ON COLLINEATION GROUPS THAT FIX A LINE OF A FINITE PROJECTIVE PLANE

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1. A number of results have been obtained recently concerning the structure of the collineation group G of a finite projective plane Π (see [10], [11], and [12]). One situation that has not been handled in those papers is the case where G fixes a line of Π. The purpose of this paper is to give the structure of G when G fixes a line q of Π and the following is satisfied.

Hypothesis H. (i) q has $n + 1$ points, n odd.
(ii) G contains no Baer involutions.
(iii) G contains at most one involutory homology with given center and axis.

Throughout, if X is a collineation group of Π we let $\bar{X}$ represent the action of X on q as a permutation of the points of q. In the following theorem let $\hat{X}$ be the subgroup of central collineations in X with axis q.

THEOREM 1. *If G is a collineation group of a finite projective plane Π fixing a line q of Π and satisfying Hypothesis* H, *then G is solvable or G contains a normal subgroup H such that G/H is solvable and such that one of the following holds:*

(i) *H has a normal subgroup N such that N is solvable and $H/N \cong PSL_2(r)$, r odd.*
(ii) *$\bar{H}/O(\bar{H}) \cong Psp(4, r)$, $\hat{H}$ solvable.*
(iii) *$\bar{H}/Z(\bar{H}) \cdot O(\bar{H}) \cong A_5 \cdot E_{16}$.*
(iv) *$\bar{H}/Z(\bar{H}) \cdot O(\bar{H}) \cong G_1 \times G_2$ where G_i is isomorphic to $PSL_2(r)$ (r odd) or A_7.*

$A_5 \cdot E_{16}$ is explained below. Collineation groups of the above type are known to exist except for (iii) and (v) for $G \cong A_7$.

The results of [2] are a special case of Theorem 1. Instances where H(iii) does not hold have been handled in [12]. Cases where G does not fix a line of Π have been considered in [10] and [12]. Nothing is known in general if H(ii) does not hold.

We list several recent results in group theory but first we give some definitions.

DEFINITION 1.1. Let S be a Sylow 2-subgroup of a group G. G is said to be fusion simple if:
(1) G has no subgroups of index 2.
(2) Every involution of $Z(S)$ is conjugate in G to another involution in S.
(3) $O(G) = \langle 1 \rangle$.

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DEFINITION 1.2. A 2-subgroup S is said to be of sectional 2-rank at most m if for every subgroup T of S and every homomorphic image R of T, the elementary abelian subgroups of R have order at most 2^m.

DEFINITION 1.3. A group G is said to be quasisimple if $G = G'$ and $G/Z(G)$ is simple.

THEOREM 2 [7]. *If G is a finite simple group of sectional 2-rank at most 4, then G is isomorphic to one of the groups in the following list.*

(I) Odd characteristic. $L_2(q)$, $L_3(q)$, $U_3(q)$, $G_2(q)$, $D_4^2(q)$, $Psp(4, q)$, *q odd,* $G_2^1(q)$, *q an odd power of* 3, $L_4(q)$, $q \not\equiv 1 \pmod 8$ *or* $U_4(q)$, $q \not\equiv 5 \pmod 8$, $L_5(q)$, $q \equiv 3 \pmod 4$, *and* $U_5(q)$, $q \equiv 1 \pmod 4$.

(II) Even characteristic. $L_2(8)$, $L_2(16)$, $L_3(4)$, $U_3(4)$, *or* $Sz(8)$.

(III) Alternating. A_7, A_8, A_9, A_{10}, *or* A_{11}.

(IV) Sporadic. M_{11}, M_{12}, M_{22}, M_{23}, J_1, J_2, J_3, M^c, *or* L.

(The explanation of the symbols can be found in [7] and [13].)

COROLLARY B. *If G is a quasisimple group of 2-rank 2 with $O(G) = 1$, then either G is simple or $G \cong Sp(4, q)$, q odd.*

(2-rank 2 means the Sylow 2-subgroups of G contain elementary abelian subgroups of order at most 4.)

