AN ARITHMETIC SQUARE FOR VIRTUALLY NILPOTENT SPACES[1]

BY

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1. Introduction

1.1 *The arithmetic square.* The aim of this paper is to generalize Sullivan's observation ([6, 3.58] and [4, p. 192]) that the homotopy type of a simply connected finite complex is determined by *primary* information, *rational* information and certain *coherence* data.

If X is a connected space and A and B are abelian groups, denote (see 3.1) by X_A and *A-localization* of X and by $X_{A,B}$ the *B-localization* of X_A in the sense of Bousfield. One can then (see again 3.1) form the commutative *arithmetic square*

$$\begin{array}{ccc} & X_Z & \rightarrow & X_P \\ & \downarrow & & \downarrow \\ X_Q \sim & X_{Z,Q} & \rightarrow & X_{P,Q} \end{array}$$

in which Z denotes the integers, Q the rationals, $P = \oplus Z/p$ (where the direct sum is taken over all primes p) and the bottom map is induced by the top map. Our main result now states (see 4.1) that *this arithmetic square is, up to homotopy, a fibre square if X is a virtually nilpotent space*, i.e., (see Section 2) if

(i) each Postnikov stage P_nX has a finite covering space which is nilpotent, or equivalently

(ii) π_1X has a nilpotent subgroup of finite index and, for every integer $n > 1$, π_1X has a subgroup of finite index which acts nilpotently on π_nX.

Obvious examples of virtually nilpotent spaces are spaces with a *finite* (or even *trivial*) *fundamental group* and *nilpotent* spaces, but the Klein bottle, for instance, which has neither of these properties, is also virtually nilpotent. Other examples are discussed in Section 2.

1.2 *Application.* If X is a nilpotent space, then (1.5 and 3.3) $X_Z \sim X$ and the above result thus implies that, up to homotopy, every nilpotent space X can be reconstructed from its *primary localization* (the nilpotent space X_P), its *rational localization* (the nilpotent space X_Q) and its *coherence map* (the map $X_Q \rightarrow X_{P,Q}$); in particular no further finiteness conditions on X are necessary, so that the alleged counter example of [4, p. 195] is incorrect. This also suggests the question when spaces U and V and a map $V \rightarrow U_Q$ are, up to homotopy,

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the primary and rational localizations and the coherence map of a nilpotent space. It turns out (see 4.4) that this is the case *if and only if*

(i) the space U is P-Bousfield (i.e., $U_P \sim U$) and nilpotent,

(ii) the space V is Q-Bousfield and nilpotent, and

(iii) the homotopy inverse limit [4, Chapter X] of the diagram $V \to U_Q \leftarrow U$ is connected, i.e., every element $y \in \pi_1 U_Q$ can be written in the form $y = uv$, where u and v are in the image of $\pi_1 U$ and $\pi_1 V$ respectively. One can make similar statements (see also 4.4) for spaces which are Z-Bousfield and virtually nilpotent.

1.3 *Application.* Another manner in which the arithmetic square can be used is due to the fact that we prove more than was mentioned in 1.1, namely that (see 3.3) for a virtually nilpotent space X, all the spaces in the arithmetic square (except possibly X_Z) can, up to homotopy, be obtained with the use of the *completion* functors of [4]. Thus in this case the arithmetic square provides a way of getting a hold on X_Z by means of spaces which are often easier to understand than X_Z itself. For instance, use of the arithmetic square facilitates Bousfield's calculations [1] of $\pi_* X_Z$ when $X = RP^2$, the real projective plane. In fact this paper is the result of our attempts at understanding these calculations of Bousfield.

1.4 *A generalization.* One can generalize the above results by replacing everywhere Z by an arbitrary subring $R \subset Q$ and P by $P \otimes R$. Note that $P \otimes R = \oplus Z/p$, where the direct sum is taken over the primes p for which $1/p$ is not in R.

1.5 *Organization of the paper.* We start in Section 2 and Section 3 with a brief discussion of (virtually) nilpotent spaces, localizations and completions. At the end of Section 3 we formulate our first results on localizations of virtually nilpotent spaces, Section 4 deals extensively with the arithmetic square and Section 5 contains two lemmas which seem to be of some interest in their own right. The rest of the paper is devoted to the various proofs.

1.6 *Notation and terminology.* Throughout the paper we will mean by a *space* a *simplicial set with base point.* If the reader prefers he can, of course, use CW-*complexes with base point* instead, but then he may have to take some extra care whenever infinite products come in.

