ON THE TRANSFORMATION OF log η

BY

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1. The Dedekind η-function is defined by

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}).$$

It has a well-defined logarithm which transforms according to the rules

$$\log \eta(z + 1) = \log \eta(z) + \pi i/12, \tag{1}$$

$$\log \eta(-1/z) = \log \eta(z) + \tfrac{1}{2} \log (z/i). \tag{2}$$

The first of these rules is obvious and many proofs have been given for the second. One of the neatest proofs, which has been given by Weil [8], is to note that the Mellin transform of

$$f(z) = \pi i z/12 - \log \eta(z) \tag{3}$$

is

$$\Phi(s) = (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s + 1) \tag{4}$$

where $\zeta(s)$ is the Riemann zeta-function. The functional equation for $\zeta(s)$ implies that $\Phi(s) = \Phi(-s)$. Also, $\Phi(s)$ is regular except for simple poles at $s = \pm 1$ and a double pole at $s = 0$. The formula (2) follows from these facts by applying the residue theorem.

Since the transformations $z \mapsto z + 1$ and $z \mapsto -1/z$ generate the full modular group, repeated applications of (1) and (2) gives a formula relating $\log \eta(\sigma z)$ and $\log \eta(z)$ whenever $\sigma \in SL(2, \mathbf{Z})$. Dedekind was the first to prove that if

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with } a, b, c, d \in \mathbf{Z},\ c > 0,\ ad - bc = 1,$$

then

$$\log \eta \left(\frac{az + b}{cz + d}\right) = \log \eta(z) + \tfrac{1}{2} \log \left(\frac{cz + d}{i}\right) + \frac{a + d}{12c} \pi i - \pi i s(d, c) \tag{5}$$

where $s(d, c)$ is the Dedekind sum

$$s(d, c) = \sum_{\lambda \bmod c} ((\lambda d/c))((\lambda/c)) \tag{6}$$

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and

$$((y)) = \begin{cases} y - [y] - 1/2 & \text{if } y \notin \mathbf{Z} \\ 0 & \text{if } y \in \mathbf{Z} \end{cases}$$

is the usual "sawtooth" function of period 1.

Since Dedekind, several other proofs of (5) have been given and the sums $s(d, c)$ have been much studied and generalized (see [5, pp. 438–446]). One recent proof, given by Goldstein and de La Torré [2] used the functional equations of certain congruence zeta-functions to generalize the $\zeta(s)\zeta(s+1)$ proof of (2) to a proof of (5).

In this note a proof of (5) is given in which the role of $\zeta(s)\zeta(s+1)$ is essentially played by $L(s, \chi)L(s+1, \chi)$ for various Dirichlet characters χ. The idea is that the general transformation law (5) for $\log \eta(z)$ should be a consequence of "twisted" functional equations related to (2). The approach seems to be at least superficially similar to that used in Weil's theorem [9] (or see [6, Chapter V]). It is also related to some recent joint work of the author and L. Goldstein [4] (see especially Examples 3.2, 3.3, 3.4).

In the course of the proof some functions $f_\chi(z)$ are introduced. These turn out to be integrals of the third kind for $H^*/\Gamma(N)$ and may be of some interest in themselves. They transform according to the rule

$$f_\chi(z) = \chi(d)^2 f_\chi(\sigma z) - \pi i \sum_{t \bmod N} \chi(t) s(d - tc, Nc)$$

for all $\sigma \in \Gamma_0^0(N)$. The derivatives $f_\chi'(z)$ are related to division values of Weierstrass $\wp$-functions. This subject will be explored in another paper.

In giving yet another proof (and not a particularly short one at that) for a well-known formula, one feels obligated to say something about the motivation behind the proof. In this case the idea is to show that the formula (5) is equivalent to a collection of functional equations for Dirichlet series with Euler products. The existence of an Euler product seems to make the technique susceptible to an adelic interpretation and in this way it may be generalized to various analogs of $\log \eta$ for number fields. These functions play a central role in the development of class number formulas over certain algebraic number fields (see [3]).