COROLLARY C. *If G is a nonsolvable fusion-simple group of sectional 2-rank at most 4, then one of the following holds:*

(i) *G' is simple.*

(ii) *G' is the direct product of two simple groups of sectional 2-rank 2.*

(iii) *G' is the direct product of a simple group of sectional 2-rank 2 and $Z_2n \times Z_2n$ for some n.*

(iv) *G' is a nontrivial extension of E_8 or E_{16} by A_5, A_6, A_7, or $L_3(2)$.*

THEOREM 3 ([14], Four Generator Theorem). *Let S be a 2-group with no normal abelian subgroup of rank 3. Then every subgroup of S can be generated by four (or fewer) elements.*

A subgroup of rank 3 is elementary abelian subgroup of order 8.

In this paper small letters p, q will represent lines, letters A, B will represent points. AB is the line joining A and B, pq the intersection of p and q. PIq means P lies on the line q. An involution is a group element of order 2. Throughout, q will be the line fixed by G. $\bar{x}$ will be the action of the collineation x on the fixed line q of Π. The group theoretic notation is standard and can be found in [6], [9], and [13].

The geometric notation is also standard and is found in [3]. A well-known theorem on collineations of projective planes states that the fixed point set of a collineation or order 2 is either (i) a line, or (ii) a line with the point not on it, or (iii) a square root subplane. In this paper we are assuming all involutions are of type (ii). [9, Theorem 20.9.7, p. 405.]

In Section 2 we prove the main result of this paper and in Section 3 we prove results related to the structure of the 2-subgroups of collineation groups of finite planes. These latter results are interesting in their own right and hopefully will prove useful in the general study of collineation groups of finite projective planes.

I would like to thank the referee for suggestions that considerably shortened the proof of Theorem 1.

2. In this section we give the proof of Theorem 1. The connection between Theorems 2 and 3 follows easily from the following two lemmas.

LEMMA 2.1 [3, 3.18, p. 120]. *Let a and b be involutory homologies with centers A and B and axis p and q, respectively. If $pIB \neq AIq \neq p$ then ab is an involutory homology with axis AB and center pq. a, b, and ab are the only involutory homologies having their respective centers and axis.*

LEMMA 2.2. *If G is a collineation group satisfying Hypothesis* H, *then an elementary abelian* 2*-subgroup of G has order at most* 4, *i.e., G has* 2*-rank at most* 4.

Proof. Assume T is an elementary abelian 2-subgroup of G of order 8. Let $T = \langle a, b, c\rangle$ where $|a| = |b| = |c| = 2$ and T is abelian. Let a and b be as in Lemma 2.1 with A and B their centers. Let C be the center of ab. Since c centralizes $\langle a, b\rangle$, c must fix the set $\{A, B, C\}$ a triangle. Since c is an involutory homology, c must fix each of the points A, B, C. One of these points must be the center of c and the line through the remaining two must be its axis. By Lemma 2.1, $c \in \langle a, b\rangle$ contradicting $|T| = 8$. This proves the lemma.

Lemma 2.2 implies that G has 2-rank 2. This immediately implies that G has normal 2-rank 2, i.e., if $S \in Syl_2(G)$ and $T \lhd S$, T elementary abelian, then $|T|$ is at most 4. Theorem 3 now implies that G has sectional 2-rank at most 4, so we can apply Theorem 2 and its corollaries to G.

Before proceeding with the application of Theorem 2, we handle some trivial cases. Throughout this section, q will be the line Π fixed by G.

The following lemma shows that we can assume G contains involutions with axis q.

