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THE ASYMPTOTIC BEHAVIOR OF THE PRINCIPAL EIGENVALUE OF SINGULARLY PERTURBED DEGENERATE ELLIPTIC OPERATORS[1]

BY

ALLEN DEVINATZ AND AVNER FRIEDMAN

1. Introduction

Let

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} \tag{1.1}$$

be an elliptic operator in a bounded domain Ω, degenerating on $\partial\Omega$ in such a way that

$$\sum_{i,j=1}^{n} a_{ij} v_i v_j = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

$$\sum_{i=1}^{n} \left(b_i - \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial x_j} \right) v_i \geq 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where $v = (v_1, \ldots, v_n)$ is the inward normal on $\partial\Omega$. In general, the Dirichlet problem for this operator has no solution since the corresponding solution of the stochastic differential system

$$d\xi_i = \sum_{j=1}^{n} \sigma_{ij}(\xi)\, dw_j + b_i(\xi)\, dt,$$

where $\frac{1}{2} \sum_{k=1}^{n} \sigma_{ik} \sigma_{kj} = a_{ij}$ does not exit from Ω in finite time (see, for example, [4] for more details).

For any $\varepsilon > 0$, consider the elliptic operator $\varepsilon\Delta + L$ and denote by λ_ε the principal eigenvalue; i.e., λ_ε is the smallest real number such that there exists a solution u_ε for the problem

$$\varepsilon\Delta u_\varepsilon + Lu_\varepsilon = -\lambda_\varepsilon u_\varepsilon \text{ in } \Omega, \qquad u_\varepsilon = 0 \text{ on } \partial\Omega. \tag{1.4}$$

As is well known such an eigenvalue always exists. We are concerned in this paper with the asymptotic behavior of λ_ε as $\varepsilon \to 0$.

Several earlier papers consider the problem of estimating λ_ε when λ_ε is the principal eigenvalue for the operator

$$\varepsilon \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}, \tag{1.5}$$

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and (a_{ij}) is nondegenerate in $\bar{\Omega}$. For example, in [3] and [6] it is shown that under suitable conditions on the b_i the principal eigenvalue for (1.5) decays as $\exp(-c/\varepsilon)$ as $\varepsilon \to 0$, where c is a positive constant. Other estimates were obtained in [2] for other varying conditions on the functions b_i. In [1] asymptotic estimates for λ_ε were obtained for $n = 1$ and when the principal coefficient degenerates on the boundary of the domain. This is a problem of some importance in the study of population genetics. The papers [3], [6], [7], and [8] employ probabilistic tools; i.e., the Ventcel–Freidlin estimates [9], whereas in [1] and [2] the methods are entirely analytic.

In Section 2 of this paper we derive, under suitable conditions on the a_{ij} and the b_i, estimates for the principal eigenvalue of (1.4) of the form

$$\lambda_\varepsilon \leq e^{-c/\varepsilon^\mu} \quad (c > 0, 0 < \mu < 1), \tag{1.6}$$

$$\lambda_\varepsilon \leq \varepsilon^\nu \quad (\nu > 0), \tag{1.7}$$

and in Section 3 we derive estimates of the form

$$\lambda_\varepsilon \geq \exp(-c'/\varepsilon^{\mu'}) \quad (c' > 0, 0 < \mu' < 1), \tag{1.8}$$

$$\lambda_\varepsilon \geq \varepsilon^{\nu'} \qquad (\nu' > 0). \tag{1.9}$$

The numbers μ, μ' can be made as close to each other as we wish (under appropriate assumptions), as is also the case for the numbers ν, ν'. Thus our estimates from above and below are not too crude.

We note that in the estimates (1.6) and (1.8) the numbers μ and μ' do not attain the value one as we mentioned is the case in [3] and [6]. However, as (1.5) shows, the perturbed operators in the latter papers converge to first order differential operators, as $\varepsilon \to 0$, whereas (1.1) and (1.4) show the important difference that the perturbed operators of this paper converge to second order differential operators.

The results of Sections 2 and 3 extend in an obvious way to the case where λ_ε is the principal eigenvalue for the operator

$$\varepsilon \sum_{i,j=1}^{n} c_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} + L, \tag{1.10}$$

where (c_{ij}) is nondegenerate in $\bar{\Omega}$; in fact all of the estimates remain the same. This is no longer true when the matrix (c_{ij}) is degenerate on $\partial\Omega$. In Section 4 we consider this case for $n = 1$, assuming that the degeneracy on the boundary is rather "mild" so that (1.10) does in fact have a principal eigenvalue in the sense of [1].

A probabilistic interpretation of the results of this paper is given at the end of Section 3. We should like to point out that the probabilistic methods of [3], [6], [8], and [9] cannot apply to the present problem since the Ventcel–Freidlin estimates are lacking here; the reason being that, as ε varies, the diffusion processes ξ^ε corresponding to $\varepsilon\Delta + L$ are not absolutely continuous with respect to each other (the processes corresponding to (1.5) are absolutely continuous with respect to each other).

2. Upper bounds

In this section we shall obtain upper bounds for the principal eigenvalue λ_ε of the problem (1.4) as $\varepsilon \to 0$, where L is given by (1.1). We shall suppose here and in the next section that Ω is a bounded domain in R^n with a C^3 boundary. We shall make the following assumptions for the estimates of this section:

(A_1) *$a_{ij} \in C^1(\Omega)$ and b_i is Hölder continuous in $\bar{\Omega}$.*

(A_2) *a_{ij} is positive definite in Ω and positive semidefinite in $\bar{\Omega}$.*

(A_3) *Let $\rho(x)$ denote the distance from $x \in \Omega$ to $\partial\Omega$. Then there is a $\rho_1 > 0$ such that if $0 \le \rho \le \rho_1$,*

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial \rho(x)}{\partial x_i} \frac{\partial \rho(x)}{\partial x_j} \le a\rho^k, \quad a > 0,\ k > 0, \tag{2.1}$$

$$\sum_{i=1}^{n} \left(b_i(x) - \sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} \right) \frac{\partial \rho(x)}{\partial x_i} \ge \hat{b}\rho^\beta, \quad \hat{b} > b > 0,\ \beta \ge 0. \tag{2.2}$$