The author would like to thank the referee for his careful reading of this paper and his useful suggestions for simplifying and shortening the proofs of Propositions 2 and 3. In particular, he pointed out that Lemma 3 in a recent paper [7] of Shimura could be used. The author also would like to thank Larry Goldstein for his helpful comments.

2. Let $f(z)$ be the function defined by (3), let c be a positive integer and let χ be a Dirichlet character mod c. Define

$$f_\chi(z) = \sum_{r \bmod c} \chi(r) f((z + r)/c) \tag{7}$$

The transformation law (5) is equivalent to the assertion of the following theorem.

THEOREM 1. *If χ is a Dirichlet character* mod c, *then*

$$f_\chi(z) = \chi(-1)f_{\bar{\chi}}(-z^{-1}) + \pi i \sum_{\nu \bmod c} \chi(\nu)s(\nu, c) + \delta_\chi(z) \tag{8}$$

where

$$\delta_\chi(z) = \begin{cases} \phi(c)\left(\dfrac{\pi i}{12c}(z + z^{-1}) + \frac{1}{2}\log\dfrac{z}{i}\right) & \text{if } \chi \text{ is principal} \\ 0 & \text{if } \chi \text{ is nonprincipal.} \end{cases} \tag{9}$$

Most of this note is devoted to a proof of Theorem 1. But before proceeding with this proof, it will be shown that Theorem 1 implies the transformation law (5). The converse is also easy and may be seen by simply reversing the steps.

As a first step, observe that the character sum in (7) may be inverted to give

$$f\left(\frac{z + n}{c}\right) = \frac{1}{\phi(c)} \sum_{\chi \bmod c} \bar{\chi}(n) f_\chi(z) \tag{10}$$

where the sum is over all characters χ mod c and n is any integer which is relatively prime to c. Next let

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \quad (c > 0).$$

Then, by (10),

$$f\left(\frac{z - d}{c}\right) - f\left(\frac{-z^{-1} + a}{c}\right) = \frac{1}{\phi(c)} \sum_{\chi \bmod c} (\bar{\chi}(-d)f_\chi(z) - \chi(a)f_{\bar{\chi}}(-z^{-1})).$$

Since $ad = 1 + bc \equiv 1 \pmod{c}$, $\chi(a) = \chi(-1)\bar{\chi}(-d)$. Thus,

$$f\left(\frac{z - d}{c}\right) - f\left(\frac{-z^{-1} + a}{c}\right) = \frac{1}{\phi(c)} \sum_{\chi \bmod c} \bar{\chi}(-d)(f_\chi(z) - \chi(-1)f_{\bar{\chi}}(-z^{-1})).$$

But by Theorem 1, the above expression is equal to

$$\begin{aligned} \frac{1}{\phi(c)}\Bigg[\pi i \sum_{\nu \bmod c} \sum_{\chi \bmod c} \bar{\chi}(-d)\chi(\nu)s(\nu, c) \\ + \phi(c)\left(\frac{\pi i}{12c}(z + z^{-1}) + \tfrac{1}{2}\log\left(\frac{z}{i}\right)\right)\Bigg] \end{aligned} \tag{11}$$

$$= -\pi i s(d, c) + \frac{\pi i}{12c}(z + z^{-1}) + \tfrac{1}{2}\log\left(\frac{z}{i}\right).$$

Now let $\tau = (z - d)/c$. Then

$$\sigma\tau = \frac{a\tau + b}{c\tau + d} = \frac{-z^{-1} + a}{c} \quad \text{and} \quad z = c\tau + d.$$

Plug into the above formula to get

$$f(\tau) - f(\sigma\tau) = -\pi i s(d, c) + \frac{\pi i}{12c}(d + a) + \frac{\pi i}{12}(\tau - \sigma\tau) + \tfrac{1}{2}\log\left(\frac{c\tau + d}{i}\right).$$

Since $\log \eta(\tau) = \pi i\tau/12 - f(\tau)$, the transformation law (5) follows immediately.