LEMMA 2.3. *Assume G is a collineation group satisfying Hypothesis* H *and fixing the line q of Π. If every involutory homology of G acts non-trivially on q, then G is solvable or G has a normal subgroup H such that G/H is solvable and H has a normal subgroup N such that H/N is isomorphic to A^7 or $SL_2(q)$.*

Proof. Let $S \in Syl_2(G)$. Let $d \in Z(S)$, $|d| = 2$. Let d fix points L and M on q. Let $e \in S - \{d\}$, $|e| = 2$. If e fixes L and M, since no two involutions can have the same center and the element, $a = de$ must have a center off q, and its axis must be q since a fixes L and M. This contradicts the hypothesis of our lemma. e commutes with d, so e fixes $\{L, M\}$. We have just seen that e does not fix this set pointwise so $e(L) = M$. From this last result we easily see that $ed \neq de$. d must be the only involution in S so S is cyclic or generalized quaternion. By [5] if $\hat{G} = G/O(G)$, $\hat{d} \in Z(\hat{G})$. The Sylow 2-subgroups of $\bar{G} = \hat{G}/\langle \hat{d} \rangle$ are cyclic or dihedral. If the Sylow 2-subgroups are cyclic, it is well known that $\bar{G}$ is solvable, hence G is. If the Sylow 2-subgroups are dihedral the lemma follows from Theorem 2.

We have just seen that if the condition of Lemma 2.3 holds then Theorem 1 follows. We can now assume that there are involutory homologies in G having axis q. The following lemma shows that we can assume G fixes a point P of Π. The following is in [1; Satz 4.3.2.]

LEMMA 2.4. *Let H be the subgroup of G generated by all homologies with axis q. Then H contains an abelian normal subgroup T of elations with axis q and T is transitive on the set of centers of homologies of H.*

We are assuming $H \neq 1$, so that a is an involutory homology in H with center P and axis q. $G = G_pT$, T as above. Since T is abelian, from the statement of Theorem 1, we see there is no loss in assuming $G = G_p$. Now, since $G = G_p$, G fixes P and q and $a \in Z(G)$, since by Hypothesis H, a is the only involution with center P and axis q.

We now will show that we can assume $G = G'$. Assume G contains a proper normal subgroup H with G/H solvable. H satisfies Hypothesis H and $|H| < |G|$, so by induction we can assume Theorem 1 true for H. Clearly, Theorem 1 now holds true for G since G/H is solvable. We see there is no loss in assuming $G = G'$. Lemmas 2.3 and 2.4 have shown us we can assume there is an $a \in Z(G)$, $|a| = 2$. There is no loss in assuming $O(G) = 1$. Theorem 1 follows from the following group theoretic result.

THEOREM 6. *If G is a finite group of 2-rank at most 2, such that $G = G'$, $O(G) = 1$, $2 \mid |Z(G)|$, then one of the following holds:*

(a) *$G/N \cong PSL_2(q)$, q odd, N a solvable subgroup of G.*

(b) *$G/Z(G) \cong A_5 \cdot E_{16}$ or $A_6 \cdot E_{16}$.*

(c) *$G \cong Sp(4, q)$, q odd.*

(d) *$G/Z(G) \cong G_1 \times G_2$ where G_i is isomorphic to $PSL_2(q)$ (q odd) or to A_7.*

Proof. Assume G has 2-rank 1. If $S \in Syl_2(G)$, S is cyclic or generalized quaternion. Using $G' = G$ and the argument used in the latter part of the proof of Lemma 2.3 we get G/N isomorphic to $PSL(2, q)$ (q odd) or to A_7, N a solvable normal subgroup of G. This is (a) in the statement of the theorem.

By Lemma 3.2 of [2], there is a normal 2-subgroup N of G such that $\bar{G} = \bar{G}/N$ is fusion simple (if necessary we consider groups of odd order fusion simple). G is sectional 2-rank at most four by Lemma 2.2, so Corollary C of [7] applies to $\bar{G}$.

The group $O_2(\bar{G})$ need not be 1, so we will let M be the maximal normal 2-subgroup of G, with $M \trianglerighteq N$ (N as above), G/N fusion simple. We will concentrate our attention on M.

