A STABLE RANGE FOR HOMOLOGY LOCALIZATION[1]

BY

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1. Introduction

A. K. Bousfield has recently shown how to construct a canonical *integral homology localization* $X_{\mathbf{Z}}$ for any space X. The aim of this paper is to show that in a natural range of dimensions the homotopy groups of $X_{\mathbf{Z}}$ are related in a stable way to the homotopy groups of X itself. In one case this relationship is direct enough to give a novel form of the Whitehead theorem.

Our technique is to construct a first quadrant spectral sequence which converges to $\pi_*(X_{\mathbf{Z}})$. We assume that the homotopy groups $\pi_i X$ are *nilpotent* $\pi_1 X$-modules [1, 4.2] for $2 \leq i \leq n$ $(n \geq 1)$, and then show that in dimensions less than $2n$ the E^2-term of this spectral sequence depends only on $\pi_1 X$ and on the action of $\pi_1 X$ upon the individual higher homotopy groups of X. Moreover, the influence of a given $\pi_1 X$-module on the tractable part of E^2 is both additive in nature and independent of the particular dimension in which the module appears as a higher homotopy group. This is what *stability* means.

Sometimes the spectral sequence allows some homotopy groups of $X_{\mathbf{Z}}$ to be computed explicitly. For instance:

1.1 THEOREM. *Let X be a connected space with finite skeleta. Suppose that $\pi_1 X$ is a nilpotent group and that $\pi_1 X$ acts nilpotently on $\pi_i X$ for $2 \leq i \leq n$ $(n \geq 1)$. Then there are natural isomorphisms*

$$\pi_i(X_{\mathbf{Z}}) \approx \pi_i X,\ i \leq n \quad \textit{and} \quad \pi_i(X_{\mathbf{Z}}) \approx (\pi_i X)^\wedge,\ n < i \leq 2n - 1.$$

Here $(\pi_i X)^\wedge$ denotes the *lower central series completion* (4.3) of $\pi_i X$ with respect to the action of $\pi_1 X$. The space X has *finite skeleta* if it has a finite number of simplices or cells in each dimension.

Since the integral homology localization functor converts homology equivalences into homotopy equivalences, we immediately obtain:

1.2 COROLLARY. *Suppose that X and Y are spaces as in* 1.1, *and that $f\colon X \to Y$ is a map which induces an isomorphism on integral homology. Then f induces isomorphisms*

$$\pi_i X \approx \pi_i Y,\ i \leq n \quad \textit{and} \quad (\pi_i X)^\wedge \approx (\pi_i Y)^\wedge,\ n < i \leq 2n - 1.$$

Organization of the paper. In Section 2 we prove a technical lemma which is at the foundation of everything that follows, in Sections 3 and 4 we define and

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partially calculate some key algebraic functors, and in Section 5 we introduce our spectral sequence and present the main result.

Notation and terminology. The notation $X_{\mathbf{Z}}$ is used for what in [1] is called the $H_*(\text{-}; \mathbf{Z})$*-localization* of the space X. In addition, $H_*(\text{-}; \mathbf{Z})$*-local* spaces, that is, spaces such that $X_{\mathbf{Z}}$ is homotopy equivalent to X, are called *Bousfield* spaces. Similarly, $H\mathbf{Z}$*-local* groups and π-modules [1, Section 5] are called *Bousfield* groups and *Bousfield* π-modules.

The word space can be taken as a synonym for *pointed Kan complex* [8] or *pointed* CW*-complex*, depending on the preference of the reader. Unless otherwise specified, all homology groups are taken with untwisted coefficients in the integers $\mathbf{Z}$.

2. The double fiber lemma

Suppose that the diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f_1} \\ Y & \xrightarrow[f_2]{} & B \end{array}$$

is a fiber square of connected spaces, and let W' be the homotopy pullback of the localized square

$$\begin{array}{ccc} W' & \to & X_{\mathbf{Z}} \\ \downarrow & & \downarrow \\ Y_{\mathbf{Z}} & \to & B_{\mathbf{Z}} \end{array}$$

There is a natural map $W_{\mathbf{Z}} \to W'$.

