GEOMETRICAL PROPERTIES DETERMINED BY THE HIGHER DUALS OF A BANACH SPACE

BY

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1. Introduction

Let X be a Banach space and X^*, X^{**}, X^{***}, and $X^{(4)}$ the successive dual spaces. We denote by J_0, J_1, and J_2 the natural embeddings of X, X^*, and X^{**} into X^{**}, X^{***}, and $X^{(4)}$ respectively. When no confusion can result we shall omit these maps and write, for example, $x \in X^{**}$ to mean $J_0(x) \in X^{**}$.

Among the consequences of a result of Dixmier [5] is the fact that if $x^{**} \in X^{**}$ then

$$\|J_2(x^{**}) - J_0^{**}(x^{**})\| \geq \operatorname{dist}(x^{**}, X).$$

It is easy to verify that

$$J_2(x^{**})|_{X^*} = J_0^{**}(x^{**})|_{X^*}$$

and so if $x^{**}(x^*) = 1$ where $\|x^{**}\| = \|x^*\| = 1$ and $x^{**} \notin X$, then $J_1(x^*)$ has two distinct norming elements in $X^{(4)}$. Since by a famous theorem of James [8], [9] such an x^{**} and x^* must exist if X is not reflexive we have that if X is not reflexive then X^{***} is not smooth and $X^{(4)}$ is not rotund.

It is clear from this formulation of Dixmier's theorem that reflexivity is implied by a condition on X weaker than X^{***} smooth, involving only the behavior of X^* in X^{***} and X in X^{**}. This suggests the possibility of studying geometrical properties of Banach spaces determined by viewing the spaces as subspaces of their second duals. In this paper we define several such properties and explore the connections among them and other geometrical notions.

Section 2 contains various background information. In particular we examine Dixmier's theorem in a little more detail with a view to motivating the definitions which are taken up in the later sections.

Since our properties are defined using the second and third dual spaces, the fact that B (the unit ball of X) is weak* dense in B^{**} (the unit ball of X^{**}) and Helly's theorem are used extensively. Particularly useful is the elegant combination of Goldstine's theorem and Helly's theorem expressed in the principle of local reflexivity [12]. A version of this result due to Dean [3] is stated in Section 2.

One of the principal themes of this work is the exploitation of the observation that there is a dichotomy of geometrical conditions determined by which

Received January 13, 1976.

variable of the difference quotients defining the derivative of the norm is uniformized. Thus dualities of geometrical conditions fall into two groups such as "Fréchet differentiable norm" and "locally uniformly rotund" on the one hand and "uniformly Gateaux differentiable" and "uniformly convex in each direction" on the other. A tool for obtaining such pairs of dualities are elementary inequalities which were used by Lindenstrauss [11] in studying (UR) and (US) spaces. These are given in Section 2.

A second main idea is that uniform conditions on the geometry of X induce uniqueness conditions on the properties of X in X^{**}. Both local reflexivity and the inequalities are used to obtain results of this sort.

In Section 3 we define three related conditions: *X very smooth*, *X very rotund*, and *X Hahn-Banach smooth* all of which are taken more or less directly from the statement of Dixmier's theorem. The main results concern uniform geometric conditions which imply these properties and the connection between these properties and reflexivity and the Radon-Nikodym property.

Scattered throughout this section are various remarks and examples concerning the extent to which the implications can or cannot be reversed. However, we break with custom and do not collect what we have been unable to prove or give a counter-example for in a section of problems and questions. The open problems and missing counter examples are obvious.

It should be mentioned that these properties are all "isometric" conditions (i.e., not preserved under isomorphisms) although they do imply isomorphic conditions like the Radon-Nikodym property. It is possible to define an isomorphic smoothness condition dual to the Radon-Nikodym property [17] and we hope to deal with the "second dual" version of this elsewhere.

In Section 4 we switch to the "other variable" to define conditions *X extremely smooth* and *X extremely rotund* and carry out roughly the same program as in the previous section for these properties. Results here, however, are somewhat less complete probably reflecting the comparative dearth of information about properties related to uniform Gateaux differentiability of the norm.

In Section 5 we change the point of view slightly and consider a condition on X which is equivalent to strong rotundity properties in X^*. This is a strengthening of Vlasov's necessary and sufficient condition on X for X^* to be rotund [19]. We characterize this property in terms of rotundity of the dual and the smoothness properties of Section 3. Combining the results of this section with those of Section 3 we obtain a necessary and sufficient condition for a Banach space to have a separable dual space.

In all we consider five new geometrical properties and the relations among them and various previously defined properties. In an effort to minimize the possibility of confusion we summarize some of our main theorems in Section 6 in the form of charts of implication. In some sense these charts (especially the second) can be considered to be one of the main results of this paper. We wish to thank the referee for suggesting this form of presentation and for improving Lemma 5.2.

2. Preliminaries

It will be convenient for us to regard an arbitrary Banach space as a subspace of its second dual. Thus, in particular $X^* \subseteq X^{***}$ and in fact $X^{***} = X^* \oplus X^\perp$ as is well known. Moreover, for $x \in X^*$ and $x^\perp \in X^\perp$ we have $\|x^* + x^\perp\| \geq \|x^*\|$ because X^* is the range of the contractive projection $J_1 J_0^*$ on X^{***}.

Passing to dual spaces we have that $X^{(4)} = X^{**} \oplus X^{*\perp}$ where X^{**} is the range of $J_2 J_1^*$. However, we can also write

$$X^{(4)} = X^{\perp\perp} \oplus X^{*\perp} \quad \text{where } X^{\perp\perp} \text{ is the range of } J_0^{**}J_1^* = (J_1 J_0^*)^*.$$

Dixmier [5] proved that $X^{**} \cap X^{\perp\perp} = X$ so that, if X is not reflexive, $X^{*\perp}$ is the null manifold of two distinct contractive projections. This implies that $X^{(4)}$ is not rotund [1]. In fact, for any norm-1 $x^{**} \in X^{**}$, the line segment joining the norm-1 vectors $J_0^{**}(x^{**})$ and $J_2(x^{**}) = x^{**}$ lies on the surface of the unit ball of $X^{(4)}$. As was mentioned earlier it can be shown that $J_0^{**}(x^{**}) - x^{**} \in X^{*\perp}$ and if $x^{**} \notin X$, the vectors are distinct.