Assume M has a characteristic elementary abelian subgroup E with $|E| > 2$. Since G has 2-rank 2, $|E| = 4$. Let $a \in Z(G)$, $|a| = 2$. We can assume $E = \langle a, b\rangle$, $|b| = 2$. Since E is characteristic in M and $G \trianglelefteq M$ we have $G \trianglerighteq E$. Thus $G \trianglerighteq C_G(E)$. $|G\colon C_G(E)| \leq 6$ and since $G = G'$, we must have $E \subseteq Z(G)$. There are no involutions c in $G - E$, for then $\langle c, E\rangle$ would be an elementary abelian group in G of order 8, which is impossible. G contains at most three involutions, so the theorem follows from the argument given in Section 3 of [2].

From now on we can assume M has no characteristic elementary abelian subgroup of order 4. From [6, Theorem 4.9, p. 198] we have that M is the central product of groups E and R where $E = 1$, or E is extraspecial and either R is cyclic, quaternion, dihedral, or semidihedral. $E/Z(E)$ is elementary abelian of order 4^k for some k [6, Theorem 5.2(iii), p. 204]. M has sectional 2-rank at most 4, $M = ER$ so $|E/Z(E)| \leq 4$ and $|E| \leq 8$. Let $\bar{M} = M/\Phi(M)$. If G acts trivially on $\bar{M}$, then $\tilde{G} = G/C_G(M)$ is a 2-group. Since $G = G'$, this implies $G = C_G(M)$ and M is abelian. M has no characteristic elementary abelian subgroups of order 4 so M is cyclic. We have that either M is cyclic or G acts nontrivially on $\bar{M}$.

We now take up the case where G acts nontrivially on $\bar{M}$. G has sectional 2-rank at most 4, so $|\bar{M}| \leq 16$. Clearly $|\bar{M}| \geq 8$ since $G = G'$. Assume $|\bar{M}| = 8$. $M = ER$ and hence $|E| = 8$ and R is cyclic. Clearly G must act irreducibly on M since $G = G'$, but this is clearly seen to be impossible if E is extraspecial of order 8 and R is cyclic. Now assume that $|\bar{M}| = 16$ and G acts irreducibly on $\bar{M}$. Since $M = ER$, M must be extraspecial. In the characteristic 2 case, the outer automorphism group of M is contained in one of the 4-dimensional orthogonal groups over $GF(2)$. This latter fact implies $\tilde{G} = C_G(M) \cong A_5$ and part (b) of the theorem holds.

We now assume M is cyclic, so N is cyclic, G/N is fusion simple. In this case $M = N$. Also $2 \mid |N|$. We apply Corollary C of [7]. If (i) holds, then G is a quasisimple group. Corollary B of [7] now applies. G is not simple because $2 \mid |Z(G)|$, so $G \cong Sp(4, q)$ and (c) of the theorem holds.

If (ii) holds, $\bar{G} = G/N \cong L_1 \times L_2$ where L_1 and L_2 are simple groups of sectional 2-rank 2. Let L be a subgroup of G such that $L \supset N$ and $L/N \cong L_1$. If L has 2-rank 2 and $N \cap L' \neq 1$, then $L \cong Sp(4, q)$ by Corollary B. This is impossible since $Sp(4, q)$ has sectional 2-rank 4 and L_1 does not. Thus if L has 2-rank 2, we must have $L' \cap N = 1$. If L' has 2-rank 2, then L has an elementary abelian group of order 8 which is impossible. Thus L' has 2-rank 1, implying that its Sylow 2-subgroups are quaternion. But [5] implies there is a $z \in Z(L')$, which is impossible since $L' \cong L_1$, which is simple. We now have that L has 2-rank 1 and as was seen in the first part of the proof of this theorem, this implies L_1 isomorphic to $PSL_2(q)$ (q odd) or to A_7. The same holds for L_2, so part (d) of the theorem holds. Part (iii) of Corollary C cannot hold since we are assuming $G = G'$. If (iv) holds, then as we saw above, part (b) of the theorem holds.