The symbol $\sim$ will, as usually, mean *has the same homotopy type as* and a space X will be called *A-Bousfield* if $X_A \sim X$.

2. Virtually nilpotent spaces

We start our brief discussion of virtually nilpotent spaces with a review of a definition [4, p. 59].

2.1 *Nilpotent spaces.* A space X is called *nilpotent* if it is *connected* and

(i) $\pi_1 X$ is a *nilpotent group*, and
(ii) for every integer $n > 1$, $\pi_n X$ is a *nilpotent* $\pi_1 X$-*module*, i.e., $\pi_n X$ has a finite $\pi_1 X$-filtration for which $\pi_1 X$ acts trivially on the successive quotients.

This is equivalent to requiring that each map $P_n X \to P_{n-1} X$ $(n \geq 1)$ in the Postnikov decomposition of X is, up to homotopy, a finite composition of principal fibrations.

We now define in a similar manner:

2.2 *Virtually nilpotent spaces.* A space X is called *virtually nilpotent* if it is *connected* and
(i) $\pi_1 X$ is a *virtually nilpotent group*, i.e., $\pi_1 X$ has a nilpotent normal subgroup of finite index, and
(ii) for every integer $n > 1$, $\pi_n X$ is a *virtually nilpotent* $\pi_1 X$-*module*, i.e., $\pi_1 X$ has a normal subgroup of finite index which acts nilpotently on $\pi_n X$.
This implies that, for every integer $k > 1$, there is a nilpotent normal subgroup of $\pi_1 X$ of finite index which acts nilpotently on $\pi_i X$ for $1 < i \leq k$. It may however *not* be possible to find a nilpotent normal subgroup of $\pi_1 X$ of finite index which acts nilpotently on *all* $\pi_n X$ $(n > 1)$.

An immediate consequence is:

2.3 PROPOSITION. *A space X is virtually nilpotent if and only if each Postnikov stage $P_n X$ has a finite regular covering space which is nilpotent.*

2.4 *Remark.* The words *normal* in 2.2 and *regular* in 2.3 could have been omitted, because of the algebraic fact that *every subgroup of finite index contains a normal subgroup of finite index.*

2.5 *Remark.* Virtually nilpotent groups and π-modules (which were implicitly defined in 2.2) behave a lot like nilpotent groups and π-modules. For instance

(i) *every subgroup, quotient group and central extension of a virtually nilpotent group is virtually nilpotent*, and
(ii) *in a short exact sequence $0 \to M' \to M \to M'' \to 0$ of π-modules, the module M is virtually nilpotent if and only if M' and M'' are so.*

This follows immediately from the corresponding properties of nilpotent groups and π-modules.

2.6 *Examples.* (i) *Simply connected spaces.*
More generally,
(ii) *nilpotent spaces*, and
(iii) *spaces with a finite fundamental group*, such as, for instance, the *real projective spaces RP^n.*

(iv) $K(\pi, 1)$'s, where π is a *supersolvable group*, i.e., π has a finite filtration by normal subgroups for which the successive quotients are cyclic groups. The *Klein bottle* is such a space. To show that these spaces are indeed virtually nilpotent one uses induction as follows. Let

$$1 \longrightarrow C \longrightarrow E \xrightarrow{f} D \longrightarrow 1$$

be a short exact sequence of groups in which C is cyclic and D is virtually nilpotent and let $N \subset D$ be a nilpotent subgroup of finite index. As the group of automorphisms of C is finite, there is a subgroup $M \subset N$ of finite index which acts trivially on C, and from this one readily deduces that $f^{-1}M \subset E$ is a nilpotent subgroup of finite index.

(v) *Supersolvable spaces*, i.e., connected spaces X such that $\pi_1 X$ is a supersolvable group and, for every integer $n > 1$, $\pi_n X$ has a finite $\pi_1 X$-filtration for which the successive quotients are cyclic. The argument is essentially the same as in (iv).

3. Localizations and completions

In order to state and prove our main results and put them in their proper perspective we need the notions of localizations and completions of spaces.