Hence, if X is not reflexive there are vectors x^{**} and $x^{*\perp} \neq 0$ with

$$\|x^{**} + x^{*\perp}\| = \|x^{**}\| = 1$$

so that $x^{**}|_{X^*}$ has two distinct Hahn-Banach extensions. Moreover, from James' theorem [9] we can even find vectors such that

$$\|x^{**} + x^{*\perp}\| = 1 = (x^{**} + x^{*\perp})(x^*) = x^{**}(x^*) = \|x^{**}\| = \|x^*\|,$$

so that $J_1(x^*) \in X^{***}$ is a norming element for both x^{**} and $J_0^{**}(x^{**})$. The exclusion of this condition will provide the starting point for our definition in Section 3.

The principle of local reflexivity is due to Lindenstrauss and Rosenthal and states, roughly, that the natural map $J_0\colon X \to X^{**}$ can be inverted on finite dimensional subspaces of X^{**}. The precise statement of this result which we will use is due to Dean [3].

PROPOSITION (Lindenstrauss, Rosenthal, Dean). *Let $A \subseteq X^{**}$ and $F \subseteq X^*$ be finite dimensional subspaces and let $0 < \delta < 1$ be arbitrary. Then there exists a linear map $T\colon A \to X$ such that:*

(a) *$T(a) = a$ for all $a \in A \cap X$.*
(b) *$f(T(a)) = a(f)$ for all $a \in A$ and $f \in F$.*
(c) *For all $a \in A$ $(1 - \delta)\|a\| \leq \|T(a)\| \leq (1 + \delta)\|a\|$.*

In order to discuss various smoothness conditions it will be convenient to use the following notation. For $x, y \in X$ let

$$\rho(x, y) \equiv \|x + y\| + \|x - y\| - 2.$$

In this terminology a *smooth* Banach space is one for which

$$\lim_{\lambda \to 0} \rho(x, \lambda y)/\lambda = 0 \quad \text{for all } \|x\| = 1 \text{ and } y \in X.$$

The notion of smoothness can be strengthened in at least two ways. We say that X has *Fréchet differentiable norm* if for each $\|x\| = 1$,

$$\lim_{\|y\| \to 0} \rho(x, y)/\|y\| = 0,$$

in other words the rate of convergence is independent of the direction y. If, on the other hand, we uniformize in the first variable we say that X has a *uniformly Gateaux differentiable norm* if for each $\|y\| = 1$,

$$\lim_{\lambda \to 0} \sup_{\|x\| = 1} \rho(x, \lambda y)/\lambda = 0.$$

The following inequalities are elementary and follow from rearranging terms. For $x, y \in X$ and $\|x^*\| = \|y^*\| = 1$,

$$\rho(x, y) \geq (x^* + y^*)(x) + (x^* - y^*)(y) - 2.$$

For each $x^*, y^* \in X^*$ and $\varepsilon > 0$ there exist $\|x\| = \|y\| = 1$ such that

$$\rho(x^*, y^*) \leq x^*(x + y) + y^*(x - y) + \varepsilon - 2.$$

These basic inequalities will appear in various guises in the sequel.

3. Very smooth spaces

The discussion of Dixmier's Theorem at the beginning of the previous section suggests the following definitions:

(a) A Banach space X is said to be *Hahn-Banach smooth* if in X^{***}, $\|x^* + x^\perp\| = \|x^*\| = 1$ implies that $x^\perp = 0$. In other words $x^* \in X^{***}$ is the unique Hahn-Banach extension of $x^*|_X$.

(b) X is said to be *very smooth* if each $x \in S$ has a unique norming element in X^{***}.

(c) X is said to be *very rotund* if no $x^* \in S^*$ is simultaneously a norming element for some $x \in S$ and $x^{**} \in S^{**}$ where $x^{**} \neq J_0(x)$.

The following implications are obvious and summarize the observations made earlier.

LEMMA 1. (1) *If X^* is very rotund then X is very smooth.*

(2) *If X^* is smooth then X is very rotund.*

(3) *If X^* is very smooth or Hahn-Banach smooth then X is reflexive.*

Generalizing somewhat a result of [1] we have:

THEOREM 2. *For X a Banach space the following are equivalent:*

(1) *X is very smooth.*

(2) *For each $x \in S$ and $y^{**} \in S^{**}$ $\lim_{\lambda \to 0} \rho(x, \lambda y^{**})/\lambda = 0$.*

(3) *Each closed subspace of X is the range of at most one contractive projection in X^{**}.*

Proof. Suppose that X is very smooth. Since the norm is a convex function for each $x \in S$ and $y^{**} \in S^{**}$ we have that

$$n^+(x, y^{**}) \equiv \lim_{\lambda \to 0^+} \frac{\|x + \lambda y^{**}\| - 1}{\lambda} \text{ exists.}$$

Moreover, a simple application of the Hahn-Banach Theorem as in [6, Ch. 5] shows that for each c, if $-n^+(x, -y^{**}) \leq c \leq n^+(x, y^{**})$ then there exists an $\|x^{***}\| = 1$ with $x^{***}(x) = 1$ and $x^{***}(y^{**}) = c$. Hence, we must have that $-n^+(x, -y^{**}) = n^+(x, y^{**})$ which is condition (2).