Let F_i be the homotopy fiber of f_i $(i = 1, 2)$. The purpose of this section is to prove:

2.1 Double Fiber Lemma. *Suppose that F_1 and F_2 are one-connected, and that $\pi_1 B$ acts nilpotently on $H_i(F_1)$ for $i \leq m$ and on $H_i(F_2)$ for $i \leq n$. Then the map $\pi_i(W_{\mathbf{Z}}) \to \pi_i(W')$ is an isomorphism for $i \leq m + n$ and an epimorphism for $i = m + n + 1$.*

Less symmetric but more useful is the following straightforward corollary:

2.2 Corollary. *Under the assumptions of* 2.1 *the relative homotopy map $\pi_i(Y_{\mathbf{Z}}, W_{\mathbf{Z}}) \to \pi_i(B_{\mathbf{Z}}, X_{\mathbf{Z}})$ is an isomorphism for $i \leq m + n + 1$ and an epimorphism for $i = m + n + 2$.*

The proof of 2.1 rests on three preliminary results.

2.3 Fiber Lemma (Bousfield). *Let $F \to E \to B$ be a fiber sequence in which B is connected. F is one-connected, and $\pi_1 B$ acts nilpotently on the homology groups $H_i(F)$ $(i \geq 0)$. Let F' be the homotopy fiber of the localized map $E_{\mathbf{Z}} \to B_{\mathbf{Z}}$. Then the natural map $F_{\mathbf{Z}} \to F'$ is a homotopy equivalence.*

The proof is a routine induction, using [1, 5.5, 8.9] and the standard spectral sequence comparison theorem [2, p. 92]. Note that the space $F_{\mathbf{Z}}$ is homotopy equivalent to F [1, 4.3].

2.4 LEMMA. *Let $f: X \to Y$ be a map of connected Bousfield spaces, and suppose that the induced map $H_i(X) \to H_i(Y)$ is an isomorphism for $i \leq n$ and an epimorphism for $i = n + 1$ $(n \geq 1)$. Then the induced map $\pi_i X \to \pi_i Y$ is also an isomorphism for $i \leq n$ and an epimorphism for $i = n + 1$.*

This is a straightforward consequence of [7] or of [1, 5.5] and the techniques of [5].

2.5 LEMMA. *Suppose that the diagram*

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f_1} \\ Y & \xrightarrow[f_2]{} & B \end{array}$$

is a fiber square. Let F_i be the homotopy fiber of f_i $(i = 1, 2)$ and suppose that F_1 and F_2 are, respectively, m-connected and n-connected $(m,n \geq 0)$. Then there is a natural exact homology sequence

$$H_{m+n+2}(W) \to H_{m+n+2}(X) \oplus H_{m+n+2}(Y) \to H_{m+n+2}(B)$$
$$\to H_{m+n+1}(W) \to \cdots \to H_0(X) \oplus H_0(Y) \to H_0(B) \to 0.$$

This can be proved by a standard application of the relative Serre spectral sequence.

Proof of 2.1. Let X_m be the mth stage in the Moore-Postnikov factorization of the map $X \to B$ [8, p. 34]. There is a fibration $X_m \to B$ with the mth Postnikov stage $P_m F_1$ as the fiber, and a map $X \to X_m$ which commutes with projection onto B and on fibers induces the usual map $F_1 \to P_m F_1$. A spectral sequence argument analogous to [2, p. 64] shows that $\pi_1(B)$ acts nilpotently *in all dimensions* on the homology of $P_m F_1$.

Similarly, if Y_n is the nth stage in the Moore-Postnikov factorization of $Y \to B$, the group $\pi_1(B)$ acts nilpotently in all dimensions on the homology of the fiber $P_n F_2$ of the natural map $Y_n \to B$.

Construct a diagram

$$\begin{array}{ccccc} W & \to & \bar{X} & \to & X \\ \downarrow & & \downarrow & & \downarrow \\ \bar{Y} & \to & \bar{B} & \to & X_m \\ \downarrow & & \downarrow & & \downarrow \\ Y & \to & Y_n & \to & B \end{array}$$

by defining the spaces $\bar{B}$, $\bar{X}$, and $\bar{Y}$ to be the homotopy inverse limits of the appropriate small squares. The upper left-hand square in this diagram is then automatically a homotopy fiber square.