3.1 *Localizations.* Given a space X and an abelian group A, one can consider all *A-homology equivalences* $X \to Y$, i.e., all maps $X \to Y$ which induce an isomorphism $H_*(X; A) \approx H_*(Y; A)$, and ask whether there is, up to homotopy, a *terminal* one among these. If there is one, it is obviously unique, up to homotopy, and is called an *A-localization of X.* The existence of A-localizations for arbitrary X has long been an open question. It was finally settled by Bousfield in [3], where he constructed a *functorial A-localization* $X \to X_A$, which is also *natural in A* in the sense that there is a functorial commutative diagram

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ X_B & \to & X_A \end{array}$$

(in which the vertical maps are the localization maps) whenever B is such that every B-homology equivalence is an A-homology equivalence. This happens, for instance, if $B = Z$. Bousfield also showed [2] that there is no loss of generality in assuming that A is either a *subring of the rationals* or a *direct sum of groups Z/p*, where p runs through a set of primes. More precisely, if $R \subset Q$ denotes the subring which contains $1/p$ for the primes p for which A is uniquely p-divisible, and if P is as in 1.1, then

$$\begin{array}{lll} X_A \sim X_R & \text{if } A \text{ is not a torsion group} \\ X_A \sim X_{P \otimes R} & \text{if } A \text{ is a torsion group.} \end{array}$$

In general, given X and A, it is quite difficult to find out what X_A looks like. This may be due to the fact that X_A is obtained from X by means of a transfinite construction. Still, for certain X and A, one can get a good hold on X_A using *completions*, which are a kind of *first approximation to localizations*.

3.2 *Completions*. Given a space X and a ring R, which is either a *subring of the rationals* or Z/p, *the integers modulo a prime p*, the R-*completion of* X is [4, Chapter I] a functorial map $X \to R_\infty X$, where the space $R_\infty X$ is the inverse limit of a tower of principal fibrations with abelian fibres, on which one has a fairly good hold. Sometimes this map is an R-homology equivalence and in that case [4, Chapter VII] it is an R-localization. This happens, for instance, when $R_\infty X$ is nilpotent (3.3 and [4, p. 24]). In particular one has for nilpotent spaces ([4, Chapter V and Chapter VI] and [3, 4.3]).

3.3 PROPOSITION. *Let X be a nilpotent space. Then*

(i) *for every subring $R \subset Q$, the space $R_\infty X$ is nilpotent (in fact $Z_\infty X \sim X$) and the map $X \to R_\infty X$ is an R-localization, and*

(ii) *for every set of primes p, the space $\prod (Z/p)_\infty X$ is nilpotent and the map $X \to \prod (Z/p)_\infty X$ is a $(\oplus Z/p)$-localization.*

For virtually nilpotent spaces one has a similar but weaker result, which will be proven in Section 6.

3.4 PROPOSITION. *Let X be a virtually nilpotent space. Then*

(i) *the space $Q_\infty X$ is nilpotent and the map $X \to Q_\infty X$ is a Q-localization, and*

(ii) *for every set of primes p, the space $\prod (Z/p)_\infty X$ is virtually nilpotent and the map $X \to \prod (Z/p)_\infty X$ is a $(\oplus Z/p)$-localization.*

If X is virtually nilpotent and $R \subset Q$ a proper subring, then (see 4.2) the space X_R is also virtually nilpotent, but the map $X \to R_\infty X$ need *not* be an R-homology equivalence and hence *not* an R-localization either (this happens, for instance, for the real projective plane and $R = Z$ [4, p. 216]). Still it is in this case possible to get a good hold on the homotopy type of X_R, because X_R fits into an arithmetic square, which (see 4.1) is, up to homotopy, a fiber square and in which (3.4) the homotopy types of the other three spaces can be described in terms of the completion functors. Before going into all this in more detail in Section 4, we state one more proposition, which will also be proven in Section 6.

3.5 PROPOSITION. *Let X be a virtually nilpotent space. Then, for every set of primes p, every element $y \in \pi_1 Q_\infty(\prod (Z/p)_\infty X)$ can be written in the form $y = uv$, where u and v are in the image of $\pi_1(\prod (Z/p)_\infty X)$ and $\pi_1 Q_\infty X$ respectively.*

4. Arithmetic square theorems

Throughout this section $R \subset Q$ will be an arbitrary but fixed subring, $P = \oplus Z/p$ (p prime) as in 1.1 and hence $P \otimes R = \oplus Z/p$, where the direct sum runs through the primes p for which $1/p$ is not in R. Our main result then is:

4.1 FIRST ARITHMETIC SQUARE THEOREM. *If X is a virtually nilpotent space, then the arithmetic square for X (see* 1.1, 3.1 *and* 1.4)

$$\begin{array}{ccc} & X_R & \rightarrow & X_{P\otimes R} \\ & \downarrow & & \downarrow \\ X_Q \sim & X_{R,Q} & \rightarrow & X_{P\otimes R,Q} \end{array}$$

is, up to homotopy, a fiber square.