Assume now that condition (2) holds so that for each $x \in S$ the map $y^{**} \to n(x, y^{**})$ is a norm one linear functional in X^{***}. By the reasoning of [1], if E: $X^{**} \to X^{**}$ is any contractive projection with range a closed subspace of X then for any $x^{**} \neq 0$,

$$n(Ex^{**})(y^{**}) = n(Ex^{**})(Ey^{**})$$

for all $y^{**} \in X^{**}$. If E and F are contractive projections with the same range $M \subseteq X$ then for all x^{**},

$$n(Ex^{**} - Fx^{**})(Ex^{**} - Fx^{**}) = 0$$

because $EF = F$ and $FE = E$.

Finally we show that if X is not very smooth then some (one dimensional) subspace of X is the range of two distinct contractive projections in X^{**}. Namely if x^{***} and y^{***} are distinct norming functionals for some $x \in S$, then

$$x^{**} \to x^{***}(x^{**})x \quad \text{and} \quad x^{**} \to y^{***}(x^{**})x$$

are the projections required. Q.E.D.

THEOREM 3. *If the norm of X is Fréchet differentiable then X is very smooth.*

Proof. From the definition of Fréchet differentiability we have that

$$\lim_{\lambda \to 0} \sup_{\|y\| = 1} \rho(x, \lambda y)/\lambda = 0 \quad \text{for all } x \in S.$$

Goldstine's Theorem and the fact that the norm in X^{**} is weak* lower semi-continuous give $\lim_{\lambda \to 0} \rho(x, \lambda y^{**})/\lambda = 0$ for all $x \in S$ and $y^{**} \in S^{**}$. Hence, from Condition 2 of the previous theorem, X is very smooth. Q.E.D.

It is known that c_0 has an equivalent Fréchet differentiable norm while l^∞ has no equivalent smooth norm [2] so that X very smooth is strictly weaker than X^{**} smooth. We shall give an example at the end of this section to show that it is possible to have X very smooth but not Fréchet differentiable. On the other hand if X is very smooth then X^{**} must possess some degree of smoothness as the following two results show.

LEMMA 4. *If X is very smooth then X^* has the Radon-Nikodym property.*

Proof. Since X is smooth for each $x \in S$ let $n(x) \in S^*$ denote the unique functional such that $n(x)(x) = 1$. It is well known that the map $n: S \to S^*$ is continuous from the norm topology of S to the weak* topology of S^*. Now since X is very smooth this map $x \to n(x)$ can be thought of as a map from S to S^{***} and the standard technique shows that n is continuous to the weak* topology of S^{***} and hence to the *weak* topology of S^*. Hence $n: S \to S^*$ is norm to weak continuous. By a result of Diestel and Faires [4] this implies that X^* has the Radon-Nikodym property. Q.E.D.

Note that this implies that while l^1 has an equivalent smooth norm it has no very smooth norm.

THEOREM 5. *If X is very smooth then for each $\varepsilon > 0$ there is a $\delta > 0$ and an $x^{**} \in S^{**}$ such that $\rho(x^{**}, \lambda y^{**})/\lambda < \varepsilon$ for all $\|y^{**}\| = 1$ and $0 < \lambda \leq \delta$.*

Proof. We shall show that if X^{**} fails the above property then the unit ball of X is not dentable [7] and so X^* fails the Radon-Nikodym property and hence, from Lemma 3, X is not very smooth.

By a result of Phelps [15] we need only show that B_{X^*} does not have slices of small diameter.

Suppose that there is an $\varepsilon > 0$ such that for any $\|x^{**}\| = 1$ and $\delta > 0$ there is a $\|y^{**}\| = 1$ and $0 < \lambda \leq \delta$ with

$$\|x^{**} + \lambda y^{**}\| + \|x^{**} - \lambda y^{**}\| \geq 2 + \lambda\varepsilon.$$

Given any $\|x^{**}\| = 1$ there is a positive sequence (λ_k) decreasing to 0 and $\|y_k^{**}\| = 1$ with

$$\|x^{**} + \lambda_k y_k^{**}\| + \|x^{**} - \lambda_k y_k^{**}\| \geq 2 + \lambda_k\varepsilon.$$

For each k let $x_k^*, y_k^* \in S^*$ where

$$(x^{**} + \lambda_k y_k^{**})(x_k^*) \geq \|x^{**} + \lambda_k y_k^{**}\| - \lambda_k^2$$

and

$$(x^{**} - \lambda_k y_k^{**})(y_k^*) \geq \|x^{**} - \lambda_k y_k^{**}\| - \lambda_k^2.$$

Clearly x_k^* and y_k^* both belong to the $2\lambda_k + \lambda_k^2$ slice of S^* determined by x^{**} and so the diameter of this slice is bounded below by $\|x_k^* - y_k^*\|$.

However

$$\begin{aligned} 2 + \lambda_k\|x_k^* - y_k^*\| &\geq x^{**}(x_k^* + y_k^*) + \lambda_k y_k^{**}(x_k^* - y_k^*) \\ &= (x^{**} + \lambda_k y_k^{**})(x_k^*) + (x^{**} - \lambda_k y_k^{**})(y_k^*) \\ &\geq \|x^{**} + \lambda_k y_k^{**}\| + \|x^{**} - \lambda_k y_k^{**}\| - 2\lambda_k^2 \\ &\geq 2(1 - \lambda_k^2) + \lambda_k\varepsilon. \end{aligned}$$

Hence every slice fails to have small diameter as required. Q.E.D.

It is immediate (see the corollary to Lemma 5.2) that if X is smooth and Hahn-Banach smooth then X is very smooth. On the other hand c_0 in the usual norm is Hahn-Banach smooth because $(l^\infty)^*$ is the l^1 sum of l^1 and $c_0^\perp$ so that Hahn-Banach smooth does not imply very smooth or even smooth.