If $\beta > 0$, then as $\rho \to 0$,

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 \rho(x)}{\partial x_i \, \partial x_j} = - \sum_{i,j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} \frac{\partial \rho(x)}{\partial x_i} + o(\rho^\beta). \tag{2.3}$$

Note that the condition (2.1) of (A_3) implies

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial \rho(x)}{\partial x_i} \frac{\partial \rho(x)}{\partial x_j} = 0, \quad x \in \partial\Omega.$$

Thus [4,208] it follows that

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 \rho(x)}{\partial x_i \, \partial x_j} = - \sum_{i,j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} \frac{\partial \rho(x)}{\partial x_i}, \quad x \in \partial\Omega.$$

Consequently, as $\rho \to 0$,

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 \rho(x)}{\partial x_i \, \partial x_j} = - \sum_{i,j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} \frac{\partial \rho(x)}{\partial x_i} + o(1). \tag{2.4}$$

Thus (2.3) of (A_3) holds for $\beta = 0$ also. Furthermore, if $0 < \beta < 1$ and the $\partial a_{ij}/\partial x_j$ are Hölder continuous (exponent $\beta' > \beta$) then (2.3) is clearly satisfied.

In order to state our first theorem, we shall, for the sake of notational simplicity, introduce certain constants involving a, b, k, and β. Let us set

$$(2.5)\quad I_{\beta,k} = \int_0^\infty \frac{y^\beta}{1+y^k}\,dy = \frac{k^\beta}{\Gamma(1/k)}\Gamma\left(1+\beta+\frac{1}{k}\right)\frac{\pi}{\sin[\pi(1+\beta)/k]}, \quad k > 1+\beta,$$

$$(2.6)\qquad c = bI_{\beta,k}/a^{(\beta+1)/k}, \quad k \geq 1+\beta,$$

$$(2.7)\qquad \mu = 1 - \frac{\beta+1}{k}, \qquad k \geq 1+\beta,$$

$$(2.8)\qquad \nu = \left(\frac{b}{a} - 1\right)/k, \qquad k \geq 1+\beta.$$

We then have the following:

THEOREM 2.1. *Under the hypotheses* (A_1)–(A_3) *there exists a positive constant A so that*

$$(2.9)\qquad \lambda_\varepsilon \leq Ae^{-c/\varepsilon^\mu} \quad \textit{for } 1+\beta < k,$$

$$(2.10)\qquad \lambda_\varepsilon \leq \begin{cases} A\varepsilon^\nu & \textit{if } b/a > 1 \\ A & \textit{if } b/a < 1 \end{cases} \quad \textit{for } 1+\beta = k,$$

and

$$(2.11)\qquad \lambda_\varepsilon \leq A \quad \textit{for } 1+\beta > k.$$

In the sequel we use the notation $L_\varepsilon \equiv \varepsilon\Delta + L$.

In order to obtain these bounds for λ_ε it will be necessary for us to use the following lemma from [2].

LEMMA 2.2. *Let* (A_1) *and* (A_2) *hold and let* $\Phi \in C(\bar{\Omega}) \cap C^2(\Omega)$. *If* $L_\varepsilon\Phi(x) + A_\varepsilon\Phi(x) \geq 0$ *for* $x \in \Omega$ *and some* $A_\varepsilon > 0$, *and* $\Phi(x) > 0$ *for* $x \in \Omega$, $\Phi = 0$ *on* $\partial\Omega$, *then* $\lambda_\varepsilon \leq A_\varepsilon$.

Proof of Theorem 2.1. In order to apply Lemma 2.2 we shall choose $\Phi = \exp\phi$ and shall work, at first, only in a neighborhood of $\partial\Omega$. We shall take $\phi(x) = f(\rho(x))$, where f is a twice differentiable function in some interval $(0, \rho_0)$. Notice that in order that Φ vanish on $\partial\Omega$ it is necessary that $f(\rho) \to -\infty$ as $\rho \to 0$. Computing the various derivatives of Φ we find that

$$(2.12)\qquad \frac{L_\varepsilon\Phi}{\Phi} = \sum_{i,j=1}^n [\varepsilon\delta_{ij} + a_{ij}]\left\{[f'' + (f')^2]\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} + f'\frac{\partial^2\rho}{\partial x_i\,\partial x_j}\right\} + \sum_{i=1}^n b_i\frac{\partial\rho}{\partial x_i}f'.$$

Our object is to try to make this quantity nonnegative in a neighborhood of $\partial\Omega$.

The first thing we notice is that in a neighborhood of $\partial\Omega$ we have

$$(2.13)\qquad \sum_{i,j=1}^{n} \varepsilon\delta_{ij}\frac{\partial\rho(x)}{\partial x_i}\frac{\partial\rho(x)}{\partial x_j}=\varepsilon,\quad \varepsilon\sum_{i,j=1}^{n}\delta_{ij}\frac{\partial^2\rho(x)}{\partial x_i\,\partial x_j}=\varepsilon\Delta\rho(x).$$

From the hypothesis (2.3) of (A_3), (2.12) and (2.13) we have for sufficiently small ρ, say $0\le\rho\le\rho_0$,

$$(2.14)\qquad \frac{L_\varepsilon\Phi}{\Phi}=\left[\varepsilon+\sum_{i,j=1}^{n}a_{ij}\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j}\right][f''+(f')^2] + \left[\sum_{i=1}^{n}\left(b_i-\sum_{j=1}^{n}\frac{\partial a_{ij}}{\partial x_j}\right)\frac{\partial\rho}{\partial x_i}+\varepsilon\Delta\rho+o(\rho^\beta)\right]f'.$$

Let us set

$$(2.15)\qquad h(\rho)=[-b\rho^\beta/(\varepsilon+a\rho^k)],\qquad h_\gamma(\rho)=h(\rho)+\gamma,$$

where γ is a nonnegative constant which will be chosen subsequently. Let f_γ be a solution to the Bernoulli equation

$$(2.16)\qquad f_\gamma''+(f_\gamma')^2=h_\gamma f_\gamma'.$$

Since $h<0$, from (2.14) (with f_γ replacing f), (2.15), (2.16), and (2.1) of (A_3) we have,

$$(2.17)\qquad \frac{L_\varepsilon\Phi}{\Phi}\ge\left[-b\rho^\beta+\sum_{i=1}^{n}\left(b_i-\sum_{j=1}^{n}\frac{\partial a_{ij}}{\partial x_j}\right)\frac{\partial\rho}{\partial x_i}+\varepsilon(\gamma+\Delta\rho)+o(\rho^\beta)\right]f_\gamma'.$$