Combining this with 3.4, 3.5 and the argument of [5, 7.1] one gets:

4.2 COROLLARY. *If X is a virtually nilpotent space, then so is X_R.*

One also clearly has the slightly more general statement:

4.3 COROLLARY. *If X_R is a virtually nilpotent space, then the conclusion of Theorem* 4.1 *holds.*

To prove Theorem 4.1 one factors the arithmetic square for X into a *map* $X_R \rightarrow W$ and a *commutative diagram*

$$\begin{array}{ccc} W & \rightarrow & X_{P\otimes R} \\ \downarrow & & \downarrow \\ X_{R,Q} & \rightarrow & X_{P\otimes R,Q} \end{array}$$

which is, up to homotopy, a fiber square. Then [3, 5.5, 9.1 and 12.9] we have to show that the map $X_R \rightarrow W$ is an A-homology equivalence for $A = R$ or equivalently for $A = Q$ and $A = P \otimes R$. Thus it suffices to show that the maps $W \rightarrow X_{P\otimes R}$ and $W \rightarrow X_{R,Q}$ are $P \otimes R$- and Q-homology equivalences respectively. But this follows immediately from 3.4, 3.5 and the following result which is in some sense a complement of Theorem 4.1 and which will be proven in Section 7.

4.4 SECOND ARITHMETIC SQUARE THEOREM. *Let*

$$\begin{array}{ccc} W & \rightarrow & U \\ \downarrow & & \downarrow \\ V & \rightarrow & U_Q \end{array}$$

be a fiber square of connected spaces in which

(i) *the space U is $P \otimes R$-Bousfield (i.e., $U_{P\otimes R} \sim U$) and virtually nilpotent and the map $U \rightarrow U_Q$ is a Q-localization of U, and*

(ii) *the space V is Q-Bousfield and nilpotent.*

Then the space W is R-Bousfield and virtually nilpotent and the maps $W \to U$ and $W \to V$ are $P \otimes R$- and Q-homology equivalences and hence $P \otimes R$- and Q-localizations of W respectively. Moreover, if U is nilpotent, then so is W.

4.5 *Summary.* The first arithmetic square theorm implies that the homotopy type of an R-Bousfield and (virtually) nilpotent space X is completely determined by

(i) its *$P \otimes R$-localization*, the $P \otimes R$-Bousfield and (virtually) nilpotent space $X_{P \otimes R}$,

(ii) its *Q-localization*, the Q-Bousfield and nilpotent space X_Q, and

(iii) the *coherence map* $X_Q \to X_{P \otimes R, Q}$.

The second arithmetic square theorem provides a kind of complement, as it implies that spaces U and V and a map $V \to U_Q$ are, up to homotopy, the $P \otimes R$- and Q-localizations and the coherence map of an R-Bousfield and (virtually) nilpotent space *if and only if*

(i) the space U is $P \otimes R$-Bousfield and (virtually) nilpotent,

(ii) the space V is Q-Bousfield and nilpotent and

(iii) the homotopy inverse limit [4, Chapter X] of the diagram $V \to U_Q \leftarrow U$ is connected, i.e., every element $y \in \pi_1 U_q$ can be written in the form $y = uv$, where u and v are in the image of $\pi_1 U$ and $\pi_1 V$ respectively.

We end with the observation (4.6) that not all arithmetic squares are, up to homotopy, fibre squares.

Let π be an infinite cyclic group, let $M = \text{inj lim } Q[\pi]/I^s$, where I denotes the augmentation ideal of the rational group ring $Q[\pi]$, let $n > 1$ and let Y_{2n} and Y_{2n+1} be the spaces with two nontrivial homotopy groups

$$\pi_1 Y_{2n} = \pi_1 Y_{2n+1} = \pi \quad \text{and} \quad \pi_{2n} Y_{2n} = \pi_{2n+1} Y_{2n+1} = M$$

and the given action of π on M (which is clearly not nilpotent). Then we prove in Section 8:

4.6 *Counter examples. For at least one of the two spaces Y_{2n} and Y_{2n+1}, the arithmetic square is not, up to homotopy, a fiber square.*

5. Two useful lemmas

We state and prove here two lemmas which are used in the proofs of Proposition 3.4 and Theorem 4.4, but which are also of some interest in themselves.