3. In this section we give some results on 2-subgroups of collineation groups. These results are interesting in their own right and hopefully will prove useful in further classification of collineation groups of finite projective planes.

LEMMA 3.1. *Let a, b, c, and d be involutory homologies acting on a finite projective plane Π. Let a have center P and axis q, and assume b, c, and d, centralize a, and have centers on q. Assume b, c, and d all interchange the points L and M on q. If $S = \langle b, c, d\rangle$ is a 2-group containing no Baer involutions, then the subgroup of S fixing L and M, is cyclic or generalized quaternion.*

Proof. Let x be the points of the line PL excluding $\{P, L\}$ and let y be the points of PM excluding $\{P, M\}$. Call these sets the x and y axis, respectively. Let $T = x \cup y$. S acts faithfully as a permutation group on T, with b, c, and d interchanging x and y. Since $|b| = 2$ we can order the points of T so that b has the permutation matrix

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

(I the $(n-1) \times (n-1)$ identity matrix, n the order of Π). For any collineation z in S_{LM} let $x(z)$ and $y(z)$ represent the action of z on the x and y axis, respectively. Since $|c| = |d| = 2$, if we let $x(bc) = C$ and $x(bd) = D$, then $y(bc) = C^{-1}$ and $y(bd) = D^{-1}$, respectively. Since S is a 2-group, $|bc|$, $|bd|$, $|C|$, $|D|$ are all powers of 2.

Let $G = \langle bc, bd\rangle$. Assume C and D commute. Let $e \in G$, $e = (bc)^i(bd)^j$, $|e| = 2$. $x(e) = C^iD^j$, $y(e) = C^{-i}D^{-j}$, $(x(e))^{-1} = D^{-j}C^{-i}$, and $D^{-j}C^{-i} \sim C^{-i}D^{-j}$. Inverses and conjugates have the same order, so $|x(e)| = |y(e)| = 2$. e is a homology since $|e| = 2$, and it acts nontrivially on the x and y axis. e fixes P, L, M so the axis of e is q and from Lemma 2.1, $e = a$. Thus G contains one element of order two, is abelian, and hence cyclic. Thus the lemma is proved and from now on we assume C and D do not commute.

We have $ab = ba$, so if we let $x(a) = A$, then $y(A) = A$.

By the argument of the previous paragraph the element of order two in $\langle bc\rangle$ and $\langle bd\rangle$ is a. Thus $A \in \langle C\rangle$ and $A \in \langle D\rangle$. Let E be the largest power of C such that $E \notin \langle C\rangle \cap \langle D\rangle$, but $E^2 \in \langle C\rangle \cap \langle D\rangle$. Let F be the corresponding power of D. E and F must exist if C and D do not commute. We can assume $E^2 = F^2$. Let $e \in \langle b, c\rangle$ with $x(e) = E$, $y(e) = E^{-1}$, and let $f \in \langle b, d\rangle$ with $x(f) = F$ and $y(f) = F^{-1}$. $E^{-1}F$ is conjugate to FE^{-1}, and FE^{-1} is $(EF^{-1})^{-1}$. Thus $E^{-1}F$ and EF^{-1} fix the same number of points. If E and F commute, then $e^{-1}f$ would have order two and so $e^{-1}f = a$ by the argument we used above. But then $f = ea$, and $F = EA \in \langle C\rangle$, a contradiction to the choice of F. Thus E and F do not commute.