Choose γ so that $\gamma+\Delta\rho\ge 0$; then if $f_\gamma'\ge 0$ we have arrived at the fact that

$$(2.18)\qquad \frac{L_\varepsilon\Phi}{\Phi}\ge 0,\quad 0\le\rho\le\rho_0.$$

In order to solve (2.16) we make the standard transformation $f_\gamma'=1/g$. This leads to the first order linear equation $g'+h_\gamma g=1$ which has the solution

$$\left(\exp\int_0^\rho h_\gamma\right)g(\rho)-g(0)=\int_0^\rho\left(\exp\int_0^\sigma h_\gamma\right)d\sigma.$$

Choose $g(0)=0$ so that $f_\gamma'(0)=\infty$. Thus we have

$$(2.19)\qquad f_\gamma'(\rho)=\left[\int_0^\rho\left(\exp\left(-\int_\sigma^\rho h_\gamma\right)\right)d\sigma\right]^{-1}=\frac{\exp\int_0^\rho h_\gamma}{\int_0^\rho(\exp\int_0^\sigma h_\gamma)\,d\sigma}.$$

Notice that $f_\gamma'(\rho)>0$ and $f_\gamma'(\rho)\to+\infty$ as $\rho\to 0$. Integrating (2.19) and normalizing $f_\gamma(\rho)$ in any way (i.e., choosing the constant of integration in any way) we always get $f_\gamma(\rho)\to-\infty$ as $\rho\to 0$.

We have shown that for ρ sufficiently small, $0\le\rho\le\rho_0$, we have for every constant $A>0$, $L_\varepsilon\Phi+A\Phi>0$. We must now investigate the range where $\rho\ge\rho_0$. Toward this end let $\zeta(\rho)$ be a C^2 function such that $\zeta(\rho)\equiv 1$ for

$0 \le \rho \le \rho_0/2$, and $\zeta(\rho) = 0$ for $\rho \ge \rho_0$. Let us now redefine Φ and set $\Phi = e^{f_\gamma \zeta}$. This new function coincides with the old Φ in the range $0 \le \rho \le \rho_0/2$, and is identically 1 for $\rho \ge \rho_0$. Thus in the range $\rho \ge \rho_0$ we have $L_\varepsilon \Phi/\Phi = 0$, and in the range $\rho_0/2 \le \rho \le \rho_0$ we have (2.14) with $f_\gamma \zeta$ replacing f.

In order to estimate the right hand side of (2.14), with f replaced by $f_\gamma \zeta$, it is necessary to get estimates on f_γ and its first two derivatives in the range $\rho_0/2 \le \rho \le \rho_0$. In order not to carry along the annoying factor γ in the computations we note that if f is a solution to (2.16) with h replacing h_γ, then there is a positive constant C so that $(1/C)f' \le f'_\gamma \le Cf'$. Thus it is enough to estimate f' in the range $\rho_0/2 \le \rho \le \rho_0$. We use (2.19), with h_γ replaced by h. We first consider the case $1 + \beta < k$. We have

$$\int_0^\rho h = -b \int_0^\rho \frac{\tau^\beta}{\varepsilon + a\tau^k}\, d\tau \tag{2.20}$$
$$= -\frac{b}{a^{(\beta+1)/k}} \frac{1}{\varepsilon^{1-(\beta+1)/k}} \int_0^{\rho/(\varepsilon/a)^{1/k}} \frac{y^\beta}{1+y^k}\, dy$$
$$= -\frac{b}{a^{(\beta+1)/k}} \frac{1}{\varepsilon^{1-(\beta+1)/k}} I_{\beta,k}(\varepsilon, \rho).$$

Note that

$$I_{\beta,k}(\varepsilon, \rho) \to I_{\beta,k} = \int_0^\infty \frac{y^\beta}{1+y^k}\, dy \quad \text{as } \varepsilon \to 0,$$

uniformly for $\rho_0/2 \le \rho \le \rho_0$. In the latter range we also have

$$\int_0^\rho \left(\exp \int_0^\sigma h\right) d\sigma = \int_0^\rho \frac{1}{h} \frac{d}{d\sigma}\left(\exp \int_0^\sigma h\right) d\sigma \tag{2.21}$$
$$= -\int_0^\rho \frac{\varepsilon + a\sigma^k}{b\sigma^\beta} \frac{d}{d\sigma}\left(\exp \int_0^\sigma h\right) d\sigma$$
$$\ge \frac{\varepsilon}{b\rho_0^\beta}\left[1 - \exp \int_0^\rho h\right]$$
$$\ge C\varepsilon,$$

where C is a positive constant. Now $I_{\beta,k}(\varepsilon, \rho)$ may be made as close to $I_{\beta,k}$ as we wish by taking ε sufficiently small. Also, b in (2.20) may be replaced by any constant b', $b < b' < \hat{b}$. Putting these observations together we conclude from (2.20), (2.21) that

$$f'(\rho) \le Ce^{-c/\varepsilon^\mu}, \tag{2.22}$$

where C is another positive constant. Note that in (2.22) the notation (2.6), (2.7) has been used.

By properly normalizing f_γ, say by taking $f_\gamma(\rho_0/2) = 0$, we find that by integrating (2.22) (with f_γ replacing f) between $\rho_0/2$ and ρ, we get the inequality

(2.22) for $f_\gamma(\rho)$ in the range $\rho_0/2 \le \rho \le \rho_0$. Since

$$|f_\gamma''(\rho)| \le |h_\gamma| f_\gamma' + (f_\gamma')^2 \tag{2.23}$$

we get the same inequality for $|f_\gamma''(\rho)|$ in the same range. Consequently, from the right hand side of (2.14), with $f_\gamma \zeta$ replacing f, we see that for $\rho_0/2 \le \rho \le \rho_0$,

$$\frac{L_\varepsilon \Phi}{\Phi} = O(e^{-c/\varepsilon^\mu}).$$

Thus there is a positive constant A so that in all of Ω we have

$$L_\varepsilon \Phi + A e^{-c/\varepsilon^\mu} \Phi \ge 0.$$

An application of Lemma 2.2 gives the inequality (2.9) of Theorem 2.1.