5.1 Nilpotent Action Lemma. *Let $X \to K(\phi, 1)$ and $X \to K(\psi, 1)$ be fibrations with fibers F and G respectively. If F is nilpotent and the induced map $\pi_1 X \to \phi \times \psi$ is onto, then the group ψ*

(i) *is nilpotent, and*

(ii) *acts nilpotently on the groups $H_i(G; A)$ for every coefficient group A.*

Proof. Conclusion (i) follows readily from the fact that the fibration $X \to K(\psi, 1)$ restricts to a fibration $F \to K(\psi, 1)$ with connected fiber. To prove (ii) observe that the fibration $X \to K(\psi, 1)$ is the composition

$$X \to K(\phi, 1) \times K(\psi, 1) \to K(\psi, 1)$$

of two fibrations with connected fibers. In the first of these ψ acts nilpotently on the A-homology of the fiber because [4, p. 60 and p. 62] this is true for the induced fibration $F \to K(\psi, 1)$, in view of the fact that the latter is a fibration of nilpotent spaces with connected fiber. In the second fibration ψ obviously acts trivially on the A-homology of the fiber, and the desired result now follows by the argument of [4, p. 64].

5.2 PRE-NILPOTENCY LEMMA. *Let $R \subset Q$ be a subring, let ϕ be an R-perfect group (i.e., $H_1(\phi; R) = 0$) and let $Y \to K(\phi, 1)$ be a fibration with nilpotent fiber N. Then $R_\infty Y$ is nilpotent and hence so is Y_R.*

Proof. It suffices (3.2 and 3.3) to show that *Y is R-homology equivalent to a nilpotent space.* We do this in three steps.

First we apply the fiber-wise R-completion of [4, Chapter I] to the fibration $Y \to K(\phi, 1)$ and denote the resulting fibration by $Y' \to K(\phi, 1)$. *The map $Y \to Y'$* then *is an R-homology equivalence*, because (3.3) the map $N \to R_\infty N$ is so.

Next we note that (3.3) $R_\infty N$ is nilpotent and that in fact [4, p. 133] the group $\nu = \pi_1 R_\infty N$ is R-nilpotent, i.e., ν is nilpotent and its lower central series quotients are R-modules. If $\pi = \pi_1 Y'$ and ψ is the R-completion of π in the sense of [4, Chapter IV], then the 5-term exact sequence of the Serre spectral sequence together with the fact that $H_1(\phi; R) = 0$ yields that the map $H_1(\nu; R) \to H_1(\pi; R)$ is onto and from this and [4, p. 30] it follows that the map $\nu \to \psi$ is also onto. This in turn implies that the obvious map $\pi \to \phi \times \psi$ is onto. Thus (5.1) the fiber M of the fibration $Y' \to K(\psi, 1)$ is connected and ψ acts nilpotently on its R-homology. Moreover ψ is nilpotent and, as ψ is the R-completion of $\pi = \pi_1 Y'$, it follows [3, 7.5] that the map $Y' \to K(\psi, 1)$ induces an isomorphism $H_1(Y'; R) \approx H_1(\psi; R)$ and an epimorphism $H_2(Y'; R) \to H_2(\psi; R)$. A Serre spectral sequence argument now yields that $H_0(\psi; H_1(M; R)) = 0$. But this, together with the nilpotency of the action of ψ on $H_1(M; R)$, readily implies that $H_1(M; R) = 0$ and hence [4, p. 206] the map $M \to R_\infty M$ is an R-homology equivalence. If we apply the fiber-wise R-completion to the fibration $Y' \to K(\psi, 1)$ and denote the resulting fibration by $Y'' \to K(\psi, 1)$, then *the map $Y' \to Y''$ is* clearly also *an R-homology equivalence.*

Finally we observe [4, p. 206] that $\pi_1 R_\infty M = 1$ and [4, p. 133] hence $H_*(M; R) \approx H_*(R_\infty M; Z)$. Thus the group ψ acts nilpotently on the Z-homology of the simply connected space $R_\infty M$. In view of [4, p. 63], ψ therefore acts nilpotently on the E_1-term and hence all of the integral homotopy spectral sequence of $R_\infty M$ [4, p. 283]. As $R_\infty M$ is simply connected, this spectral sequence converges strongly to the homotopy groups of $R_\infty M$. From this it follows that ψ acts nilpotently on the homotopy groups of $R_\infty M$ and, as ψ itself is nilpotent, this finally implies that *the space Y'' is nilpotent.*

6. Proof of Propositions 3.4 and 3.5

Proof of Proposition 3.4. As it suffices to show that 3.4 holds for each Postnikov stage $P_n X$, *it is no real restriction if we assume the existence of a fibration $X \to K(\phi, 1)$ for which the group ϕ is finite and the fiber N is nilpotent.*

Under this assumption, (i) is an immediate consequence of 3.3 and 5.2.

