \ldots, K_r \subset G$, $\Phi \subset M_s(G)$, and $E \subset \mathbf{Z}^r_+$ be as in Theorem 2.4. Since Φ is a separable subset of $\mathbf{M}_s(G)$, there exists a nonzero measure $\mu_0 \in \mathbf{M}_s^+(G)$ such that every element of Φ is absolutely continuous with respect to μ_0. Let D be a σ-compact subset of G which carries μ_0 and has zero Haar measure. We write $D = \bigcup_{n=1}^{\infty} D_n$, where the D_n are compact subsets of D such that $\emptyset \neq D_n \subset D_{n+1}$ for all $n = 1, 2, \ldots$. For each natural number n, we shall construct $2^n r$ functions $f_{1b}, f_{2b}, \ldots, f_{rb} \in L^1_+ \cap L^\infty(G)$, $b \in \{0, 1\}^n$, as follows.

First choose $2r$ functions $f_{10}, f_{11} \in L^\infty_+(K_1), \ldots, f_{r0}, f_{r1} \in L^\infty_+(K_r)$ such that $\|f_{ji}\|_1 = 1$ for all j and i, and such that

$$(\operatorname{supp} f_{ji}) \cap (\operatorname{supp} f_{kl}) = \emptyset$$

unless $(j, i) = (k, l)$. Suppose that n is a natural number, and that the functions $f_{jb} \in L^\infty_+(K_j)$ have been defined for all $j = 1, 2, \ldots, r$, and all $b \in \{0, 1\}^k$, $k \leq n$. We assume that all the f_{jb} have L^1-norm 1 and disjoint supports. Setting

$$Y_n = \bigcup_{k=1}^{n} \bigcup_{j=1}^{r} \{\gamma \in \Gamma : |\hat{f}_{jb}(\gamma)| \geq n^{-1} \text{ for some } b \in \{0, 1\}^k\},$$

we repeatedly apply Lemmas 6.3 and 3.1 to find $2^{n+1} r$ functions $f_{jbi} \in L^\infty_+(K_j)$ which satisfy the following conditions for $j, k \in \{1, 2, \ldots, r\}$, $b, c \in \{0, 1\}^n$, and $i, l \in \{0, 1\}$:

(7.1.1) $\operatorname{supp} f_{jb0}$ and $\operatorname{supp} f_{jb1}$ are disjoint subsets of $\operatorname{supp} f_{jb}$;

(7.1.2) $\|f_{jb0}\|_1 = \|f_{jb1}\|_1 = 1$;

(7.1.3) $m_G[D_n + \sum_{j=1}^{r} (\operatorname{supp} f_{jbi})_{m_j}] < 4^{-n}$ for all

$$(m_1, \ldots, m_r) \in E \cap \{0, 1, \ldots, n\}^r;$$

(7.1.4) $|\hat{f}_{jb}(\gamma) - \hat{f}_{jbi}(\gamma)| < 2^{-n}$ for all $\gamma \in Y_n$;
(7.1.5) $\|f_{1b}^{(n_1)} * \cdots * f_{rb}^{(n_r)} - f_{1bi}^{(n_1)} * \cdots * f_{rbi}^{(n_r)}\|_1 < 2^{-n}$ for all

$$(n_1, \ldots, n_r) \in \{0, 1, \ldots, n\}^r \backslash E;$$

(7.1.6) $\|f_{jb} * f_{kc} - f_{jbi} * f_{kcl}\|_p < 2^{-n}$ for all $p \in [1, n]$ unless $(b, i) = (c, l)$.

This completes the induction.