It remains to obtain the inequalities (2.10) and (2.11). If $1 + \beta = k$, then we have

$$h = -[b\rho^{k-1}/(\varepsilon + a\rho^k)], \quad k \ge 1. \tag{2.24}$$

In order to obtain estimates on $f'(\rho)$ it is slightly more convenient to look at it in the form

$$f'(\rho) = \left[\int_0^\rho \left(\exp\left(-\int_\sigma^\rho h\right)\right) d\sigma\right]^{-1}$$

Using (2.24) we now have

$$-\int_\sigma^\rho h\, d\tau = \frac{b}{ka} \int_\sigma^\rho \frac{ka\tau^{k-1}\, d\tau}{\varepsilon + a\tau^k} = \ln \left[\frac{\varepsilon + a\rho^k}{\varepsilon + a\sigma^k}\right]^{b/ka}, \tag{2.25}$$

so that

$$\int_0^\rho \left(\exp\left(-\int_\sigma^\rho h\right)\right) d\sigma = (\varepsilon + a\rho^k)^{b/ka} \int_0^\rho \frac{d\sigma}{(\varepsilon + a\sigma^k)^{b/ka}}. \tag{2.26}$$

An easy estimation of the integral on the right shows that for $\rho_0/2 \le \rho \le \rho_0$,

$$\int_0^\rho \frac{d\sigma}{(\varepsilon + a\sigma^k)^{b/ka}} \ge \begin{matrix} C/\varepsilon^{(b/a-1)/k}, & b/a > 1, \\ C, & b/a < 1, \end{matrix} \tag{2.27}$$

where we have taken C as a positive generic constant. Correspondingly, for $\rho_0/2 \le \rho \le \rho_0$,

$$f'(\rho) \le \begin{matrix} C\varepsilon^{(b/a-1)/k}, & b/a > 1, \\ C, & b/a < 1. \end{matrix} \tag{2.28}$$

We have not considered the case $b/a = 1$ since if this is the case we can increase b slightly without violating (A_3). By suitably normalizing f_γ, say $f_\gamma(\rho_0/2) = 0$, an integration of (2.28) shows that these inequalities persist for f_γ. Also (2.23) shows that they hold for $|f_\gamma''|$ as well. Thus proceeding as before we have the estimates (2.10).

Finally if $\beta + 1 > k$, it is easily established that for $\rho_0/2 \le \rho \le \rho_0$,

$$\int_0^\rho \left(\exp \int_0^\sigma h\right) d\sigma \ge C > 0,$$

where C is independent of ε. Since $\exp \int_0^\rho h \le C$, independent of ε, we conclude (using (2.19)) that $f'_\gamma(\rho) \le C$, and then also that $|f''_\gamma| \le C$ and $f_\gamma(\rho) < C$, provided f_γ is properly normalized. Proceeding as previously, the inequality (2.11) is established. The proof of Theorem 2.1 is complete.

Remark. The proof of Theorem 2.1 remains valid if the two conditions (2.2), (2.3) are replaced by the single condition

$$\sum_{i,j=1}^n a_{ij} \frac{\partial^2 \rho}{\partial x_i \, \partial x_j} + \sum_{i=1}^n b_i \frac{\partial \rho}{\partial x_i} \ge \hat{b} \rho^\beta.$$

Furthermore, we do not need to assume, in this case, that a_{ij} is continuously differentiable but just Hölder continuous. Since the condition (1.3) is a standard condition for the nonsolvability of the Dirichlet problem, we have preferred to state (A_3) in such a way so as to expose the condition (2.2) which is a quantitative version of (1.3).

3. Lower bounds

In this section we shall obtain lower bounds for the principal eigenvalue of the boundary value problem considered in the last section. In addition to the conditions imposed on Ω in that section we shall make the following assumption:

(B_0) *The distance function ρ can be extended from a neighborhood of $\partial\Omega$ to a function $\rho \in C^2(\bar{\Omega})$ which has only one critical point $x_0 \in \Omega$ at which the Hessian $(\partial^2 \rho / \partial x_i \, \partial x_j)$ is negative definite.*

A word is in order concerning this assumption. If Ω is a ball, then it is clear that a function satisfying (B_0) may be constructed. This is also true if Ω is C^2-diffeomorphic to a ball, say $\{|y| < 1\}$. Indeed, let $r(x)$ be the distance from $x \in \Omega$ to $\partial\Omega$, $\tilde{\rho}$ a function which satisfies (B_0) for the ball $\{|y| < 1\}$ (with grad $\tilde{\rho}(0) = 0$), and ϕ a diffeomorphism onto this ball. Let $r_0 > 0$ be sufficiently small so that $r(x)$ is a C^2 function for $0 \le r \le r_0$, and the latter region does not contain $\phi^{-1}(0)$. Let ψ be a C^∞ function on the real line so that $\psi(t) \equiv 1$ for $t \le r_0/2$, $\psi(t) = 0$ for $t \ge r_0$ and $\psi'(t) \le 0$. Set

$$\rho(x) = r(x)\psi \circ r(x) + [\tilde{\rho} \circ \phi(x) + \gamma](1 - \psi \circ r(x)),$$

where γ is a constant to be chosen in a moment. Clearly ρ coincides with r for $0 \le r \le r_0/2$, and coincides with $\tilde{\rho} \circ \phi + \gamma$ for $r \ge r_0$. Function ρ has a critical point at $x_0 = \phi^{-1}(0)$, and no other critical points in the region $0 \le r \le r_0/2$, and $r \ge r_0$. To see that there are no critical points in $r_0/2 < r < r_0$ we differentiate in the inward normal direction to $\partial\Omega$; i.e., we differentiate with respect to r. We get

$$\frac{d\rho}{dr} = \psi + (1-\psi)\frac{d\tilde{\rho}\circ\phi}{dr} - [\tilde{\rho}\circ\psi + \gamma - r]\psi'.$$

Choose γ large enough so that the term in brackets on the right is nonnegative. Since the first sum on the right is positive and $\psi' \le 0$ in $r_0/2 \le r \le r_0$, we see that ρ does not have a critical point in the given region.