In a recent paper Namioka and Phelps [13] discuss a property they call (**). For every net (x_α^*) in X^*, if

$$(x_\alpha^*) \xrightarrow{*} x^* \quad \text{and} \quad \|x_\alpha^*\| \longrightarrow \|x^*\|$$

then $(x_\alpha^*) \to x^*$. In other words weak* and norm convergence coincide on the unit sphere of X^*. Namioka and Phelps show that if X^* satisfies (**) then X is an Asplund space and hence X^* has the Radon-Nikodym property, and they ask whether property (**) in X implies that X has an equivalent Fréchet differentiable norm. It is clear that if X does *not* have a Fréchet differentiable norm and X^* has property (**) then X is not smooth because in a smooth space the map $x \to n(x)$ is norm to weak* continuous. Some information in the positive direction is given by:

THEOREM 6. *If X^* has property* (**) *then X is Hahn-Banach smooth.*

Proof. Suppose that in X^{***} we have $\|x^*\| = 1 = \|x^* + x^\perp\|$ where $\|x^\perp\| \geq a > 0$.

Let E be the two dimensional subspace of X^{***} spanned by x^* and $x^\perp$ and let (F_α) denote the net of finite dimensional subspaces of X directed by inclusion. From local reflexivity we have for each α a linear map $T_\alpha\colon E \to X^*$ such that for each α:

(a) $T_\alpha x^* = x^*$.
(b) Since $x^\perp \in X^\perp$, $x_\alpha^*(x_\alpha) = x^*(x_\alpha)$ for each $x_\alpha \in F_\alpha$ where $x_\alpha^* = T_\alpha(x^* + x^\perp)$.
(c) For each $e \in E$,

$$(1 - 1/\dim(F_\alpha))\|e\| \leq \|T_\alpha e\| \leq (1 + 1/\dim(F_\alpha))\|e\|.$$

Combining (a), (b), and (c) we have that

$$(x_\alpha^*) \xrightarrow{*} x^* \quad \text{and} \quad \|x_\alpha^*\| \longrightarrow \|x^*\|$$

while $\|x_\alpha^* - x^*\| = \|T_\alpha(x^\perp)\|$ remains bounded away from zero. This contradicts (**) in X^*. Q.E.D.

It is known that if X^* is separable then X has an equivalent norm in which X^* has property (**) and is rotund and hence X has Fréchet differentiable norm. The preceding lemma shows that in this norm X is also Hahn-Banach smooth. This will be used in Section 5 to establish part of a necessary and sufficient condition for a separable space to have a separable dual space.

Passing now to rotundity we say that X is weakly locally uniformly rotund (wLUR) if for any sequence $(x_k) \subseteq S$ and $x \in S$, if $\|x_k + x\| \to 2$ then $(x_k) \to x$ weakly. A result corresponding to Theorem 3 for the property "very rotund" is a consequence of:

LEMMA 7. *If X is* wLUR *then for any*

$$\|x^{**}\| = \|J_0(x)\| = 1 \quad \text{if } \|x^{**} + J_0(x)\| = 2$$

then $x^{**} = J_0(x)$.

Proof. If $\|x^{**} + J_0(x)\| = 2$ where $x^{**} \neq J_0(x)$ then from local reflexivity we have a sequence of linear maps T_k: {span $J_0(x), x^{**}$} $\to X$ such that $T_k J_0(x) = x$ for each k and

$$1/(1 - \varepsilon_k) \geq \|T_k x^{**}\| \geq 1/(1 + \varepsilon_k)$$

where (ε_k) is a positive sequence decreasing to zero. While for some (fixed) $y^* \in S^*$ and $a > 0$,

$$y^*(T_k x^{**} - x) = (x^{**} - J_0(x))(y^*) \geq a > 0$$

since $x^{**} \neq J_0(x)$.

Let $x_k \equiv T_k x^{**}/\|T_k x^{**}\|$; then we have $\|x_k + x\| \to 2$ while $y^*(x_k - x) \geq a \geq 0$ contradicting wLUR in X. Q.E.D.

COROLLARY. *If X is* (wLUR) *then X is very rotund.*

For any set Γ, $c_0(\Gamma)$ has an equivalent norm which is (wLUR) but for Γ uncountable $l^1(\Gamma)$ has no smooth norm so that implication (2) of Lemma 1 cannot be reversed.

A Banach space Y is said to have property (H) if for all sequences (y_k), $y_k \rightharpoonup y$ weakly and $\|y_k\| \to \|y\|$ imply that $y_k \to y$, i.e., norm and weak convergence of sequences coincide on the surface of the unit ball. Any smooth reflexive space X, such that X^* fails property (H) is very smooth, but the norm in X is not Fréchet differentiable. Mark Smith of the University of Illinois, has kindly supplied the following example of this phenomenon.

Let $\|\cdot\|_2$ denote the usual norm on l^2 and consider the new norm defined by

$$\|x\|_0 \equiv \max\{\tfrac{1}{2}\|x\|_2, \|x\|_\infty\}.$$

Clearly this is equivalent to $\|\cdot\|_2$ because $\frac{1}{2}\|\cdot\|_2 \leq \|\cdot\|_0 \leq \|\cdot\|_2$. For (α_k) an element of l^2 let $T((\alpha_k)) \equiv (\alpha_k/k)$ so that T: $l^2 \to l^2$ is a 1-1 continuous linear map. Hence the equivalent norm $|||x||| \equiv \|x\|_0 + \|Tx\|_2$ is strict convex [2].

We shall show that $(l^2, |||\cdot|||)$ fails property (H) so that

$$X \equiv (l^2, |||\cdot|||)^*$$

is very smooth but not Fréchet differentiable. Let $x = (1, 0, \ldots, 0, \ldots)$ and for each k let $x_k = (1, 0, \ldots, 0, 1, 0, \ldots)$ (1 in the kth place). Then $|||x_k||| = 1 + (1 + 1/k^2)^{1/2}$ and $|||x||| = 2$ so that $|||x_k||| \to |||x|||$. Also $x_k \rightharpoonup x$. However, for each x, $|||x_k - x||| = 1 + 1/k$ which contradicts (H).