Since e and f centralize $e^2 = f^2$, $\langle e, f\rangle \trianglerighteq \langle e^2\rangle$. Since $f^2 \in \langle e^2\rangle$, $\langle e, f\rangle/\langle e^2\rangle$

is a dihedral group. Let $g = ef$. $\langle e, f\rangle = \langle e, g\rangle$, and every element in $\langle e, g\rangle$ can be written in the form $e^i g^j$. Suppose $e^i g^j$ has order two. Then $x(e^i g^j) = E^i(EF)^j$ and $y(e^i g^j) = E^{-i}(E^{-1}F^{-1})^j$ have order dividing 2. These two matrices must have the same order. For we have $E^{-1}F^{-1} = E^{-1}(EF)^{-1}E$. Thus

$$E^{-i}(E^{-1}F^{-1})^j = E^{-i}(E^{-1}F^{-1})^j = E^{-i}E^{-1}(EF)^{-j}E.$$

This matrix is conjugate to $((EF)^{-j}E^{-1})E^{-i}E = (EF)^{-j}E^{-i}$. This matrix in turn is $[E^i(EF)^j]^{-1}$. Thus $e^i g^j$ has order two on both the x and y axis, and since there are no Baer involutions, again by Lemma 2.1, $e^i g^j = a$. Thus $\langle e, g\rangle$ has only one element of order two, so it is cyclic or generalized quaternion. If the group is cyclic, then E and F commute, which we have shown is not the case.

We now have to handle the possibility of G being larger than $\langle e, f\rangle$. $e^2 = f^2$ so $|e| = |f| = 4$. Let $h \in \langle g\rangle$, $|h| = 4$. Then $\langle e, h\rangle$ is quaternion. Assume there exists an $r \in \langle bc\rangle$ with $r^2 = e$. $\langle e\rangle \lhd \langle r, h\rangle$ and $r^2 \in \langle e\rangle$, $h^2 \in \langle e\rangle$ so $\langle r, h\rangle/\langle e\rangle$ is dihedral. We have $r^{-1}(rh)r = (rh)^{-1}e^i$ for some i. $h^2 = e^2$ and $r^{-2} = e^{-1}$ yield $i = -1$. We compute directly that $(rh)^{-1}e(rh)^2 = e^{-1}$ so $r^{-1}(rh)^2 r = (rh)^{-2}$. We also see $(rh)^2 \in C(e)$. Let $k = (rh)^2$. Since $e^{-1} = ea$, $E^{-1} = EA$. Also $F^{-1} = FA$ so $x(g^i) = y(g^i)$ for all $g^i \in \langle g\rangle$. Let $x(h) = y(h) = H$. Let $x(k) = K$, $x(r) = R$ so $K = RHRH$. But we also have $y((rh)^2) = R^{-1}HR^{-1}H = RHRH$, because $R^2 = E$, $H^{-1}EH = E^{-1}$, and $H^2 = E^2$. Thus $y((rh)^2) = K$. We have shown $r^{-1}kr = k^{-1}$ so every element in $\langle r, k\rangle$ can be written as $r^i k^j$. Assume $|r^i k^j| = 2$. Assume $R^i K^j = 1$. Then $K^j = R^{-i}$. r centralizes e and inverts k, so $|r^i k^j| = 2$ implies $|K^j| = |R^{-i}| \leq 2$. But then $y(r^i k^j) = R^{-i}K^j = R^i K^j = x(r^i k^j) = 1$ which is impossible. Then $|R^i K^j| = 2$. By the same argument $|R^{-i}K^j| = 2$ implying $r^i k^j = a$. Thus $\langle r, k\rangle$ is cyclic or generalized quaternion. Since $e \in Z\langle r, k\rangle$ and $|e| = 4$, $\langle r, k\rangle$ must be cyclic. r inverts k so $|k| \leq 2$. It is clear that $|rh| > 2$, so $|k| = 2$, and $|rh| = 4$. Also $(rh)^2 = e^2 = h^2$. This and $(rh)^4 = 1$ gives $h^{-1}rh = r^{-1}$, so $\langle r, h\rangle$ is generalized quaternion.