Let $j \in \{1, 2, \ldots, r\}$ and $b = (b_1, b_2, \ldots) \in \mathbf{D}(2)$ be given. Writing $b(n) = (b_1, \ldots, b_n) \in \{0, 1\}^n$ for all $n = 1, 2, \ldots$, we infer from (7.1.1), (7.1.2), and (7.1.4) that the sequence $(f_{jb(n)})_{n=1}^{\infty}$ converges weak* to a probability measure $\mu_{jb} \in \mathbf{M}(K_j)$. The supports of the measures $\mu_{jb}, j \in \{1, 2, \ldots, r\}$ and $b \in \mathbf{D}(2)$, are disjoint by (7.1.1), and the closure of their union has zero Haar measure by (7.1.3). (Notice that the dominative set E contains the fundamental unit vectors $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$.) Moreover, (7.1.1) and (7.1.3) show that

$$m_G\left[D + \sum_{j=1}^{r} (\operatorname{supp} \mu_{jb})_{m_j}\right] = 0 \quad \text{for all } b \in \mathbf{D}(2) \text{ and } (m_j) \in E.$$

Since every $\nu \in \Phi$ is carried by $\mathbf{D}$, it follows that the measure

$$\nu * \mu_{1b}^{m_1} * \cdots * \mu_{rb}^{m_r}$$

is singular with respect to m_G for all b and (m_j) as above. This establishes part (b). Finally parts (c) and (d) are easily seen from (7.1.5) and (7.1.6), which completes the proof of Theorem 2.4.

To prove Corollary 2.5, it suffices to put $K_1 = \cdots = K_r = K$, $\Phi = \{\delta(0)\}$, and $E = \{0, 1, \ldots, n-1\}^r$ in Theorem 2.4 (see also the above proof).

7.2. Theorem 2.4 holds for every nondiscrete locally compact (not necessarily, abelian) group. Moreover, in this case, the orders of the convolution products in parts (b) and (c) may be arbitrary. We omit the details.

8. Remarks

Theorem 2.4 has various refinements. We state three of them without proofs in 8.1, 8.2, and 8.3.

8.1. When G is metrizable, the measures μ_{jb} in Theorem 2.4 can be so chosen as to satisfy the following additional condition: if $(n_1, \ldots, n_r)$ and $(2n_1, \ldots, 2n_r)$ are in E, and if $\mu_1, \ldots, \mu_r \in \mathbf{M}(G)$ satisfy $\mu_j \ll \mu_{jb}$ for all $j = 1, 2, \ldots, r$ and some $b \in \mathbf{D}(2)$, then

$$\|\nu * \mu_1^{n_1} * \cdots * \mu_r^{n_r}\| = \|\nu\| \, \|\mu_1\|^{n_1} \cdots \|\mu_r\|^{n_r}$$

for all $\nu \in \Phi$ (cf. Lemma 3 of [6]).

8.2. Given $\mu \in \mathbf{M}(G)$, let $\tilde{\mu}$ be the measure defined by the requirement $\tilde{\mu}(K) = \overline{\mu(-K)}$ for all Borel sets K in G. It is possible to construct the measures μ_{jb} in Theorem 2.4 so that the $\nu_{jb} = (\mu_{jb} + \tilde{\mu}_{jb})/2$ have properties (b), (c), and (d) of Theorem 2.4. An interesting consequence of this fact is that there exists a singular measure $\mu \in L^{1/3}(G)$ such that $\mu^2 \notin \mathbf{M}_a(G)$ but $\mu * \tilde{\mu} \in \mathbf{M}_a(G)$. To see this, we apply Theorem 2.4 and the above fact with $r = 2$ and $E = \{0, 1\}^2$ to find two probability measures $\nu_1, \nu_2 \in L^{1/2}(G) \cap \mathbf{M}_s(G)$ such that $\tilde{\nu}_1 = \nu_1$, $\tilde{\nu}_2 = \nu_2$, and $\nu_1 * \nu_2 \in \mathbf{M}_s(G)$. Then $\mu = \nu_1 + i\nu_2$ satisfies $\mu^3 \in \mathbf{M}_a(G)$, $\mu^2 \notin \mathbf{M}_a(G)$, and $\mu * \tilde{\mu} \in \mathbf{M}_a(G)$.