4. Extremely smooth spaces

As was mentioned earlier, some of the results of this section are similar to those of Section 3, except that here we consider uniformizing the Gateaux derivative in the first variable.

A Banach space is said to be *extremely smooth* if whenever $x^{***}(x^{**}) = y^{***}(x^{**}) = 1$ where $x^{**} \in S^{**}$ and $x^{***}, y^{***} \in S^{***}$ then $x^{***} - y^{***} \in X^\perp$

A Banach space is said to be *extremely rotund* if every finite dimensional subspace of X is a Chebyshev subspace of X^{**}.

Note that if X is extremely smooth then X^* is rotund so that l^1 has an equivalent norm which is smooth but not extremely smooth [10], [18].

A characterization of the property "extremely smooth" similar to that given by Theorem 3.2 for "very smooth" is the following:

THEOREM 1. *The following are equivalent:*

(1) *X is extremely smooth.*

(2) *For each $x^{**} \in S^{**}$ and $y \in S$, $\lim_{\lambda \to 0} \rho(x^{**}, \lambda y)/\lambda = 0$.*

(3) *If two contractive projections on X^{**} have the same range then they agree on X.*

Proof. The equivalence between conditions (1) and (2) is proved by the appropriate modification of the corresponding part of Theorem 3.2 and will be omitted.

Hence, assume that X is extremely smooth and that E and F are contractive projections on X^{**} with the same range. If $n(Ex^{**}) \in X^{***}$ is any norming functional for Ex^{**} then so is $E^*n(Ex^{**})$ and so these functionals must agree on X. Thus for $y \in X$, $n(Ex^{**})(y) = n(E^{**})(Ey)$. A simple calculation now shows that for any $x \in X$,

$$\|Ex - Fx\| = n(Ex - Fx)(Ex - Fx) = 0.$$

On the other hand, if X is not extremely smooth let x^{***} and y^{***} be norming elements for some $x^{**} \in S^{**}$ which differ on X. Then the maps

$$y^{**} \to x^{***}(y^{**})x^{**} \quad \text{and} \quad y^{**} \to y^{***}(y^{**})x^{**}$$

will contradict condition (3). Q.E.D.

A consequence of extremely smooth concerning the norm to weak* continuity of the map n can be obtained with the help of the following lemma which, in fact, characterizes dual spaces of extremely smooth spaces.

LEMMA 2. *X is extremely smooth iff for all $x^{**} \in S^{**}$ and all sequences (x_k^*), $(y_k^*) \in S^*$, $\lim x^{**}(x_k^*) = 1 = \lim x^{**}(y_k^*)$ implies that*

$$x_k^* - y_k^* \xrightarrow{*} 0.$$

Proof. If for some $x^{**} \in S^{**}$ and $y \in S$ there are norm-1 sequences (x_k^*) and (y_k^*) with $\lim x^{**}(x_k^*) = 1 = \lim x^{**}(y_k^*)$ while $|(x_k^* - y_k^*)(y)|$ remains bounded away from zero, then if x^{***} and y^{***} in X^{***} are weak* limit points of (x_k^*) and (y_k^*) we have $x^{***}(x^{**}) = 1 = y^{***}(x^{**})$ while

$$|(x^{***} - y^{***})(y)| > 0.$$

For the converse, suppose that X is not extremely smooth and let (λ_k) decrease to zero and let x^{**}, y and ε satisfy

$$\rho(x^{**}, \lambda_k y)/\lambda_k \geq \varepsilon > 0.$$

For each k let (x_k^*) and (y_k^*) be chosen so that x_k^*, $y_k^* \in S^*$ and

$$(x^{**} + \lambda_k y)(x_k^*) \geq \|x^{**} + \lambda_k y\| - \lambda_k^2$$

and

$$(x^{**} - \lambda_k y)(y_k^*) \geq \|x^{**} - \lambda_k y\| - \lambda_k^2.$$

Clearly $\lim_k x^{**}(x_k^*) = 1 = \lim_k y^{**}(x_k^*)$. However, using the idea of the inequalities of Section 2 we have

$$\begin{aligned} 2 + \lambda_k \varepsilon - 2\lambda_k^2 &\leq \|x^{**} + \lambda_k y\| + \|x^{**} - \lambda_k y\| - 2\lambda_k^2 \\ &\leq x^{**}(x_k^* + y_k^*) + \lambda_k(x_k^* - y_k^*)(y) \\ &\leq 2 + \lambda_k(x_k^* - y_k^*)(y). \end{aligned}$$

Hence $(x_k^* - y_k^*)(y)$ remains bounded away from zero. Q.E.D.

A similar characterization of very smooth can also be given where the condition becomes: If $x_k^*(x), y_k^*(x) \to 1$ then $x_k^* - y_k^* \rightharpoonup 0$. This is essentially proved in Theorem 3.5 and reflects the fact that for very smooth spaces the norming map is norm to weak continuous. The corresponding idea for extremely smooth spaces is given by the following:

COROLLARY. *If X is extremely smooth then the norming map $n\colon S \to S^*$ has an extension $\bar{n}\colon S^{**} \to B^*$ which is continuous from the norm topology of S^{**} to the weak* topology of S^*.*

Proof. If $x^{**} \in S^{**}$ let $(x_k^*) \subseteq S^*$ be any sequence such that $x^{**}(x_k^*) \to 1$. Define $\bar{n}(x^{**}) = x^*$ where $x^* \in B^*$ is any weak* limit point of (x_k^*). The previous lemma shows that $\bar{n}(x^{**})$ is well defined since x^* is unique.