Suppose $|g| > |h|$. We can assume $g^2 = h$. r normalizes $\langle e, h\rangle$ since $[\langle r, h\rangle : \langle e, h\rangle] = 2$ and g normalizes $\langle e, h\rangle$ by similar reasoning. r and g generate distinct nontrivial automorphisms of $\langle e, h\rangle$ of order two. But the automorphism group of the quaternion group $\langle e, h\rangle$ is isomorphic to A_3. Thus $3 \mid |S|$ which is impossible. Thus we conclude that if $r \notin \langle e\rangle$, then $|g| = 4$ and $\langle e, f\rangle$ is a quaternion group. Also, $\langle r, h\rangle$ is a generalized quaternion group containing $\langle e, f\rangle$.

Now assume $bd \notin \langle f\rangle$. We can assume $(bd)^2 = f$, and using the same argument we used for r, we get bd acting on $\langle e, f\rangle$ as an automorphism of order two, distinct from the action of r or $\langle e, f\rangle$. This is impossible. Thus if $r \notin \langle e\rangle$, we get $bd \in \langle f\rangle$.

The last obstacle is to show $\langle r, f\rangle = \langle bc, bd\rangle$ is generalized quaternion. This is the case if $(bc)^2 \in \langle e\rangle$. If $(bc)^2 \notin \langle e\rangle$, we can proceed by induction, assuming $\langle r^2, f\rangle$ is generalized quaternion, so that $\langle r, f\rangle/\langle r^2\rangle$ is dihedral, and then we can use the arguments that we used for $\langle r, h\rangle$ to show $\langle r, f\rangle$ is generalized quaternion.

All this is under the assumption $r \in \langle e\rangle$. If $r \notin \langle e\rangle$ then either $bd \in \langle f\rangle$ or $bd \notin \langle f\rangle$. The former is trivial and the latter would just involve repeating the argument we used for r. Thus the lemma is proved.

LEMMA 3.2. *Let S be a 2-group satisfying Hypothesis* H, *and containing an involution d with center L and axis MP, P not on q. Then:*

(i) *$Q = C_S(b) \cap S_{LM}$ is cyclic or generalized quaternion.*

(ii) *If $c \in S$, $|c| = 2$, $c(M) = L$ and if b and c are chosen so that $|bc|$ is maximal then $Q \subset N\langle b, c\rangle$.*

(iii) *There are at most* 4 *classes of involutions in S interchanging L and M, all involutions of S are in $\langle b, c, Q\rangle$ and $[S: \langle b, c, d, Q\rangle] \leq 4$.*

Proof. Since b interchanges L and M, the only involution in S_{LM} centralizing b is a. This gives (i).

Let $t \in Q$, $|t| = 4$. $db = bda$ so $(bdt)^2 = 1$ since $t^2 = a$. Let $c \in S$, $|c| = 2$, $(L)c = M$ such that $|bc|$ is maximal. By Lemma 3.1, $\langle b(bdt), bc\rangle$ is cyclic or generalized quaternion. $\langle bc\rangle \lhd \langle dt, bc\rangle$ because $|bc|$ is maximal. Thus $dt \in N(\langle bc\rangle)$. If Q is generalized quaternion, it is generalized by its elements of order 4, so $Q \subset N(\langle b, c\rangle)$. If Q is cyclic, let $Q = \langle r\rangle$ and let $r^{-1}cr = e$. $\langle be, bc\rangle$ is cyclic or generalized quaternion. If it is cyclic, then $be \in \langle bc\rangle$ and $Q \subset N\langle bc\rangle$. If $\langle be, bc\rangle$ is generalized quaternion, since $|be| = |bc|$ and $|bc|$ is maximal, $be \in \langle bc\rangle$ or $|be| = 4$. In the former case $Q \subset N\langle b, c\rangle$. In the latter case $(be)^{-1}bbe = ba$ and $dbe \in C(b) \cap S_{LM}$. $|dbe| = 4$, so let $t = dbe$, $t \in \langle r\rangle$. By the same reasoning $|dbc| = 4$, $dbc \in C(b) \cap S_{LM}$ and we can assume $dbc = t$. Thus $c = e$ and (ii) follows.