Theorem 1 will be proved by proving a functional equation for the Mellin transform of $f_\chi(z)$. This functional equation is a direct consequence of the functional equation of $L(s, \chi)$.

If χ is a Dirichlet character defined mod c, the Gauss-Ramanujan sums $\tau_k(\chi; c)$ are defined by

$$\tau_k(\chi; c) = \sum_{r \bmod c} \chi(r) e^{2\pi i k r/c} \quad (k \in \mathbf{Z}). \tag{12}$$

The function $f(z)$ has a Fourier expansion

$$f(z) = \sum_{k=1}^{\infty} \sigma_{-1}(k) e^{2\pi i k z} \tag{13}$$

where $\sigma_{-1}(k) = \sum_{d|k} d^{-1}$. It is now an easy consequence of the definition (7) that $f_\chi(z)$ has the Fourier expansion

$$f_\chi(z) = \sum_{k=1}^{\infty} \tau_k(\chi; c)\sigma_{-1}(k) e^{2\pi i k z/c}. \tag{14}$$

It is now easy to write down the Mellin transform of $f_\chi(z)$. It is

$$F(s, \chi) = (2\pi/c)^{-s}\Gamma(s)\phi(s, \chi) \tag{15}$$

where $\phi(s, \chi)$ is the Dirichlet series

$$\phi(s, \chi) = \sum_{k=1}^{\infty} \tau_k(\chi; c)\sigma_{-1}(k) k^{-s}. \tag{16}$$

By Mellin inversion,

$$f_\chi(z) = \frac{1}{2\pi i}\int_{\operatorname{Re} s = \sigma > 1} (z/i)^{-s} F(s, \chi)\, ds. \tag{17}$$

Thus, Theorem 1 is a routine consequence of the following theorem about $F(s, \chi)$ (see, for example, [4] or [6]).

THEOREM 1′. (a) $F(s, \chi) = \chi(-1)F(-s, \bar{\chi})$.

(b) $$F(s, \chi) - \pi i \sum_{\nu \bmod c} \chi(\nu) s(\nu, c) \cdot \frac{1}{s}$$

$$- \delta_\chi \phi(c)\left(-\frac{1}{2s^2} + \frac{\pi}{12c}\left(-\frac{1}{s+1} + \frac{1}{s-1}\right)\right)$$

is an entire function and is bounded (uniformly) in vertical strips. ($\delta_\chi = 0$ *if* χ *is not principal and* $\delta_\chi = 1$ *if* χ *is principal. Thus, usually,* $F(s, \chi)$ *only has a single simple pole at* 0.)

Theorem 1′ will be proved by expressing $\phi(s, \chi)$ in terms of Dirichlet L-series for which the functional equation and the residues are well known. Since the theorem is needed for arbitrary characters, and the functional equation of the L-series is nice only for primitive characters, it is necessary to relate χ to the primitive character inducing it.

Let χ *be a character* mod c. *Then* $c = \mathscr{f}N$ *where* $\mathscr{f}$ *is the conductor of the primitive character* χ_1 *which induces* χ mod c *and*

$$\chi(m) = \chi_1(m) \sum_{d \mid (m, N)} \mu(d) \quad (m \in Z) \tag{18}$$

where μ *is the Möbius function. This notation is used for the rest of this paper.*

The proof of Theorem 1′ will be broken up into three propositions. Proposition 1 is the crucial one and will be proved following the statement of the three propositions.

PROPOSITION 1. *Let* χ *be a primitive character modulo its conductor* $\mathscr{f}$. *Let* $\tau(\chi) = \tau_1(\chi, \mathscr{f})$ *denote the usual Gauss sum attached to* χ.