For the continuity statement assume that $x_k^{**} \to x^{**}$ and $y \in X$ is arbitrary. Let $\varepsilon_k \searrow 0$. Using the Bishop-Phelps theorem [14] we may assume that for each k there is a vector $y_k^{**} \in S^{**}$, close to x_k^{**} such that $y_k^{**}(y_k^*) = 1$ where $y_k^* = \bar{n}(y_k^{**}) \in S^*$. Now

$$\begin{aligned} x_k^{**}(y_k^*) &= y_k^{**}(y_k^*) - (y_k^{**} - x_k^{**})(y_k^*) \\ &\geq 1 - \|y_k^{**} - x_k^{**}\| \end{aligned}$$

and so, using the lemma again, y_k^{**} may be chosen so that

$$|(x_k^* - y_k^*)(y)| < \varepsilon_k \quad \text{where } x_k^* = \bar{n}(x_k^{**}).$$

Clearly we may assume that $y_k^{**} \to x^{**}$ and so $x^{**}(y_k^*) \to 1$. To complete the argument note that for $x^* = \bar{n}(x^{**})$,

$$|(x^* - x_k^*)(y)| \leq |(x^* - y_k^*)(y)| + |(y_k^* - x_k^*)(y)|. \qquad \text{Q.E.D.}$$

THEOREM 3. *If the norm functional on X is uniformly Gateaux differentiable then X is extremely smooth.*

Proof. Use Theorem 1 and the same argument as Theorem 3.3. Q.E.D.

Recall that a Banach space is said to be uniformly convex in each direction if for any norm-1 sequences (x_n) and (y_n), if $\|x_n + y_n\| \to 2$ and $x_n - y_n \to z$ then $z = 0$.

THEOREM 4. *If X is uniformly convex in each direction then X is extremely rotund.*

Proof. Suppose some finite dimensional subspace of X is not a Chebychev subspace of X^{**}. This implies that for some $\|x^{**}\| = 1$ and $x \neq 0$ we have

$$\left\|\frac{x^{**} + (x^{**} - x)}{2}\right\| = 1 = \|x^{**} - x\|.$$

Using local reflexivity we can produce sequences $\|x_n\| \to 1$ and $\|x_n - x\| \to 1$ with $\|x_n + (x_n - x)\| \to 2$. This contradicts uniform convexity in each direction. Q.E.D.

For any set Γ, $l^1(\Gamma)$ has an equivalent norm which is uniformly convex in every direction while for Γ not finite $l^\infty(\Gamma)$ cannot be made even smooth. Thus if X^* is extremely smooth then (since X^{**} is rotund) X is extremely rotund, but the implication cannot be reversed.

5. Property V

It is often quite easy to specify conditions on X^* which guarantee that X possesses some desired geometrical property. On the other hand conditions on X which are necessary and sufficient for a geometrical property in X^* lie somewhat deeper. For example it is immediate that if X^* is rotund then X is smooth, but the implication cannot be reversed [10], [18]. In fact X^* is rotund iff every two dimensional factor space of X is smooth [2]. L. P. Vlasov [19] has translated this fact into the following useful form:

THEOREM 1 (Vlasov). *X^* is rotund iff for every nested sequence*

$$B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n \subseteq B_{n+1} \subseteq \cdots$$

of open balls in X with radii increasing and unbounded, the set Cl $(\bigcup B_n)$ *is either all of X or a half space.*

We generalize this to the following stronger property: X is said to have property V if there *do not* exist open balls

$$B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n \subseteq B_{n+1} \subseteq \cdots$$

with radii increasing and unbounded, and norm-1 functionals x^* and (y_k^*) such that for some constant c

$$x^*(b) > c \quad \text{for all } b \in \bigcup B_n,$$

$$y_k^*(b) > c \quad \text{for all } b \in B_n \text{ where } n \leq k,$$

and dist (conv $(y_1^*, y_2^*, \ldots)$, $x^*) > 0$.

Note that if X fails property V and y^{***} is a weak* limit point of (y_k^*) in X^{***}, then $y^{***} \neq x^*$ but $y^{***}(b) \geq c$ for all $b \in \bigcup B_n$. By Vlasov's reasoning [19] this gives a nonsmooth factor space X^{**}/L where $L \subseteq X$. We shall give a more precise version of this fact in Lemma 3. First, however, we consider a result in the other direction.

LEMMA 2. *If X has property* V *then X is Hahn-Banach smooth.*

Proof. Suppose that in x^{***} we have $\|x^* + x^\perp\| = 1 = \|x^*\|$ where $x^\perp \neq 0$. We shall use local reflexivity (or really just Helly's Theorem) repeatedly to obtain a sequence of balls and functionals contradicting property V.

Assume that $\|x^\perp\| = 4a$ where $a > 0$ and let $x^{**} \in S^{**}$ where $x^\perp(x^{**}) > \|x^\perp\|/2 = 2a$.

Since $\|x^*\| = 1$ there is a sequence $(x_n) \subseteq S$ with $x^*(x_n) > 1 - \delta_n$ where $\delta_n \searrow 0$, $0 < \delta_1 < a$ and $\sum \delta_n < 1$. Notice that we also have $(x^* + x^\perp)(x_n) > 1 - \delta_n$.

Using Helly's Theorem we produce a sequence (u_k^*) where for each k,

$$u_k^*(x_n) = (x^* + x^\perp)(x_n) > 1 - \delta_n$$

for all $n \leq k$ and $x^{**}(u_k^*) = (x^* + x^\perp)(x^{**})$ and $1 \leq \|u_k^*\| < 1 + \delta_k$.

We adopt the notation $B(r; x)$ for the open ball of radius r centered at the point x. Let $y_n^* \equiv u_n^*/\|u_n^*\|$ and

$$B_n \equiv B\left(\sum_1^n (1 + \delta_i); x_1 + \cdots + x_n\right).$$

Clearly $r_n \nearrow \infty$ and from the triangle inequality $B_n \subseteq B_{n+1}$, for each n.