We will prove (ii) only for "the set of all primes," as the proof of the other cases is essentially the same. Let J be the set of primes which divide the order of $H_1(\phi; Z)$ and let $R = Z[J^{-1}] \subset Q$. Then (3.3, 5.2 and [4, p. 184 and p. 188]) the spaces $R_\infty X$ and $\prod (Z/q)_\infty X \sim \prod (Z/q)_\infty R_\infty X$ are nilpotent, the map $X \to \prod (Z/q)_\infty X$ is a $(\oplus Z/q)$-localization and $\tilde{H}_*(\prod (Z/q)_\infty X; Z/p) = 0$ for all $p \in J$, where the direct products and the direct sum run through the primes q which are not in J.

To complete the proof suppose $p \in J$ and let $\Gamma^p\phi \subset \phi$ be the maximal (Z/p)-perfect subgroup. Then it is not hard to see that $\phi_p = \phi/\Gamma^p\phi$ is a finite p-group. Furthermore, if M is the fiber of the fibration $X \to K(\phi_p, 1)$ and $R' \subset Q$ is the largest subring not containing $1/p$, then N is the fiber of the fibration $M \to K(\Gamma^p\phi, 1)$ and, as $H_1(\Gamma^p\phi; R') = 0$, it follows from 3.3, 5.2 and [4, p. 188] that the spaces $R'_\infty M$ and $(Z/p)_\infty M \sim (Z/p)_\infty R'_\infty M$ are nilpotent. In the fibration $X \to K(\phi_p, 1)$ the finite p-group ϕ_p must [4, p. 215] act nilpotently on the (Z/p)-homology of M and as the maps

$$M \to (Z/p)_\infty M \quad \text{and} \quad K(\phi_p, 1) \to (Z/p)_\infty K(\phi_p, 1)$$

are (Z/p)-localizations (every finite p-group is nilpotent), it follows readily from [4, p. 62] that the map $X \to (Z/p)_\infty X$ is also a (Z/p)-localization and that the space $(Z/p)_\infty X$ is virtually nilpotent. Furthermore $\tilde{H}_*((Z/p)_\infty X; Z/r) = 0$ for all primes $r \neq p$, because the same holds for $(Z/p)_\infty M$ and $(Z/p)_\infty K(\phi_p, 1)$ [4, p. 184]. Application of the Kunneth theorem now completes the proof of Proposition 3.4.

Proof of Proposition 3.5. If $\pi_1 X$ is *abelian*, this is an easy calculation and for *nilpotent* $\pi_1 X$ the desired result then follows from [4, p. 130 and p. 170]. Finally one gets the general case by a careful analysis of the above proof of Proposition 3.4, which yields the exactness of the sequences

$$\pi_1 Q_\infty N \to \pi_1 Q_\infty X \to 1,$$

$$\pi_1(\prod (Z/p)_\infty N) \to \pi_1(\prod (Z/p)_\infty X) \to \prod \phi_p \to 1,$$

$$\pi_1 Q_\infty(\prod (Z/p)_\infty N) \to \pi_1 Q_\infty(\prod (Z/p)_\infty X) \to 1.$$

7. Proof of Theorem 4.4

Clearly V and U_Q are R-Bousfield and so [4, p. 188] is U and [3, Section 12] hence W. Moreover the argument of [5, 7.1] shows that W is (virtually) nilpotent.

To prove the rest of the theorem we show that there exists an integer $n \geq 1$ and a factorization

$$\begin{array}{ccccccccccccc} W & \to & A_n & \to \cdots \to & A_i & \to & A_{i-1} & \to \cdots \to & A_1 & = & U \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ V & \to & B_n & \to \cdots \to & B_i & \to & B_{i-1} & \to \cdots \to & B_1 & = & U_Q \end{array}$$

of the fiber square such that

(i) each small square is, up to homotopy, a fiber square,
(ii) each B_i is Q-Bousfield and nilpotent,
(iii) each map $A_i \to B_i$ is a Q-homology equivalence,
(iv) each map $A_i \to A_{i-1}$ is a $P \otimes R$-homology equivalence, and
(v) the map $\pi_1 V \to \pi_1 B_n$ is onto,

and then proceed as follows. Let F be the homotopy fiber of the maps $V \to B_n$ and $W \to A$. Then F is connected and [4, p. 60, p. 62] $\pi_1 B_n$ and $\pi_1 A_n$ act nilpotently on the Q-homology of F. As the map $A_n \to B_n$ is a Q-homology equivalence, so is, by the Quillen-Zeeman comparison argument of [4, p. 92], the map $W \to V$. Thus the latter is a Q-localization of W. To prove that the map $W \to U$ is a $P \otimes R$-localization of W, we observe that [4, Chapter V] F has uniquely divisible Z-homology groups. Hence $\tilde{H}_*(F; P) = 0$ and thus the map $W \to A_n$ is a $P \otimes R$-homology equivalence.