8.3. Let $q = q(G)$ denote the largest element of $\{2, 3, \ldots, \infty\}$ such that every neighborhood of $0 \in G$ contains an element of order q. If G is metrizable, then the measures μ_{jb} in Theorem 2.4 can be so constructed as to satisfy both the conditions stated in 8.1 and 8.2 ,and also the following one: suppose (i) s is a natural number, (ii) x_{jk} ($k = 1, \ldots, s$) are different s elements of supp μ_{jb} for all $j = 1, 2, \ldots, r$ and some $b \in \mathbf{D}(2)$, and (iii) (m_{jk}) is a nonzero $r \times s$ matrix of integers, each with modulus $<q$, such that

$$(\textstyle\sum_k |m_{1k}|, \ldots, \sum_k |m_{rk}|) \in E;$$

then $\sum_{jk} m_{jk}x_{jk} \neq 0$.

8.4. There exists a nonzero measure $\mu \in L^{1/5}(G)$ such that μ and $\nu_1 * \nu_2$ are mutually singular for all $\nu_1, \nu_2 \in \mathbf{M}_0(G)$, where $\mathbf{M}_0(G) = \{\nu \in \mathbf{M}(G)\colon \hat{\nu} \in \mathbf{C}_0(\Gamma)\}$. A sketch of the proof of this fact is as follows.

First assume that G is metrizable. By Remark 8.3 with $r = 1$ and $E = \{0, 1, 2, 3, 4\}$, there exists a probability measure $\mu \in L^{1/5}(G)$ such that if $x_1, \ldots, x_4$ are different four elements of $K = \text{supp}\, \mu$, then $x_1 \pm x_2 \pm x_3 \pm x_4 \neq 0$. To force a contradiction, suppose that $(\nu_1 * \nu_2)(K) \neq 0$ for some continuous measures ν_1 and ν_2 in $\mathbf{M}(G)$. Then an easy application of the Fubini theorem yields four elements a_1, a_2, b_1, b_2 of G such that $a_i + b_j \in K$ for all $i, j \in \{1, 2\}$ and $a_i + b_j \neq a_k + b_l$ unless $(i, j) = (k, l)$. But we have $(a_1 + b_1) + (a_2 + b_2) - (a_1 + b_2) - (a_2 + b_1) = 0$, which gives us the desired contradiction. Therefore we have shown that $\mu \perp \nu_1 * \nu_2$ for all continuous measures ν_1 and ν_2. To prove the general case, notice that if $\nu \in \mathbf{M}_0(G)$ and H is a closed nonopen subgroup of G, then we have $|\nu|(H) = 0$, as is easily seen. It therefore suffices to consider any nondiscrete metrizable quotient of G. We do not know if there is a nonzero measure $\mu \in L^{1/2}(G)$ with the above property.

8.5. Let $(n_k)_1^\infty$ be a sequence of natural numbers, and D a σ-compact subset of G with zero Haar measure. Then there exists a sequence $(\mu_k)_1^\infty$ of probability measures in $\mathbf{M}(G)$ such that (a) $\mu_k^{n_k+1} \in \mathbf{M}_a(G)$ for all $k = 1, 2, \ldots$, (b) the infinite convolution product $\mu_1^{m_1} * \mu_2^{m_2} * \cdots$ converges weak* to a probability measure $\mu_m \in \mathbf{M}(G)$ whenever $m = (m_k)$ is a sequence of non-negative integers such that $m_k \leq n_k$ for all k, and (c) $m_G[D + (\operatorname{supp} \mu_m)] = 0$ for every sequence m as in (b). This can be proved by modifying the proof of Theorem 2.4.

8.6. The methods in Section 6, combined with Cohen's idea in [1], yield this result: let $1 \leq p \leq q < \infty, f \in L^p \cap L^q(G)$, n a natural number, and $\varepsilon > 0$; then there exist $g \in L^p \cap L^q(G)$ and a probability measure $\mu \in L^{1/n}(G)$ such that (a) $f = g * \mu$, (b) $m_G[\operatorname{supp}(\mu^{n-1})] = 0$, and (c) $\|f - g\|_p + \|f - g\|_q < \varepsilon$. A similar assertion holds even if the space $L^p \cap L^q(G)$ is replaced by any one of the following spaces; $A(G)$, $\mathbf{C}_0(G)$, and $PF(G) = [\mathbf{C}_0(\Gamma)]^\wedge$—the space of all pseudofunctions on G.

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Tokyo Metropolitan University
Setagaya, Tokyo