It follows immediately from 2.3 that in the partially localized diagram

$$\begin{array}{ccccc} W' & \to & (\bar{X})_{\mathbf{Z}} & \to & X_{\mathbf{Z}} \\ \downarrow & & \downarrow & & \downarrow \\ (\bar{Y})_{\mathbf{Z}} & \to & (\bar{B})_{\mathbf{Z}} & \to & (X_m)_{\mathbf{Z}} \\ \downarrow & & \downarrow & & \downarrow \\ Y_{\mathbf{Z}} & \to & (Y_n)_{\mathbf{Z}} & \to & B_{\mathbf{Z}} \end{array}$$

all of the small squares are also homotopy fiber squares.

There is an obvious map between the above two diagrams, which induces a homomorphism between the exact homology sequences that result from an application of 2.5 to the upper left-hand squares. A five lemma argument now shows that the map $W \to W'$ induces an isomorphism $H_i(W) \to H_i(W')$ for $i \leq n + m + 1$. It follows that the map $W_{\mathbf{Z}} \to W'$ has this same homological property. Since W' is a Bousfield space [1, 12.9], an application of 2.4 completes the proof.

3. Functors for the E^2-term

Let σ be a fixed group, and let σ' be the $H\mathbf{Z}$-localization of σ [1, 5.1]. In this section we introduce an interesting sequence (S_k) of additive functors from the category of σ-modules to the category of σ'-modules. If the space X has fundamental group σ, these functors can be applied to the higher homotopy groups of X to give part of the E^2-term of a spectral sequence converging to $\pi_*(X_{\mathbf{Z}})$.

Construction and additivity. Let K be the space $K(\sigma, 1)$. For any σ-module M and integer $n \geq 2$, let $L(M, n)$ be the split fibration over K, with fiber $K(M, n)$, which is determined by the action of σ on M. There is a natural projection $L(M, n) \to K$ and a natural section $K \to L(M, n)$.

We define $S_k^n(M)$ to be the relative homotopy group $\pi_{n+k+1}(K_{\mathbf{Z}}, L(M, n)_{\mathbf{Z}})$. Since the fiber of the map $L(M, n) \to K$ is $(n - 1)$-connected, so is the fiber of $L(M, n)_{\mathbf{Z}} \to K_{\mathbf{Z}}$ (2.4); thus these groups vanish if $k < 0$.

3.1 Proposition. *There are natural suspension maps* $S_k^n(M) \to S_k^{n+1}(M)$ $(n \geq 2)$ *which are isomorphisms if* $k \leq 2n - 2$ *and epimorphisms if* $k \leq 2n - 1$.

This is proved below.

The *stable* groups $S_k^n(M)$ $(k \leq 2n - 2)$ will be denoted $S_k(\sigma; M)$ or simply $S_k(M)$. They depend only on the group σ and the σ-module M.

The fundamental group $\pi_1(L(M, n)_{\mathbf{Z}})$ is naturally isomorphic to σ' [1, 7.3], so that the groups $S_k^n(M)$ come equipped by construction with an action of σ'. This action is compatible with suspension and induces a σ'-action on the stable groups $S_k(M)$. It is not hard to prove from [1, 5.5, 8.6, and 8.7] that the groups $S_k(M)$ are *Bousfield* σ'-modules.

3.2 Proposition. *The family* (S_k) *forms an exact connected sequence of additive functors from the category of* σ*-modules to the category of Bousfield*

σ'-modules. In particular, a short exact sequence $0 \to M' \to M \to M'' \to 0$ of σ-modules gives rise to a natural long exact sequence

$$\to S_k(M') \to S_k(M) \to S_k(M'') \to S_{k-1}(M') \to \cdots \to S_0(M'') \to 0$$

of σ'-modules.

Determination of the functor S_0. There are obvious maps

$$M = \pi_{n+k+1}(K, L(M, n)) \to S_0^n(M) \quad (n \geq 2),$$

which are compatible with suspension and induce a stable σ-map $M \to S_0(M)$. Since the induction functor $\mathbf{Z}[\sigma'] \otimes_{\mathbf{Z}[\sigma]}$- is left adjoint to the restriction functor from the category of σ'-modules to the category of σ-modules, this σ-map can be extended to a unique σ'-map $\mathbf{Z}[\sigma'] \otimes_{\mathbf{Z}[\sigma]} M \to S_0(M)$.

Let E denote the $H\mathbf{Z}$-localization functor on the category of σ'-modules [1, 5.3].