To see that the Hessian of ρ is negative definite at x_0, let Df designate the differential of f. Noting that $\tilde{\rho}$ has a critical point at $\phi(x_0)$, we have

$$D_x^2\rho(x_0) = D_y^2(\tilde{\rho}(0))[D_x\phi(x_0)]^2.$$

Since the Hessian $D_y^2\tilde{\rho}$ is nonsingular at 0 and $D_x\phi$ is nonsingular at x_0, it follows that the Hessian $D_x^2\rho(x_0)$ is nonsingular. Finally since ρ has a local maximum at x_0 it follows that $D_x^2\rho(x_0)$ is negative definite.

In addition to the assumption (B_0) and the assumptions (A_1) and (A_2) of Section 2 we shall now make the following assumption:

(B_3) *For all* $0 \le \rho \le \rho_1$, ρ_1 *sufficiently small,*

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} \ge a\rho^k, \quad a > 0,\ k \ge 0, \tag{3.1}$$

$$\sum_{i=1}^{n}\left(b_i - \sum_{j=1}^{n}\frac{\partial a_{ij}}{\partial x_i}\right)\frac{\partial\rho}{\partial x_j} \le \tilde{b}\rho^\beta, \quad 0 < \tilde{b} < b,\ \beta \ge 0, \tag{3.2}$$

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2\rho}{\partial x_i\,\partial x_j} = -\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial x_j}\frac{\partial\rho}{\partial x_i} + o(\rho^\beta), \quad \beta > 0,\ \rho \to 0 \tag{3.3}$$

and, if $\beta = 0$,

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} = 0 \quad \text{on } \partial\Omega. \tag{3.4}$$

Using the notation (2.5)–(2.8) we have the following

THEOREM 3.1. *Under the hypotheses* (A_1), (A_2), (B_0), *and* (B_3) *there exists a positive constant* A *so that*

$$\lambda_\varepsilon \ge Ae^{-c/\varepsilon^\mu} \quad \text{for } 1 + \beta < k, \tag{3.5}$$

$$\lambda_\varepsilon \ge \begin{matrix} A\varepsilon^\nu & \text{if } b/a > 1 \\ A & \text{if } b/a < 1 \end{matrix} \quad \text{for } 1 + \beta = k, \tag{3.6}$$

and

$$\lambda_\varepsilon \ge A \quad \text{for } 1 + \beta > k. \tag{3.7}$$

In order to obtain these bounds for λ_ε it will now be necessary for us to use the following lemma from [2].

LEMMA 3.2. *Let* (A_1) *and* (A_2) *hold and let* $\Phi \in C(\bar{\Omega}) \cap C^2(\Omega)$. *If* $L_\varepsilon\Phi(x) + A\Phi(x) \le 0$ *for* $x \in \Omega$ *and some* $A > 0$ *and* $\Phi(x) \ge 1$ *in* Ω, *then* $\lambda_\varepsilon \ge A$.

Proof of Theorem 3.1. In order to apply Lemma 3.2 we shall, as in the last section, choose $\Phi = e^\phi$, where $\phi(x) = f(\rho(x))$ and f is twice differentiable in $[0, \infty)$. Let us first work in a neighborhood of $\partial\Omega$. For ρ sufficiently small, say $0 \le \rho \le \rho_0$, we have the equation (2.14) which we repeat for the reader's convenience:

$$(3.8)\quad \frac{L_\varepsilon\Phi}{\Phi} = \left[\varepsilon + \sum_{i,j=1}^{n} a_{ij}\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j}\right][f'' + (f')^2] + \left[\sum_{i=1}^{n}\left(b_i - \sum_{j=1}^{n}\frac{\partial a_{ij}}{\partial x_j}\right)\frac{\partial\rho}{\partial x_i} + \varepsilon\Delta\rho + o(\rho^\beta)\right]f'.$$

We note that this formula is true for $\beta \ge 0$.

Let us now set

$$(3.9)\qquad h_\gamma = h - \gamma = -\left[\frac{b\rho^\beta}{\varepsilon + a\rho^k} + \gamma\right],$$

where γ is a nonnegative constant which is to be determined. We shall again solve Bernoulli's equation

$$(3.10)\qquad f_\gamma'' + (f_\gamma')^2 = h_\gamma f_\gamma'.$$

This time we shall take $f_\gamma'(0) = \varepsilon$ so that we get a solution

$$(3.11)\quad f_\gamma'(\rho) = \frac{\exp\int_0^\rho h_\gamma}{\varepsilon + \int_0^\rho(\exp\int_0^\sigma h_\gamma)\,d\sigma} = \frac{1}{\varepsilon\exp(-\int_0^\rho h_\gamma) + \int_0^\rho(\exp(-\int_\sigma^\rho h_\gamma)\,d\sigma}.$$

We see that f_γ is a C^2 function in $[0, \infty)$. If f is a solution to $f'' + (f')^2 = hf'$, with $f'(0) = \varepsilon$, it is clear from (3.11) that there is a positive constant C so that $(1/C)f' \le f_\gamma' \le Cf'$. Thus estimates on f' will give estimates on f_γ'. Clearly f' may be written in the form (3.11) with f' replacing f_γ' and h replacing h_γ.

Let us first take the case $\beta + 1 < k$. Using C as a positive generic constant, it is clear that

$$(3.12)\qquad \varepsilon + \int_0^\rho\left(\exp\int_0^\sigma h\right)d\sigma \le C.$$

Further, $I_{\beta,k}(\varepsilon, \rho) \le I_{\beta,k}$, so that following the computations of Section 2 we get

$$(3.13)\qquad \exp\int_0^\rho h \ge Ce^{-c/\varepsilon^\mu}.$$

Recall that the constants c and μ are given by (2.6) and (2.7).