(a) $\phi(s, \chi) = \tau(\chi)L(s, \bar{\chi})L(s + 1, \bar{\chi})$.
(b) $F(s, \chi) = \chi(-1)F(-s, \bar{\chi})$.
(c) *If* χ *is nonprincipal* ($\mathscr{f} > 1$) *then* $F(s, \chi)$ *is regular except possibly for a simple pole at* $s = 0$ *and*

$$\operatorname*{Res}_{s=0} F(s, \chi)\, ds = \pi i L(0, \chi)L(0, \bar{\chi}).$$

(d) *If* χ *is principal* ($\mathscr{f} = 1$) *then*

$$F(s) + \frac{1}{2s^2} + \frac{\pi}{12}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

is entire.

PROPOSITION 2. *Let*

$$H(s, \chi) = \sum_{de \mid c/\mathscr{f}} \mu(d)\mu(e)\chi_1(d)\bar{\chi}_1(e)\sigma_1(c/\mathscr{f}de)d^{-s}e^s$$

where $\sigma_1(k) = \sum_{d \mid k} d$. *Then*

(a) $\phi(s, \chi) = (c/\mathscr{f})^{-s}\phi(s, \chi_1)H(s, \chi)$ *and*
(b) $F(s, \chi) = F(s, \chi_1)H(s, \chi)$.

PROPOSITION 3. (a) *If* χ *is nonprincipal* ($\mathscr{f} > 1$), *then*

$$\sum_{v \bmod c} \chi(v)s(v, c) = H(0, \chi)L(0, \chi_1)L(0, \bar{\chi}_1).$$

(b) *If χ is principal* ($f = 1$), *then* $H(0, \chi) = \phi(c)$ *and* $H(\pm 1, \chi) = \phi(c)/c$.

It is easy to see that Theorem 1′ is a direct consequence of Propositions 1, 2, and 3. First, Theorem 1′(a) follows from Propositions 1(b) and 2(b) upon noting that

$$H(s, \chi) = H(-s, \bar{\chi}) \quad \text{and} \quad \chi(-1) = \chi_1(-1).$$

Second, Theorem 1′(b) follows from Proposition 2(b) and Propositions 2(c) and 3(a) (respectively 2(d) and 3(b)) if χ is nonprincipal (respectively χ is principal).

The proofs of Propositions 2 and 3 are purely combinatorial in nature. These propositions merely serve the purpose of reducing the proof of Theorem 1′ to the case of a primitive character which is handled by Proposition 1. Nevertheless, these proofs are fairly complicated and will be given in Section 3. On the other hand, the proof of Proposition 1 is quite easy. It depends directly on the functional equation of the Dirichlet L-series $L(s, \chi)$ for a primitive character $\chi \bmod f$. This functional equation is stated below, for convenience, in an asymmetrical form.

$$L(1 - s, \bar{\chi}) = \frac{i^{\varepsilon}2^{\varepsilon-1}}{\tau(\chi)} \left(\frac{2\pi}{f}\right)^{-s} \frac{\Gamma((s + \varepsilon)/2)\Gamma((-s + \varepsilon)/2)}{\Gamma(-s + \varepsilon)} L(s, \chi) \tag{19}$$

where $\varepsilon = 0$ or 1, respectively, according as $\chi(-1) = +1$ or -1, respectively.