The idea behind the construction of the above factorization is to try and "kill off" the cokernel of the map $\pi_1 Y \to \pi_1 U_Q$ by killing off the cokernel of its abelianization $H_1(V; Z) \to H_1(U_Q; Z)$. This does not succeed right away, but it succeeds eventually.

Suppose we already constructed the spaces A_i and B_i for $1 \leq i \leq k$ in such a manner that (i), (ii), (iii) and (iv) are satisfied. If the map $\pi_1 V \to \pi_1 B_k$ is onto, we are done. Otherwise let

$$\psi_k = \operatorname{coker}\,(H_1(V; Z) \to H_1(B_k; Z))$$

and define A_{k+1} and B_{k+1} as the covering spaces of A_k and B_k which correspond to the kernels of the maps $\pi_1 A_k \to \psi_k$ and $\pi_1 B_k \to \psi_k$. Clearly $\pi_1 B_k \to \psi_k$ is onto and, as every element of $\pi_1 B_k$ is the product of an element in the image of $\pi_1 V$ and one in the image of $\pi_1 A_k$, so is the map $\pi_1 A_k \to \psi_k$. Thus the resulting square is a fiber square. Furthermore it follows from the fact that B_k and $K(\psi_k, 1)$ are nilpotent and have uniquely divisible homotopy groups, that B_{k+1} has the same properties.

To prove (iii) and (iv) for $i = k + 1$ we observe, as at the beginning of Section 6, that it is no real restriction to assume the existence of a fibration

$U \to K(\phi, 1)$ for which the group ϕ is finite and the fiber N is nilpotent, and construct the diagram of fiber squares

$$\begin{array}{ccccccccccc} N_{k+1} & \to & N_k & \to \cdots \to & N_i & \to & N_{i-1} & \to \cdots \to & N_1 & = & N \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ A_{k+1} & \to & A_k & \to \cdots \to & A_i & \to & A_{i-1} & \to \cdots \to & A_1 & = & U \end{array}$$

in which each N_{i+1} is the covering space of N_i corresponding to the kernel of the map $\pi_1 N_i \to \psi_i$, which is onto because ψ_i has no subgroups of finite index. One then readily verifies that for $1 \leq i \leq k$,

(vi) each N_{i+1} is nilpotent,
(vii) each map $\pi_1 A_i \to \phi \times \psi_i$ is onto, and
(viii) N_{i+1} is the fiber of the fibration $A_{i+1} \to K(\phi, 1)$.

Thus (5.1) the group ψ_k acts nilpotently on the Q-homology of A_{k+1} and B_{k+1} and condition (iii) for $i = k + 1$ now follows from the case $i = k$ by the argument of [4, Chapter III, Section 7]. Furthermore we note (5.1) that ψ_k acts nilpotently on the P-homology of A_{k+1}. From this and the fact that

$$\tilde{H}_*(K(\psi_k, 1); P) = 0$$

it is not hard to deduce that the map $A_{k+1} \to A_k$ is a P-homology equivalence.
It remains to show (v), but this follows from:

7.1 LEMMA. *Let ν_1 be a group, $\mu \subset \nu_1$ a subgroup and ν_i $(i \geq 1)$ the associated relative derived series subgroups of ν_1, i.e.,*

$$\nu_{i+1} = \ker (\nu_i \to \operatorname{coker} (\mu/\Gamma_2\mu \to \nu_i/\Gamma_2\nu_i))$$

where Γ_2 denotes the commutator subgroup. If ν_1 is nilpotent, then there is an integer $n \geq 1$ such that $\nu_j = \mu$ for $j \geq n$.

This can be proved by verifying inductively that each ν_i is contained in the subgroup of ν_1 generated by μ and $\Gamma_i\nu_1$, the ith term of the lower central series of ν_1.

8. Proof of counter examples 4.6

The key to the proof of 4.6 is an algebraic lemma.

8.1 LEMMA. *Let π and M be as in 4.6 and let $\sigma = \pi \otimes Q$. Then the π-module structure on M can be extended to a unique σ-module structure and this extension has the property that the induced epimorphism*

$$H_0(\pi; M \otimes M) \to H_0(\sigma; M \otimes M)$$

is not an isomorphism. (Here the tensor product is taken over Z and the π- and σ-actions are the diagonal ones.)