From (3.8) and (3.1) of (B_3) we get

$$(3.14)\quad \frac{L_\varepsilon\Phi}{\Phi} \le \left[-b\rho^\beta + \sum_{i=1}^{n}\left(b_i - \sum_{j=1}^{n}\frac{\partial a_{ij}}{\partial x_j}\right)\frac{\partial\rho}{\partial x_i} + \varepsilon(\Delta\rho - \gamma) + o(\rho^\beta)\right]f_\gamma'.$$

Choose γ sufficiently large so that $\Delta\rho - \gamma \le 0$. From (3.12) and (3.13) it follows that $f'_\gamma \ge Ce^{-c/\varepsilon^\mu}$. From this estimate, ($B_3$) and (3.14) we see that for $0 \le \rho \le \rho_0$, there exists a positive constant A so that

$$L_\varepsilon \Phi + Ae^{-c/\varepsilon^\mu}\Phi \le 0. \tag{3.15}$$

Now let r be a small positive number so that for $|\rho(x) - \rho(x_0)| < r$,

$$\sum_{i,j=1}^{n} (\varepsilon\delta_{ij} + a_{ij})\frac{\partial^2 \rho}{\partial x_i\, \partial x_j} \le -C < 0, \quad \sum_{i=1}^{n} b_i \frac{\partial \rho}{\partial x_i} \le C/2. \tag{3.16}$$

Since $f''_\gamma + (f'_\gamma)^2 \le 0$, it follows from (2.12) and (3.16) that

$$\frac{L_\varepsilon \Phi}{\Phi} \le -\frac{C}{2} f'_\gamma.$$

This shows that (3.15) is satisfied in $|\rho(x) - \rho(x_0)| < r$.

Finally we must consider the region $\{\rho \ge \rho_0,\ |\rho - \rho(x_0)| \ge r\}$. By ($B_0$), ρ has only one critical point so that in the region under consideration $|\text{grad}\ \rho| \ge C > 0$. Thus

$$\sum_{i,j=1}^{n} (\varepsilon\delta_{ij} + a_{ij})\frac{\partial \rho}{\partial x_i}\frac{\partial \rho}{\partial x_j} \ge C|\text{grad}\ \rho|^2 \ge C > 0,$$

where C is a generic constant. We have

$$\frac{L_\varepsilon \Phi}{\Phi} = \left[h_\gamma \sum_{i,j=1}^{n} (\varepsilon\delta_{ij} + a_{ij})\frac{\partial \rho}{\partial x_i}\frac{\partial \rho}{\partial x_j} + \sum_{i,j=1}^{n} (\varepsilon\delta_{ij} + a_{ij})\frac{\partial^2 \rho}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial \rho}{\partial x_i}\right] f'_\gamma.$$

Now choose γ so large that the term in brackets is $\le -C < 0$. Using the fact that $f'_\gamma \ge Ce^{-c/\varepsilon^\mu}$, we see that once again we have (3.15). An application of Lemma 3.2 gives the inequality (3.5) of Theorem 3.1.

To obtain the inequalities (3.6) we take $\beta + 1 = k$ and h_γ as in (3.9). As above it is enough to obtain an estimate for f' which is a solution to (3.10) with h replacing h_γ. This time we have

$$-\int_\sigma^\rho h = \ln\left[\frac{\varepsilon + a\rho^k}{\varepsilon + a\sigma^k}\right]^{b/ka},$$

so that

$$\int_0^\rho \exp\left(-\int_\sigma^\rho h\right) = (\varepsilon + a\rho^k)^{b/ka} \int_0^\rho \frac{d\sigma}{(\varepsilon + a\sigma^k)^{b/ka}}.$$

These are exactly the formulas (2.25) and (2.26) which we have repeated for the convenience of the reader. To get lower bounds, we want the inequality in (2.27) reversed. It is easily seen that

$$\int_0^\rho \frac{d\sigma}{(\varepsilon + a\sigma^k)^{b/ka}} \le \begin{matrix} C/\varepsilon^{(b/a-1)/k}, & b/a > 1, \\ C, & b/a < 1. \end{matrix}$$

Further, we have

$$\varepsilon \exp\left(-\int_0^\rho h\right) = (\varepsilon + a\rho^k)^{b/ka}\varepsilon/(\varepsilon^{b/ka}).$$

Thus,

$$f'(\rho) = \frac{1}{\varepsilon \exp(-\int_0^\rho h) + \int_0^\rho (\exp(-\int_\sigma^\rho h))\, d\sigma} \geq \begin{cases} C\varepsilon^{(b/a-1)/k}, & b/a > 1, \\ C, & b/a < 1. \end{cases}$$

Arguing as before we have the inequalities (3.6) of Theorem 3.1.

Finally, if $\beta + 1 > k$, it is clear that $f'(\rho) \geq C > 0$, where C is a constant independent of ρ and ε. This gives the inequality (3.7) and the proof of Theorem 3.1 is complete.

Remark 1. Suppose $n = 1$, $\Omega = (0, 1)$ and

$$a(x)/x^k \to \alpha_1 > 0 \text{ as } x \to 0, \qquad a(x)/(1-x)^k \to \alpha_2 > 0 \text{ as } x \to 1.$$

Suppose also that $b(x)/a'(x) = \beta/k$ in a neighborhood of $x = 0$ and $x = 1$. Then

$$m \leq \lambda_\varepsilon/\varepsilon^{(\beta-1)/k} \leq M \quad \text{if } \beta > 1, \tag{3.17}$$

$$m \leq \lambda_\varepsilon/\ln(1/\varepsilon) \leq M \quad \text{if } \beta = 1, \tag{3.18}$$

$$m \leq \lambda_\varepsilon \leq M \quad \text{if } \beta < 1, \tag{3.19}$$

where m and M are positive constants. The proof of these facts is easily obtained by being a bit careful in the proofs of Theorem 2.1 and Theorem 3.1.

Remark 2. We would like to give a probabilistic interpretation for the results obtained so far. Suppose $a_{ij} = \frac{1}{2}\sum_{k=1}^n \sigma_{ik}\sigma_{kj}$ and consider the system of stochastic differential equations

$$d\xi(t) = \sigma(\xi(t))\, dw(t) + b(\xi(t))\, dt, \tag{3.20}$$

where $w(t)$ is n-dimensional Brownian motion, $b = (b_1, \ldots, b_n)$ and $\sigma = (\sigma_{ij})_{i,j=1}^n$. Let

$$\mathscr{A}(x) \equiv \sum_{i,j=1}^n a_{ij}\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j},$$

$$\mathscr{B}(x) \equiv \sum_{i=1}^n b_i\frac{\partial\rho}{\partial x_i} + \sum_{i,j=1}^n a_{ij}\frac{\partial^2\rho}{\partial x_i\,\partial x_j},$$

and suppose

$$\mathscr{A}(x) = 0 \text{ on } \partial\Omega, \qquad \mathscr{B}(x) \geq 0 \text{ on } \partial\Omega. \tag{3.21}$$

Then $\xi(t)$ remains in Ω for all $t \geq 0$, provided $\xi(0) = x \in \Omega$ (see [4]).