3.3 PROPOSITION. *For any M the above map $\mathbf{Z}[\sigma'] \otimes_{\mathbf{Z}[\sigma]} M \to S_0(M)$ extends to a natural isomorphism $E(\mathbf{Z}[\sigma'] \otimes_{\mathbf{Z}[\sigma]} M) \to S_0(M)$.*

3.4 *Remark.* The functors S_k $(k > 0)$ are *not* always the left derived functors of S_0. This is the case if and only if $S_k(F)$ vanishes whenever $k > 0$ and F is a free $\mathbf{Z}[\sigma]$-module (see 4.1).

An alternative theory. A theory parallel to ours can be constructed by using the *relative integral homology localization $L(M, n)_{\mathbf{Z}}^K$ of $L(M, n)$ over K.* By definition, $L(M, n)_{\mathbf{Z}}^K$ is a space which sits in a commutative diagram

$$\begin{array}{ccc} L(M, n) & \to & L(M, n)_{\mathbf{Z}}^K \\ & \searrow \ K \ \swarrow & \end{array}$$

in which the horizontal map is an integral homology equivalence and the map $L(M, n)_{\mathbf{Z}}^K \to K$ is an $H_*(\text{-}; \mathbf{Z})$*-fibration* [1, 10.1]. The σ-modules $S_k'(\sigma; M) = \pi_{n+k+1}(K, L(M, n)_{\mathbf{Z}}^K)$ $(k \leq 2n - 2)$ partially determine the homotopical consequences of applying the relative integral homology localization functor to an arbitrary map $X \to K$. This relative theory overlaps and is identical with ours if σ itself is a Bousfield group.

Proof of 3.1. Note that $S_k^n(M)$ is naturally isomorphic to $\pi_{n+k}(L(M, n)_{\mathbf{Z}}, K_{\mathbf{Z}})$, where the map $K_{\mathbf{Z}} \to L(M, n)_{\mathbf{Z}}$ used in the definition of this relative homotopy group is the $H_*(\text{-}; \mathbf{Z})$-localization of the section $K \to L(M, n)$.

Up to homotopy there is a fiber square

$$\begin{array}{ccc} L(M, n) & \to & K \\ \downarrow & & \downarrow \\ K & \to & L(M, n+1) \end{array}$$

where the maps $L(M, n) \to K$ are the natural projections and the maps $K \to L(M, n+1)$ are the natural sections. Localization gives the suspension map

$$\pi_{n+k+1}(K_{\mathbf{Z}}, L(M, n)_{\mathbf{Z}}) \to \pi_{n+k+1}(L(M, n+1)_{\mathbf{Z}}, K_{\mathbf{Z}}).$$

The rest of the proposition follows from 2.2, since the homotopy fiber $K(M, n)$ of the section $K \to L(M, n+1)$ is $(n-1)$-connected.

Proof of 3.2. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of σ-modules. For any integer $n \geq 2$, the space $L(M, n)$ fibers naturally over $L(M'', n)$ with fiber $K(M', n)$. This fibration is in general nonorientable and is classified by a map $L(M'', n) \to L(M', n+1)$. Since the universal fibration over $L(M', n+1)$ with fiber $K(M', n)$ is given up to homotopy by the natural section $K \to L(M', n+1)$, up to homotopy there are fiber squares

$$\begin{array}{ccc} L(M, n) & \to & K \\ \downarrow & & \downarrow \\ L(M'', n) & \to & L(M', n). \end{array}$$

The desired long exact sequence is essentially obtained by applying 2.1 and then taking the limit of the Meyer-Vietoris homotopy sequences [2, p. 286] of the corresponding localized fiber squares as n tends to infinity over even integers. However, it is necessary to modify these Mayer-Vietoris sequences slightly so that they contain the relative homotopy groups $\pi_*(L(M, n)_{\mathbf{Z}}, K_{\mathbf{Z}}), \ldots$ rather than the absolute groups $\pi_*(L(M, n)_{\mathbf{Z}}), \ldots$. This can be done without disturbing exactness because after $H\mathbf{Z}$-localization the space $K_{\mathbf{Z}}$ is a natural retract of all the spaces in the square. The reason for using even integers in the limit is that an odd iterate of the suspension map may anticommute with the boundary homomorphisms in the Meyer-Vietoris sequences.