If now $y = x_1 + \cdots + x_n - u$ where $\|u\| \leq n + \sum_1^n \delta_i$ then

$$\begin{aligned} x^*(y) &= x^*(x_1 + \cdots + x_n - u) \\ &\geq n - \sum_1^n \delta_i \\ &\geq n - \sum_1^n \delta_i - \left(n + \sum_1^n \delta\right) \\ &> -2 \\ &> -3. \end{aligned}$$

While, for $k \geq n$,

$$\begin{aligned} y_k^*(y) &\geq \frac{u_k^*(x_1 + \cdots + x_n)}{\|u_k^*\|} - n - \sum_1^n \delta_i \\ &\geq \frac{\sum_1^n (1 - \delta_i)}{1 + \delta_k} - n - \sum_1^n \delta_i \\ &\geq \sum_1^n \frac{1 - \delta_i}{1 + \delta_i} - n - \sum \delta_i \\ &\geq \sum_1^n (1 - 2\delta_i) - n - \sum \delta_i \\ &> -3. \end{aligned}$$

Finally for each n,

$$\begin{aligned} |x^{**}(x^* - y_n^*)| &\geq |x^{**}(x^*) - (x^* + x^\perp)(x^{**})| \\ &\quad - |(x^* + x^\perp)(x^{**}) - x^{**}(y_n^*)| \\ &= |x^\perp(x^{**})| - |(x^* + x^\perp)(x^{**}) - x^{**}(u_n^*)/\|u_n^*\|\,| \\ &> 2a - (1 - 1/\|u_n^*\|)|(x^* + x^\perp)(x^{**})| \\ &\geq 2a - (1 - 1/\|u_n^*\|) \\ &> 2a - \delta_n \\ &> a. \end{aligned}$$

Q.E.D.

COROLLARY. *If X has property* V *then X is very smooth.*

Proof. From Vlasov's Theorem, property V in S implies that X^* is rotund and hence X is smooth. Hence, the only possible violation to very smooth is an equation

$$\|x^* + x^\perp\| = (x^* + x^\perp)(x) = 1 = x^*(x) = \|x^*\| = \|x\|.$$

However from the previous lemma this is impossible. Q.E.D.

The converse to Lemma 2 and the fact that V implies Vlasov's condition is:

LEMMA 3. *If X is Hahn-Banach smooth and X^* is rotund then X has property* V.

Proof. Suppose that $B_n = B(r_n; z_n)$ is a nested sequence of open balls with

$$B(r_n; z_n) \subseteq x^{*-1}(-\infty, 2) \cap \left[\bigcap_n^\infty y_k^{*-1}(-\infty, 2)\right]$$

where $z^{**}(x^* - y_k^*) \geq a > 0$ for all k.

A simple calculation shows that

$$\beta_n \equiv x^*(z_n) + r_n = \sup\{x^*(b) \mid b \in B_n\}$$

and since the balls are nested the sequence (β_n) is increasing and, since it is bounded above by 2, converges.

We may assume that

$$y_k^* \xrightarrow{*} y^*, \qquad \sup\{y^*(b) \mid b \in B_n\} \leq y^*(z_n) + r_n \equiv \alpha_n$$

and

$$\lim_k y_k^*(z_n) + r_n = \lim_k \sup\{y_k^*(b) \mid b \in B_n\} \leq 2,$$

so that (α_n) also converges.

For convenience we assume that $B_1 = B(1; 0)$ and define a sequence of vectors $w_n \equiv -z_n/(r_n - 1)$. We shall show that a weak* limit point of this sequence violates Hahn-Banach smoothness.

Clearly

$$x^*(w_n) = \frac{-x^*(z_n)}{r_n - 1} = \frac{r_n - \beta_n}{r_n - 1} \to 1$$

and likewise

$$y^*(w_n) = \frac{r_n - \alpha_n}{r_n - 1} \to 1.$$

It is elementary that, since the balls are nested, $\|z_n\| \leq r_n - 1$ and so assuming

$$w_n \xrightarrow{*} x^{**}$$

we have

$$x^{**}(x^*) = \lim_n x^*(w_n) = 1 = \lim_n y^*(w_n) = x^{**}(y^*).$$

Since X^* is rotund we must have $x^* = y^*$. Hence, we may assume that

$$y_k^* \xrightarrow{*} x^*.$$

Now let y^{***} be a weak* limit point of (y_k^*) in X^{***}. Because of the condition $z^{**}(x^* - y_k^*) \geq a > 0$ we have $y^{***} \neq x^*$. However $\|y^{***}\| = 1$ and from the previous discussion $y^{***} = x^* + x^{\perp}$. This contradicts Hahn-Banach smooth. Q.E.D.

THEOREM 4. *X has property* V *iff X is Hahn-Banach smooth and X^* is rotund.*

Proof. Combine Lemmas 2 and 3 and the observation that property V implies that X^* is rotund. Q.E.D.

THEOREM 5. *If X is separable then X^* is also separable iff X has an equivalent norm with property* V.

Proof. If X has property V then from the corollary to Lemma 2, X is very smooth and from Lemma 3.4, X^* has the Radon-Nikodym property. Since X is separable, Stegall's Theorem [16] implies that X^* is separable.

If X and X^* are both separable then X has an equivalent norm so that X^* is rotund and has property (**) [14], [20]. By Theorem 3.6, X is Hahn-Banach smooth and so, from Lemma 3, X has property V. Q.E.D.

6. Summary

In this section we summarize, in the form of charts, the connections among the various geometrical properties introduced in the previous sections. In order to avoid pages containing a galaxy of asterisks we adopt the following notational convention: If P is a geometrical property then P^k means that the kth dual of X has P. We write simply P for P^0. The usual arrow is used to indicate implication. Thus, in this notation, Dixmier's Theorem that a nonreflexive space has a nonstrictly convex fourth dual is stated $R^4 \rightarrow$ reflexive. We also employ the following abbreviations:

R—rotund (strict convex)
S—smooth
ER—extremely rotund
ES—extremely smooth
UG—uniformly Gateaux differentiable
UCED—uniformly convex in each direction
VR—very rotund
VS—very smooth
HBS—Hahn-Banach smooth
RNP—Radon-Nikodym property
wLUR—weakly locally uniformly rotund
(**)—see Section 3
F—Fréchet differentiable.