Let $\sigma^\varepsilon = (\sigma_{ij}^\varepsilon)_{i,j=1}^n$, where

$$\frac{1}{2}\sum_{k=1}^n \sigma_{ik}^\varepsilon\sigma_{kj}^\varepsilon = a_{ij} + \varepsilon\delta_{ij}, \quad \varepsilon > 0,$$

and consider the stochastic system

$$d\xi^\varepsilon(t) = \sigma^\varepsilon(\xi^\varepsilon(t))\, dw(t) + b(\xi^\varepsilon(t))\, dt. \tag{3.22}$$

Denote by τ_ε the first time $\xi^\varepsilon(t)$ hits $\partial\Omega$ and denote by τ_0 the first time $\xi(t)$ hits $\partial\Omega$. Since a.s. $\xi^\varepsilon(t) \to \xi(t)$ uniformly in finite time intervals (see [4]) we should have, at least heuristically,

(3.23) $\tau_\varepsilon \to \tau_0$ as $\varepsilon \to 0$, provided τ_0 is finite.

Now by [3] (see also [5], [6]) the principal eigenvalue λ_ε is characterized by

$$\lambda_\varepsilon = \inf\{\lambda\colon E_x[e^{\lambda\tau_\varepsilon}] < \infty \text{ for all } x \in \Omega\}.$$

Hence, as $\varepsilon \to 0$

(3.24) $\lambda_\varepsilon \to 0$ if $\tau_\varepsilon \to \infty$ and $\lambda_\varepsilon \geq C > 0$ if τ_ε remains bounded.

As mentioned before, the conditions (3.21) imply that $\tau_0 = \infty$ and hence (3.23) cannot be directly applied to conclude something about the behavior of τ_ε. In order to say something specific about the behavior of τ_ε or λ_ε as $\varepsilon \to 0$, we introduce the function

$$Q = \frac{1}{\rho}\left(\mathscr{B} - \frac{\mathscr{A}}{\rho}\right). \tag{3.25}$$

If $Q < 0$ then $\partial\Omega$ is stable with respect to the paths $\xi(t)$ (see [4]) and if $Q > 0$ then $\partial\Omega$ is unstable. Thus, if $Q > 0$, then, roughly speaking, $\xi(t)$ does not come near to the boundary as $t \to \infty$ and, consequently, we should have $\tau_\varepsilon \to \infty$ as $\varepsilon \to 0$, i.e., heuristically speaking,

(3.26) $Q > 0$ implies $\lambda_\varepsilon \to 0$;

further, the larger the Q the faster λ_ε goes to zero. Similarly we should have

(3.27) $Q < 0$ implies $\lambda_\varepsilon \geq C > 0$.

Suppose $(1/C_0)\rho^2 \leq \mathscr{A} \leq C_0\rho^2$, $(C_0 > 0)$. If $Q > C/\rho$, $C > 0$, then the first inequality in Theorem 2.1 holds, whereas if $Q > C$, $C > 0$, then the second inequalities in Theorem 2.1 hold. If in Theorem 3.1, $b/a < 1$, then $\lambda_\varepsilon \geq C > 0$; this corresponds to the case where Q is negative. The other results of Sections 2 and 3 can also be viewed in this context.

Notice that in Remark 1, if $a'(0) = 0$, $a''(0) > 0$ then $k = 2$ and $\operatorname{sgn} Q(x) = \beta - 1$ near $\partial\Omega$ so that $Q > 0$ implies (3.17), $Q = 0$ implies (3.18) and $Q < 0$ implies (3.19).

4. Further results

In this section we shall obtain asymptotic estimates for the principal eigenvalue for the problem

$$L_\varepsilon u(x) \equiv (\varepsilon a_0(x) + a_1(x))u''(x) + b(x)u'(x) = -\lambda_\varepsilon u(x), \quad x \in (0, 1) \tag{4.1}$$

$$u(0) = u(1) = 0.$$

Throughout this section we shall make the following assumptions on the coefficients:

$$a_0(x),\ a_1(x),\ b(x) \in C^1(0, 1) \cap C[0, 1],$$
$$(4.2)\qquad a_0(x),\ a_1(x) > 0 \quad \text{in } (0, 1),$$
$$a_0(x)/\rho^\alpha \to \tilde{a} > 0, \quad b(x)\frac{d\rho}{dx}\Big/\rho^\beta \to b_0 > 0 \quad \text{as } \rho \to 0,$$

where ρ is the distance from $x \in (0, 1)$ to the boundary of (0, 1),

$$(x^{-\alpha}(\varepsilon a_0 + a_1))', \quad (x^{-\alpha}(\varepsilon a_0 + a_1))'', \quad (x^{-\beta}b)'$$

are bounded in some neighborhood of 0 and $0 \le \alpha < 1 + \beta$, $\alpha < 2$, $\beta \ge 0$.

We shall first obtain upper bounds on λ_ε. Toward this end we make the following additional assumption:

$$(4.3)\qquad a_1(x) \le a\rho^k \quad \text{as } \rho \to 0,\ a > 0,\ k > \alpha.$$

In order to state the first theorem of this section we shall, for notational convenience, introduce certain constants. Let us set

$$(4.4)\qquad I_{\alpha,\beta,k} = \int_0^\infty \frac{y^{\beta-\alpha}}{1 + y^{k-\alpha}}\,dy$$
$$= \frac{(k-\alpha)^{\beta-\alpha}}{\Gamma(1/k-\alpha)}\Gamma\left(1 + \beta - \alpha + \frac{1}{k-\alpha}\right)$$
$$\times \frac{\pi}{\sin\left[\pi(1 + \beta - \alpha)/(k - \alpha)\right]}, \quad k > 1 + \beta.$$

$$(4.5)\qquad d = \frac{\tilde{b}a^{(\alpha-(\beta+1))/(k-\alpha)}}{(\tilde{a})^{(k-(\beta+1))/(k-\alpha)}} I_{\alpha,\beta,k} \qquad \zeta = \frac{k - (\beta + 1)}{k - \alpha}$$
$$\theta = \frac{(\tilde{b}/\tilde{a}) - 1}{k - \alpha}, \quad \tilde{b} < b_0,\ k \ge 1 + \beta.$$