Proof of 3.3. Let $n \geq 2$ be an integer. Consider the commutative square

$$\begin{array}{ccc} L(M, n) & \to & L(M, n)_{\mathbf{Z}} \\ \downarrow & & \downarrow \\ K & \to & K_{\mathbf{Z}} \end{array}.$$

Both horizontal maps are integral homology equivalences. Moreover, the respective fibers F_1 and F_2 of the vertical maps are both $(n-1)$-connected. The standard spectral sequence comparison theorem [2, p. 92] applies to the Serre spectral sequences of the two vertical fibrations and gives an isomorphism

$$H_0(\sigma; M) \to H_0(\sigma'; S_0(M))$$

and an epimorphism

$$H_1(\sigma; M) \to H_1(\sigma'; S_0(M)).$$

(Here $H_n(F_1)$ and $H_n(F_2)$ have been identified with M and $S_0(M)$, respectively.) Let M' be the σ'-module $\mathbf{Z}[\sigma'] \otimes_{\mathbf{Z}[\sigma]} M$. By [4, II, 5.1] the natural σ-map $M \to M'$ induces an isomorphism $H_0(\sigma; M) \to H_0(\sigma'; M')$. Consequently the

σ'-map $M' \to S_0(M)$ which extends the σ-map $M \to S_0(M)$ must be an *$H\mathbf{Z}$-map* [1, 5.3]. Since $S_0(M)$ is a Bousfield σ'-module, this implies that $S_0(M)$ is isomorphic to $E(M')$.

4. Further calculations

The purpose of this section is to calculate the functors (S_k) completely in a nontrivial case. This is accomplished by determining when the higher S_k's vanish, and then giving an explicit formula for S_0 in one of the cases in which this vanishing occurs. The notation is the same as that of Section 3.

Vanishing of the higher S_k's. Let M be a σ-module. Note that the group homomorphism $\sigma \to \sigma'$ and the module map $M \to S_0(M)$ combine to give a natural twisted homology map $H_*(K; M) \to H_*(K_{\mathbf{Z}}; S_0(M))$.

4.1 PROPOSITION. *The modules $S_k(M)$ vanish for all $k > 0$ if and only if the map $H_*(K; M) \to H_*(K_{\mathbf{Z}}; S_0(M))$ is an isomorphism.*

A σ-module is said to be *finitely generated* if it is finitely generated in the usual sense as a module over the group ring $\mathbf{Z}[\sigma]$.

4.2 PROPOSITION. *The equivalent conditions of* 4.1 *hold under any of the following assumptions.*

(a) *σ is an arbitrary group, and M is a nilpotent σ-module.*
(b) *σ is a Bousfield group, and M is a Bousfield σ-module.*
(c) *σ is a finitely generated nilpotent group, and M is a finitely generated σ-module.*

In cases (a) and (b) of 4.2 the module $S_0(M)$ is actually isomorphic to M itself. In dealing with case (c), it is possible to give $S_0(M)$ algebraically.

Lower central series completion. Let $I \subseteq \mathbf{Z}[\sigma]$ be the augmentation ideal. The *lower central series completion* $M^\wedge$ of the σ-module M is defined to be inv lim $M/I^s \cdot M$. As the inverse limit of nilpotent σ-modules, the module $M^\wedge$ itself is a Bousfield σ-module [1, 8.5].

4.3 PROPOSITION. *If σ is a finitely generated nilpotent group and M is a finitely generated σ-module, there is a natural isomorphism $S_0(M) \approx M^\wedge$.*

4.4 *Remark.* It is not hard to show using [6, 3.10] and the techniques in [6, Proof of 3.1] that the map $F \to F^\wedge$ induces an isomorphism $H_*(\sigma; F) \to H_*(\sigma; F^\wedge)$ whenever σ is a finitely generated nilpotent group and F is a *free* σ-module. It follows from [1, 7.5], 4.1 and 3.4 that for such a group σ the functor S_k $(k \geq 0)$ is naturally equivalent to the kth *left derived functor* [4, V, 5] of the lower central series completion functor on the category of σ-modules.

Proof of 4.1. Choose $n \geq 2$, and let $L(n)$ be the split fibration over $K_{\mathbf{Z}}$ with fiber $K(S_0(M), n)$ which is determined by the action of $\pi_1(K_{\mathbf{Z}}) = \sigma'$ on $S_0(M)$.