The main results of Section 4 may now be put together as follows:

$$\begin{tikzcd}
\mathrm{UG}^1 \arrow[rr, "(b)"] \arrow[d, "(a)"] & & \mathrm{UCED} \arrow[d, "(c)"] \\
\mathrm{ES}^1 \arrow[r, "(d)"] & \mathrm{R}^2 \arrow[r, "(e)"] & \mathrm{ER}
\end{tikzcd}$$

(a) Theorem 4.3.
(b) This is known. For example, see Day [2].
(c) Theorem 4.4.
(d) Lemma 4.2.
(e) Theorem 4.1.

The results of Sections 3 and 5 are interrelated as follows:

$$\begin{tikzcd}
 & & & \mathrm{wLUR}^1 \arrow[d, "(h)"] & \\
 & \mathrm{R}^3 \arrow[r, "(a)"] \arrow[d, "(e)"] & \mathrm{S}^2 \arrow[r, "(b)"] & \mathrm{VR}^1 \arrow[d, "(b)"] & \\
\mathrm{V} \arrow[r, leftrightarrow, "(d)"] & \text{HBS and } \mathrm{R}^1 \arrow[r, "(a)"] & \text{HBS and S} \arrow[r, "(f)"] & \mathrm{VS} \arrow[r, "(g)"] & \mathrm{RNP}^1 \\
 & (**)^1 \text{ and } \mathrm{R}^1 \arrow[u, "(c)"] \arrow[r, "(a)"] & (**)^1 \text{ and S} \arrow[u, "(c)"] \arrow[r, "(j)"] \arrow[d] & \mathrm{F} \arrow[u, "(h)"] & \\
 & & (**)^1 \arrow[d, "(i)"] & & \\
 & & \mathrm{RNP}^1 & &
\end{tikzcd}$$

(a) The fact that $R^{k+1} \to S^k$ is well known.
(b) $VR^{k+1} \to VS^k$ and $S^{k+1} \to VR^k$ are observations made in Section 3.
(c) Theorem 3.6 says $(**)^1 \to$ HBS.
(d) Theorem 5.4.
(e) This is immediate. Note that from this any smooth, reflexive space has property V and so the example at the end of Section 3 also shows that V does not imply Fréchet differentiability.
(f) Corollary to Lemma 5.2.
(g) Lemma 3.4.
(h) Theorem 3.3 and 3.7.
(i) This is known [13].
(j) This is an observation made in Section 3.

The observations concerning Dixmier's Theorem made in Section 2 can be stated in the present notation as $HBS^1 \to$ reflexive and $VS^1 \to$ reflexive. Also, it is obvious from Goldstine's Theorem that $(**)^2 \to$ reflexive. Hence, if all superscripts are increased by one, the two terminal nodes labeled RNP^1 become "reflexive."

References

1. H. B. Cohen and F. E. Sullivan, *Projecting onto cycles in smooth, reflexive Banach spaces*, Pacific J. of Math., vol. 34 (1970), pp. 355–364.
2. M. M. Day, *Normed linear spaces*, Springer-Verlag, Berlin, 1972.
3. D. W. Dean, *The equation $L(E, X^{**}) = L(E, X)^{**}$ and the principle of local reflexivity*, Proc. Amer. Math. Soc., vol. 40 (1973), pp. 146–148.
4. J. Diestel and B. Faires, *On vector measures*, Trans. Amer. Math. Soc., vol. 198 (1974), pp. 253–271.
5. J. Dixmier, *Sur un theoreme de Banach*, Duke Math. J., vol. 15 (1948), pp. 1057–1071.
6. N. Dunford and J. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.
7. R. Huff, *Dentability and the Radon-Nikodym property*, Duke Math. J., vol. 41 (1974), pp. 111–114.
8. R. C. James, *Characterizations of reflexivity*, Studia Math., vol. 23 (1964), pp. 205–216.
9. ———, *Reflexivity and the sup of linear functionals*, Israel J. Math., vol. 13 (1972), pp. 289–300.
10. V. Klee, *Some new results on smoothness and rotundity in normed linear spaces*, Math. Ann., vol. 139 (1959), pp. 51–63.
11. J. Lindenstrauss, *On the modulus of smoothness and divergent series in Banach spaces*, Mich. Math. J., vol. 10 (1963), pp. 241–252.
12. J. Lindenstrauss and H. P. Rosenthal, *The L_p spaces*, Israel J. Math., vol. 7 (1969), pp. 325–349.
13. I. Namioka and R. Phelps, *Banach spaces which are Asplund spaces*, Duke Math. J., vol. 42 (1975), pp. 735–750.
14. R. Phelps, *Support cones in Banach spaces and their applications*, Advances in Math., vol. 13 (1974), pp. 1–19.
15. ———, *Dentability and extreme points in Banach spaces*, J. Functional Analysis, vol. 17 (1974), pp. 78–90.
16. C. Stegall, *The Radon-Nikodym property in conjugate Banach spaces*, Trans. Amer. Math. Soc., vol. 206 (1975), pp. 213–223.

17. F. SULLIVAN, *Some geometrical relatives of the Radon-Nikodym property*, Notices Amer. Math. Soc., vol. 22 (1975), PA-567.
18. S. TROYANSKI, *Example of a smooth space whose conjugate has not strictly convex norm*, Studia Math., vol. 25 (1970), pp. 305–309.
19. L. P. VLASOV, *Approximate properties of sets in normed linear spaces*, Russian Math. Surveys, vol. 28 (1973), pp. 1–66.
20. J. H. M. WHITFIELD, *Certain differentiable functions on Banach spaces and manifolds*, Thesis, Case Institute of Technology, 1966.

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