Proof of Proposition 1. (a) Since χ is primitive, $\tau_k(\chi) = \bar{\chi}(k)\tau(\chi)$ for all integers k. Thus by (16),

$$\begin{aligned}\phi(s, \chi) &= \sum_{k=1}^{\infty} \tau_k(\chi)\sigma_{-1}(k)k^{-s} \\ &= \tau(\chi) \sum_{k=1}^{\infty} \bar{\chi}(k)\sigma_{-1}(k)k^{-s} \\ &= \tau(\chi)L(s, \bar{\chi})L(s + 1, \bar{\chi}).\end{aligned}$$

(b) By (15),

$$F(s, \chi) = \left(\frac{2\pi}{f}\right)^{-s} \Gamma(s)\phi(s, \chi) = \left(\frac{2\pi}{f}\right)^{-s} \Gamma(s)\tau(\chi)L(s, \bar{\chi})L(s + 1, \bar{\chi}).$$

Thus, by (19), applied to $L(s + 1, \bar{\chi}) = L(1 - (-s), \bar{\chi})$,

$$\begin{aligned}F(s, \chi) = \left(\frac{2\pi}{f}\right)^{-s} \Gamma(s)\tau(\chi)L(s, \bar{\chi}) \frac{i^{\varepsilon}2^{\varepsilon-1}}{\tau(\chi)} \\ \times \left(\frac{2\pi}{f}\right)^{s} \frac{\Gamma((-s + \varepsilon)/2)\Gamma((s + \varepsilon)/2)}{\Gamma(s + \varepsilon)} L(-s, \chi).\end{aligned}$$

Since $\varepsilon = 0$ or 1, $\Gamma(s + \varepsilon) = s^{\varepsilon}\Gamma(s)$ and so

$$F(s, \chi) = \frac{i^{\varepsilon}2^{\varepsilon-1}}{s^{\varepsilon}} \Gamma\left(\frac{-s + \varepsilon}{2}\right) \Gamma\left(\frac{s + \varepsilon}{2}\right) L(s, \bar{\chi})L(-s, \chi). \tag{20}$$

Since $(-1)^{\varepsilon} = \chi(-1)$, $F(s, \chi) = \chi(-1)F(-s, \bar{\chi})$.

(c) If χ is nonprincipal, $L(s, \chi)$ is entire. If $\varepsilon = 0$, the zeros of $L(s, \bar{\chi})$ and $L(-s, \chi)$ cancel the simple poles of $\Gamma(-s/2)$ and $\Gamma(s/2)$ at the even integers so that $F(s, \chi)$ is entire. If $\varepsilon = -1$, the zeros of $L(s, \bar{\chi})$ and $L(-s, \chi)$ at the odd integers cancel the poles of $\Gamma((-s + 1)/2)$ and $\Gamma((s + 1)/2)$ but an extra simple pole with residue $\pi i L(0, \chi) L(0, \bar{\chi})$ is introduced by the factor s^{-1}. Observe that if $\varepsilon = 0$, $L(0, \chi) = 0$ anyway, so the assertion follows.

(d) If χ is principal, $F(s, \chi) = \frac{1}{2}\Gamma(-s/2)\Gamma(s/2)\zeta(s)\zeta(-s)$, where $\zeta(s)$ is the Riemann zeta function. Thus (d) follows from standard facts (functional equation and $\zeta(0) = -\frac{1}{2}$) about the zeta-function. (See [7] or [6, p. I-44] or [4].)

3. In this section, Propositions 2 and 3 are proved. As mentioned above, the proofs are combinatorial rather than analytic. The formulas for non-primitive characters follow formally from those for primitive characters. The lemma below, which is used in both proofs, is a slight generalization of the lemma of Shimura [7, Lemma 3] referred to in the introduction. Indeed, in the proof of Proposition 2, Shimura's lemma appears as (22) and the analogous formula in the proof of Proposition 3 is (25).