The space $L'(n)$ is Bousfield; this can be seen by expressing it as the pullback over a map $K_{\mathbf{Z}} \to K(\sigma', 1)$ of a split fibration over $K(\sigma', 1)$, and using [1, 5.5 12.9]. There is a map of split fiber sequences

$$\begin{array}{ccccc} K(M, n) & \to & L(M, n) & \rightleftarrows & K \\ \downarrow & & \downarrow & & \downarrow \\ K(S_0(M), n) & \to & L'(n) & \rightleftarrows & K_{\mathbf{Z}} \end{array}$$

which on bases is the $H_*(\text{-}; \mathbf{Z})$-localization map and on fibers is the map induced by the natural homomorphisms $M \to S_0(M)$. It is easy to see that the modules $S_k(M)$ vanish for $0 < k < n - 1$ if and only if the induced homotopy map $\pi_i(L(M, n)_{\mathbf{Z}}) \to \pi_i(L'(n))$ is an isomorphism for $i \leq 2n - 2$ and an epimorphism for $i = 2n - 1$. By 2.4 and a simple suspension argument, this is equivalent to requiring that the map $h_i(L(M, n)) \to h_i(L'(n))$ be an isomorphism for $i \leq 2n - 2$ and an epimorphism for $i = 2n - 1$, where h_* denotes the reduced stable homotopy homology theory. Using the Serre spectral sequence and the fact that stable homotopy agrees with homotopy in the usual stable range, it is easy to carry out the calculations

$$h_i(L(M, n)) \approx h_i(K) \oplus H_{i-n}(K; M), \quad h_i(L'(n)) \approx h_i(K_{\mathbf{Z}}) \oplus H_{i-n}(K_{\mathbf{Z}}; S_0(M)) \qquad (i \leq 2n - 1)$$

This completes the proof.

Proof of 4.3. It follows from [1, 7.5] and 3.3 that if σ is nilpotent the functor S_0 is naturally equivalent to the $H\mathbf{Z}$-localization functor on the category of σ-modules. Thus 4.3 is just a special case of [3, Theorem 4].

Proof of 4.2. Part (a) follows easily from 2.3, and part (b) from [1, 5.5], Proposition 4.3 and [3, Theorem 3] show that if σ is a finitely generated nilpotent group and M is a finitely generated σ-module, the natural map $M \to S_0(M)$ induces an isomorphism $H_*(\sigma; M) \to H_*(\sigma; S_0(M))$. In view of [1, 7.5], this proves (c).

5. The spectral sequence

Let X be any connected space, and let P_nX $(n \geq 0)$ be the nth Postnikov stage of X. The Postnikov tower

$$\to P_nX \to P_{n-1}X \to \cdots \to P_1X \to P_0X \;(= *)$$

gives rise to a localized tower

$$\to (P_nX)_{\mathbf{Z}} \to (P_{n-1}X)_{\mathbf{Z}} \to \cdots \to (P_1X)_{\mathbf{Z}} \to (P_0X)_{\mathbf{Z}}.$$

Just as the homotopy inverse limit of the first tower is homotopy equivalent to X, the homotopy inverse limit of the second tower is homotopy equivalent to $X_{\mathbf{Z}}$. In fact, it follows from 2.4 that the homotopy fiber of the map $X_{\mathbf{Z}} \to$

$(P_nX)_{\mathbf{Z}}$ is $(n-1)$-connected. Similarly, the homotopy fiber of the map $(P_nX)_{\mathbf{Z}} \to (P_{n-1}X)_{\mathbf{Z}}$ is $(n-1)$-connected.

Let $P_{-1}X$ be a single point. With some reindexing, the usual construction of the homotopy spectral sequence of a tower [2, p. 258] gives:

5.1 PROPOSITION. *For any connected space X there exists a functorial first quadrant spectral sequence*

$$E^2_{i,j}(X) = \pi_{i+j+1}((P_{j-1}X)_{\mathbf{Z}}, (P_jX)_{\mathbf{Z}}) \Rightarrow \pi_{i+j}X_{\mathbf{Z}}.$$

This spectral sequence is naturally augmented by the trivial Postnikov tower spectral sequence of X.

The last statement of the proposition means that there are maps $\pi_jX \to E^2_{0,j}(X)$ $(j \geq 0)$ which pass to E^∞ and correspond there to the usual maps $\pi_jX \to \pi_jX_{\mathbf{Z}}$.