Let x be any involution of S with $(L)x = M$. $\langle bx, bc\rangle$ is cyclic or generalized quaternion. If it is cyclic, by the choice of c, $bx \in \langle bc\rangle$ and $x \in \langle b, c\rangle$. $\langle b, c\rangle$ has two classes of involutions interchanging L and M. Call these classes C_1 and C_2. If $\langle bx, bc\rangle$ is generalized quaternion then $x \in C(bc)$ and $|bx| = 4$. $bxbx = a$, so $xbx = ba$. Then $dx \in C_S(b) = \langle Q, b\rangle$ and $x \in \langle b, d, Q\rangle \cap C(bc)$. If Q is generalized quaternion let $Q = \langle e, f\rangle, f^{-1}ef = e^{-1}$, $|f| = 4$. We have seen $|bdfe^i| = 2$ since $|fe^i| = 4$. We get two classes of involutions in $\langle b, c, d, Q\rangle$:

$$C_3 = \{bdfe^{2i+1} \mid i \in Z\} \quad \text{and} \quad C_4 = \{bdfe^{2i} \mid i \in Z\}.$$

If $s \in \langle e\rangle$, $|s| = 4$, let $C = \{bds, bdsa\}$ a set of involutions. Let $u \in \langle bc\rangle$, $|u| = 4$. Q normalizes $\langle bc\rangle$ and $t \in Q$, at most inverts bc, so we can assume $s \in C(bc)$. Thus $|us| = 2$ since $u^2 = s^2 = a$. We can also assume $us = d$. But then

$$\{bds, bdsa\} = \{bus, ba\} \subseteq C_1 \cup C_2.$$

If $|x| = 2$, $(L)x = M$, $x \in \langle b, d, Q\rangle - \langle b, d, a\rangle$, $x = bdt$ for some $t \in Q$. $|x| = 2$ and $bd = bda$ implies $bdtbdt = 1$ and $at^2 = 1$, so $|t| = 4$. Thus the C_i's contain all the involutions of S interchanging L and M. If Q is cyclic $x \in \langle b, d, Q\rangle$, x as above, then $x = bdt$ or $x = bdta$, t the unique element of order four in Q. As before $\{bdt, bdta\} \subseteq C_1 \cup C_2$.

Let H be the group generated by the C_i's. $S \rhd H$ because it is generated by involutions and $H \subseteq \langle b, c, d, Q\rangle$ since we have shown the latter group contains all the involutions of S. The C_i's are complete conjugacy classes in H. Let $y \in S$, $h \in C_i$ and let $y^{-1}hy \in C_j$. Let $g^{-1}hg$ be an arbitrary element of C_i, $g \in H$. Let $y^{-1}gy = k \in H$. We have

$$y^{-1}g^{-1}hgy = k^{-1}y^{-1}hyk \in C_j,$$

so S permutes the classes C_i as sets. $N_S(C_i) = \langle b, c, d, Q\rangle$ so $[S: \langle b, c, d, Q\rangle] \leq 4$ and (iii) follows.

In the following lemma we assume the notation of Lemma 3.2.

LEMMA 3.3. *Let S be as in* 3.2 *and assume there is an involutory homology d in S with center L and axis PM. If Q is generalized quaternion, there is no homology k with center P and axis q such that* $|k| = 4$.

Proof. Assume k exists. Then $[k, i] = 1$ for all involutions i with center on q and axis through P. Thus $k \in C(b)$, $k \in Q$. $a \in Q$, and $bd = bda$ for b as in Lemma 3.2. Thus $|bdt| = 2$ for all $t \in Q$, $|t| = 4$. Hence $[k, bdt] = 1$ since bdt is an involution acting nontrivially on q. However, $[k, b] = [k, d] = 1$ so $[k, t] = 1$ for all $t \in Q$, $|t| = 4$. This is impossible if Q is generalized quaternion and $k \in Q$, $|k| = 4$.

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