LEMMA. *Let χ_1 be a primitive character with conductor f. Let N be a positive integer and let χ be the character* mod fN *induced by χ_1. Let $g(y)$ be a periodic function of y with period* 1 *and let k be an integer. Then*

$$\sum_{m \bmod fN} \chi(m) g\left(\frac{mk}{fN}\right)$$

$$= \sum_{d \mid N} \mu(d)\chi_1(d)\bar{\chi}_1\left(\frac{kd}{(kd, N)}\right)\left(k, \frac{N}{d}\right) \sum_{\beta \bmod f} \chi_1(\beta) \sum_{\alpha \bmod N/(kd, N)} g\left(\frac{\beta/f + \alpha}{N/(kd, N)}\right)$$

Proof.

$$\sum_{m \bmod fN} \chi(m) g\left(\frac{mk}{fN}\right)$$

$$= \sum_{m \bmod fN} \sum_{d \mid (N, m)} \mu(d)\chi_1(m) g\left(\frac{mk}{fN}\right)$$

$$= \sum_{d \mid N} \mu(d)\chi_1(d) \sum_{n \bmod fN/d} \chi_1(n) g\left(\frac{nk}{fN/d}\right) \quad (m = nd)$$

$$= \sum_{d \mid N} \mu(d)\chi_1(d)\bar{\chi}_1\left(\frac{kd}{(kd, N)}\right)\left(k, \frac{N}{d}\right) \sum_{n \bmod fN/(kd, N)} \chi_1(n) g\left(\frac{n}{fN/(kd, N)}\right)$$

(since χ_1 is primitive)

$$= \sum_{d \mid N} \mu(d)\chi_1(d)\bar{\chi}_1\left(\frac{kd}{(kd, N)}\right)\left(k, \frac{N}{d}\right) \sum_{\beta \bmod f} \chi_1(\beta) \sum_{\alpha \bmod N/(kd, N)} g\left(\frac{\beta/f + \alpha}{N/(kd, N)}\right)$$

$(n = \alpha f + \beta)$

Proof of Proposition 2. It is easy to see that (b) follows directly from (a). To prove (a), apply Lemma 1 with $g(y) = \exp(2\pi i y)$ to evaluate the Gauss-Ramanujan sum $\tau_k(\chi; c)$. Use the standard identity

$$\sum_{\alpha \bmod r} \exp 2\pi i \left(\frac{y+\alpha}{r}\right) = \begin{cases} e^{2\pi i y} & \text{if } r = 1 \\ 0 & \text{if } r > 1 \end{cases}. \tag{21}$$

Thus, if $\tau(\chi_1) = \tau_1(\chi_1; f)$ is the usual Gauss sum,

$$\tau_k(\chi; c) = \tau(\chi_1) \sum_{d \mid N, N \mid kd} \mu(d)\chi_1(d)\bar{\chi}_1\left(\frac{kd}{N}\right)\frac{N}{d}. \tag{22}$$

Now,

$$\begin{aligned}
\phi(s, \chi) &= \sum_{k=1}^{\infty} \tau_k(\chi; c)\sigma_{-1}(k)k^{-s} \\
&= \tau(\chi_1)N \sum_{k=1}^{\infty} \sum_{d \mid N, N \mid kd} d^{-1}\mu(d)\chi_1(d)\bar{\chi}_1(kd/N)\sigma_{-1}(k)k^{-s} \\
&= \tau(\chi_1)N \sum_{d \mid N} d^{-1}\mu(d)\chi_1(d)(N/d)^{-s} \sum_{n=1}^{\infty} \bar{\chi}_1(n)\sigma_{-1}(nN/d)n^{-s} \\
&= \tau(\chi_1)N^{1-s} \sum_{d \mid N} d^{-1}\mu(d)\chi_1(d)d^s \sum_{n=1}^{\infty} \bar{\chi}_1(n)n^{-s} \\
&\quad \times \sum_{e \mid (n, N/d)} e^{-1}\mu(e)\sigma_{-1}(n/e)\sigma_{-1}(N/de) \\
&= \tau(\chi_1)N^{1-s} \sum_{d \mid N} d^{-1}\mu(d)\chi_1(d)d^s \sum_{e \mid N/d} e^{-1}\mu(e)\sigma_{-1}(N/de)\chi_1(e)e^{-s} \\
&\quad \times \sum_{m=1}^{\infty} \bar{\chi}_1(m)\sigma_{-1}(m)m^{-s} \\
&= \tau(\chi_1)\left(\sum_{m=1}^{\infty} \bar{\chi}_1(m)\sigma_{-1}(m)m^{-s}\right) N^{-s} \sum_{de \mid N} \mu(d)\mu(e)\chi_1(d)\bar{\chi}_1(e)\frac{N}{de} \\
&\quad \times \sigma_{-1}(N/de)d^s e^{-s} \\
&= N^{-s}\phi(s, \chi_1)H(s, \chi).
\end{aligned}$$