This spectral sequence is useful only to the extent that its E^2-term can be computed. Our main result is:

5.2 PROPOSITION. *Suppose that X is a connected space and that π_1X acts nilpotently on π_jX for $1 < j \leq n$ $(n \geq 1)$. Then:*

(a) *For $i \geq 0$, $E^2_{i,0}(X)$ vanishes.*
(b) *For $i \geq 0$, $E^2_{i,1}(X)$ is naturally isomorphic to $\pi_{i+1}(K(\pi_1X, 1)_{\mathbf{Z}})$.*
(c) *If $j \geq 2$, then for $j \leq n$ or $i < n$ $E^2_{i,j}(X)$ is naturally isomorphic to $S_i(\pi_1X; \pi_jX)$.*

5.3 *Remark.* This proposition determines $E^2(X)$ outside of an infinite rectangular block whose lower left-hand corner lies at the lattice point $(n, n+1)$.

Note that according to 4.2(a) the group $S_i(\pi_1X; \pi_jX)$ vanishes whenever $i > 0$ if π_jX is a nilpotent π_1X-module. Thus the E^2-term described by 5.2 has a horizontal band of vanishing groups which is $(n-1)$ units high. If π_1X is a Bousfield group, the height of this band is increased by another unit, since

$$E^2_{i,1}(X) = \pi_{i+1}(K(\pi_1X, 1)_{\mathbf{Z}}) = \pi_{i+1}(K(\pi_1X, 1)) = 0 \quad \text{for } i > 0 \text{ [1, 5.2]}.$$

When 4.2(c) applies, there is enough additional vanishing to give:

5.4 COROLLARY. *Suppose that X is a connected space and that π_1X is a nilpotent group which acts nilpotently on π_jX for $1 < j \leq n$ $(n \geq 1)$. Suppose furthermore that π_1X is finitely generated and that π_jX is finitely generated over π_1X for $n < j \leq 2n$. Then there are natural isomorphisms $\pi_i(X_{\mathbf{Z}}) \approx \pi_1X$, $i \leq n$, and $\pi_i(X_{\mathbf{Z}}) \approx (\pi_iX)^\wedge$, $n < i \leq 2n-1$, as well as a natural epimorphism $(\pi_{2n}X)^\wedge \to \pi_{2n}(X_{\mathbf{Z}})$.*

5.5 *Remark.* In order to extract 1.1 from 5.4, it is only necessary to show that the finiteness conditions of 5.4 are automatically satisfied if X has finite skeleta. This is a routine consequence of the fact that the integral group ring of a finitely generated nilpotent group is (left and right) noetherian [3].

A slight refinement. By applying a fiberwise suspension functor to the fibration $X_{\mathbf{Z}} \to (P_n X)_{\mathbf{Z}}$ and using the Freudanthal theorem, one can improve the epimorphism in dimension $2n$ of 5.4 to an isomorphism. This gives an extra dimension in 1.1 and 1.2. The situation in dimension $2n + 1$ is more complicated. However, it is possible to show that if $\pi_{2n+1}X$ is a finitely generated $\pi_1 X$-module and the Whitehead product map $\pi_{n+1}X \otimes \pi_{n+1}X \to \pi_{2n+1}X$ is trivial, then $(\pi_{2n+1}X)^\wedge$ is a direct summand of $\pi_{2n+1}(X_{\mathbf{Z}})$.

Proof of 5.2. Parts (a) and (b) follow directly from the definition of the spectral sequence.

Let σ be the group $\pi_1 X$ and let K be $K(\sigma, 1)$. For any $j \geq 2$ there is a homotopy fiber square

$$\begin{array}{ccc} P_j X & \to & K \\ \downarrow & & \downarrow \\ P_{j-1}X & \to & L(\pi_j X, j + 1) \end{array}$$

where $L(\pi_j X, j + 1)$ is the two-stage Postnikov system of Section 3, and the map $K \to L(\pi_j X, j + 1)$ is the natural section. Localization gives a homomorphism

$$E^2_{i,j}(X) \to \pi_{i+j+1}(L(\pi_j X, j + 1)_{\mathbf{Z}}, K_{\mathbf{Z}}) \to S_i(\pi_1 X; \pi_j X)$$

where the second arrow is an iterated suspension map. The fact that this map is an isomorphism in the stated ranges follows from 2.3 if $j \leq n$ and from 3.1 and 2.2 if $j > n$. In fact, this argument additionally shows that the above map is an epimorphism if $j \geq 2$ and $i = n$.

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