Since M is a rational vector space, the tensor square $M \otimes M$ is naturally isomorphic to the direct sum $Sym^2M \oplus \Lambda^2M$, where Sym^2M and Λ^2M are the second *symmetric* and *exterior* powers of M. The following corollary is thus immediate.

8.2 COROLLARY. *At least one of the induced epimorphisms*

$$H_0(\pi; \Lambda^2M) \to H_0(\sigma; \Lambda^2M) \quad \textit{and} \quad H_0(\pi; Sym^2M) \to H_0(\sigma; Sym^2M)$$

is not an isomorphism. (Probably neither is.)

Proof of 4.6. If the first alternative of 8.2 holds, put $X = Y_{2n+1}$ and assume that the arithmetic square for X is, up to homotopy, a fiber square. An easy Serre spectral sequence argument then shows that the obvious map $X \to S^1$ is a P-homology equivalence. As [3, 5.5] $X_Z \sim X$, it is not hard to deduce from this that X_Q has only two nontrivial homotopy groups

$$\pi_1X_Q = \pi \otimes Q = \sigma \quad \text{and} \quad \pi_{2n+1}X_Q = M$$

where the action of σ on M extends the one of π. However this implies that the induced map

$$H_{4n+2}(X; Q) = H_0(\pi; \Lambda^2M) \to H_{4n+2}(X_Q; Q) = H_0(\sigma; \Lambda^2M)$$

is an isomorphism, in contradiction to our assumption.

If the second alternative of 8.2 holds, one puts $X = Y_{2n}$ and uses a similar argument.

Proof of 8.1. Identify M with the additive group of formal power series $Q[[x]]$ is one variable over Q. The action of the generator $a \in \pi$ then is given by the formula

$$a \cdot p(x) = (1 + x)p(x).$$

As for every integer $n > 0$ the power series $1 + x$ has a unique nth root with constant term 1, it follows that the π-module structure extends to a unique σ-module structure. Thus there is an element $b \in \sigma$ with $b^2 = a$ for which the action is given by the formula

$$b \cdot p(x) = (1 + x)^{1/2}p(x)$$

where $(1 + x)^{1/2}$ denotes the obvious binomial power series. Now make the invertible change of variables $u = (1 + x)^{1/2} - 1$, so that M becomes isomorphic to $Q[[u]]$ with

$$a \cdot p(u) = (1 + u)^2p(u), \qquad b \cdot p(u) = (1 + u)p(u).$$

If $\pi' \subset \sigma$ denotes the subgroup generated by b, then we will finish the proof by showing that the map $H_0(\pi; M \otimes M) \to H_0(\pi'; M \otimes M)$ has a nontrivial kernel.

If one thinks of $M \otimes M$ as the additive group of $Q[[u]] \otimes Q[[v]]$ inside $Q[[u, v]]$, then it is not hard to see that it suffices to show that

$$(1 + u)(1 + v) - 1 = u + v + uv$$

is not divisible within $Q[[u]] \otimes Q[[v]]$ by

$$(1 + u)^2(1 + v)^2 - 1 = (u + v + uv)(2 + u + v + uv)$$

or equivalently, that $(2 + u + v + uv)^{-1}$ does not lie in $Q[[u]] \otimes Q[[v]]$. And this can be proven by the identity

$$\frac{1}{2 + u + v + uv} = \frac{1}{2 + u} \sum_{k=0}^{\infty} (-1)^k \left(\frac{1 + u}{2 + u}\right)^k v^k$$

together with the algebraic fact that a formal power series $\sum C_{ij}u^i v^j$ lies in $Q[[u]] \otimes Q[[v]]$ only if the rank of the matrix (C_{ij}) (i.e., the least upper bound of the ranks of its finite submatrices) is finite.

Bibliography

1. A. K. Bousfield, *Homological localizations of spaces, groups and Π-modules*, Lecture Notes in Math., no. 418, Springer, New York, 1974, pp. 22–30.
2. ———, *Types of acyclicity*, J. Pure and Applied Algebra, vol. 4 (1974), pp. 293–298.
3. ———, *The localization of spaces with respect to homology*, Topology, vol. 14 (1975), pp. 133–150.
4. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math., no. 304, Springer, New York, 1972.
5. E. Dror, *A generalization of the Whitehead theorem*, Lecture Notes in Math., no. 249, Springer, New York, 1971, pp. 13–22.
6. D. Sullivan, *Geometric topology, part I: localization, periodicity and Galois symmetry*, M.I.T., 1970.

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