Proof of Proposition 3(a). The key fact is the following version of Dirichlet's class number formula. If χ is a *nonprincipal* character whose conductor divides c then

$$L(0, \chi) = -\sum_{\nu \bmod c} \chi(\nu)((\nu/c)). \tag{23}$$

For primitive χ with conductor c, this is an immediate consequence of the usual formula for $L(1, \chi)$ (see [1, p. 336]) and the functional equation (19) for $L(s, \chi)$. It is easy to see that it is valid even if χ is not primitive or if c is not equal to the conductor.

Now apply Lemma 1 with $g(y) = ((y))$. Instead of (21), use the easily verified identity

$$\sum_{\alpha \bmod r} \left(\left(\frac{y+\alpha}{r}\right)\right) = ((y)). \tag{24}$$

This gives

$$\sum_{m \bmod fN} \chi(m)\left(\left(\frac{mk}{fN}\right)\right) = -L(0, \chi_1) \sum_{d \mid N} \mu(d)\chi_1(d)\bar{\chi}_1\left(\frac{kd}{(kd, N)}\right)(k, N/d). \tag{25}$$

Then,

$$\begin{aligned}
&\sum_{m \bmod c} \chi(m)s(m, c) \\
&\quad = \sum_{m, k \bmod fN} \chi(m)\left(\left(\frac{mk}{fN}\right)\right)\left(\left(\frac{k}{fN}\right)\right) \\
&\quad = -L(0, \chi_1) \sum_{d \mid N} \mu(d)\chi_1(d) \sum_{k \bmod fN} \bar{\chi}_1\left(\frac{k}{(k, N/d)}\right)\left(k, \frac{N}{d}\right)\left(\left(\frac{k}{fN}\right)\right) \\
&\quad = -L(0, \chi_1) \sum_{d \mid N} \mu(d)\chi_1(d) \sum_{b \mid N/d} b \sum_{\substack{k \bmod fN \\ (k, N/d) = b}} \bar{\chi}_1\left(\frac{k}{b}\right)\left(\left(\frac{k/b}{fN/b}\right)\right) \\
&\qquad\qquad (b = (k, N/d)) \\
&\quad = -L(0, \chi_1) \sum_{bd \mid N} \mu(d)\chi_1(d)b \sum_{r \bmod fN/b} \bar{\chi}_1(r)\left(\left(\frac{r}{fN/b}\right)\right) \sum_{e \mid (r, N/bd)} \mu(e) \\
&\qquad\qquad (r = k/b) \\
&\quad = -L(0, \chi_1) \sum_{bde \mid N} \mu(d)\chi_1(d)\mu(e)\bar{\chi}_1(e)b \sum_{t \bmod fN/be} \bar{\chi}_1(t)\left(\left(\frac{t}{fN/be}\right)\right) \\
&\qquad\qquad (t = r/e) \\
&\quad = L(0, \chi_1)L(0, \bar{\chi}_1) \sum_{de \mid N} \mu(d)\mu(e)\chi_1(d)\bar{\chi}_1(e)\sigma_1\left(\frac{N}{de}\right) \qquad \text{(by (23))} \\
&\quad = L(0, \chi_1)L(0, \bar{\chi}_1)H(0, \chi).
\end{aligned}$$

Finally, the proof of 3(b) is a simple combinatorial manipulation and is omitted.

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