We then have the following

THEOREM 4.1. *Under the assumptions* (4.2) *and* (4.3) *there exists a positive constant A so that*

$$(4.6)\qquad \lambda_\varepsilon \le Ae^{-d/\varepsilon^\zeta} \quad \textit{for } 1 + \beta < k,$$

$$(4.7)\qquad \lambda_\varepsilon \le \begin{matrix} A\varepsilon^\theta & \textit{if } \tilde{b}/\tilde{a} > 1 \\ A & \textit{if } \tilde{b}/\tilde{a} < 1 \end{matrix} \quad \textit{for } 1 + \beta = k,$$

$$(4.8)\qquad \lambda_\varepsilon \le A \quad \textit{for } 1 + \beta > k.$$

Proof. From the assumptions (4.2) and (4.3) we see that

$$(\varepsilon a_0(x) + a_1(x))/\rho^\alpha \to \varepsilon\tilde{a} \quad \text{as } \rho \to 0.$$

Thus it follows from [1] that the principal eigenvalue λ_ε exists for all $\varepsilon > 0$. Further, it follows from (4.2) that for every $\gamma > 1$, there is a ρ_γ so that

$$\varepsilon a_0(x) < \gamma\varepsilon\tilde{a}\rho^\alpha \quad \text{for } 0 \le \rho \le \rho_\gamma. \tag{4.9}$$

We now proceed with the proof in a manner entirely parallel to the proof of Theorem 2.1. We shall take $\Phi = \exp\phi$ and shall work, at first, only in a neighborhood of 0 and 1. We shall take $\phi(x) = f(\rho(x))$, where f is a twice differentiable function on the real axis. Noting that $d\rho/dx = \pm 1$ in a neighborhood of the boundary of [0, 1] we have

$$\frac{L_\varepsilon\Phi}{\Phi} = [\varepsilon a_0 + a_1][f'' + (f')^2] + b\frac{d\rho}{dx}f'. \tag{4.10}$$

As in Section 2 we shall choose f so that $f' \ge 0$, $f'' + (f')^2 \le 0$, and $f(\rho) \to -\infty$ as $\rho \to 0$. Thus from (4.3), (4.9), and (4.10) we have

$$\frac{L_\varepsilon\Phi}{\Phi} \ge [\gamma\varepsilon\tilde{a}\rho^\alpha + a\rho^k][f'' + (f')^2] + b\frac{d\rho}{dx}f'. \tag{4.11}$$

Now let us take

$$h(\rho) = -\tilde{b}\rho^\beta/(\gamma\varepsilon\tilde{a}\rho^\alpha + a\rho^k) \tag{4.12}$$

and solve Bernoulli's equation

$$f'' + (f')^2 = hf'. \tag{4.13}$$

It follows from (4.3), (4.11), (4.12), and (4.13) that for ρ sufficiently small, say $0 \le \rho \le \rho_0$, $L_\varepsilon\Phi/\Phi \ge 0$.

As in Section 2, (4.13) leads to a solution

$$f'(\rho) = \left[\int_0^\rho \left(\exp\left(-\int_\sigma^\rho h\right)d\sigma\right]^{-1} = \frac{\exp\int_0^\rho h}{\int_0^\rho (\exp\int_0^\sigma h)\,d\sigma}. \tag{4.14}$$

We first consider the case $1 + \beta < k$. A simple calculation gives

$$\begin{aligned}\int_0^\rho h &= -\frac{\tilde{b}a^{(\alpha-(\beta+1))/(k-\alpha)}}{(\gamma\varepsilon\tilde{a})^{(k-(\beta+1))/(k-\alpha)}}\int_0^{\rho/(\tilde{a}\varepsilon/a)^{1/(k-\alpha)}} \frac{y^{\beta-\alpha}\,dy}{1+y^{k-\alpha}} \\ &= -\frac{\tilde{b}a^{(\alpha-(\beta+1))/(k-\alpha)}}{(\gamma\varepsilon\tilde{a})^{(k-(\beta+1))/(k-\alpha)}} I_{\alpha,\beta,k}(\varepsilon,\rho).\end{aligned} \tag{4.15}$$

In the range $\rho_0/2 \le \rho \le \rho_0$, $I_{\alpha,\beta,k}(\varepsilon,\rho) \to I_{\alpha,\beta,k}$, uniformly in ρ, where $I_{\alpha,\beta,k}$ is given by (4.4). In the same range for ρ, if ε is sufficiently small we have

$$\int_0^\rho \left(\exp\int_0^\sigma h\right)d\sigma \ge \int_0^{(\gamma\tilde{a}\varepsilon/a)^{1/(k-\alpha)}} \left(\exp\int_0^\sigma h\right)d\sigma.$$

Now, for $0 \le \sigma \le (\gamma\tilde{a}\varepsilon/a)^{1/(k-\alpha)}$ we have

$$\begin{aligned}\int_0^\sigma h &= -\int_0^\sigma \frac{\tilde{b}\tau^\beta}{\gamma\varepsilon\tilde{a}\tau^\alpha + a\tau^k}\,d\tau \\ &\ge -\frac{1}{2}\frac{\tilde{b}}{\gamma\varepsilon\tilde{a}}\int_0^\sigma \tau^{\beta-\alpha}\,d\tau \\ &= -\frac{1}{2}\frac{\tilde{b}}{\gamma\varepsilon\tilde{a}}\frac{1}{\beta+1-\alpha}\sigma^{\beta+1-\alpha}.\end{aligned}$$

Note that the last equality is possible because $\beta + 1 - \alpha > 0$. Make the transformation

$$\frac{1}{2}\frac{\tilde{b}}{\gamma\varepsilon\tilde{a}}\frac{1}{\beta+1-\alpha}\sigma^{\beta+1-\alpha} = y.$$

Then

$$\text{(4.18)} \quad \int_0^{(\gamma\tilde{a}\varepsilon)^{1/(k-\alpha)}} \left(\exp\int_0^\sigma h\right) d\sigma = C\varepsilon^{1/(\beta+1-\alpha)} \int_0^{c\varepsilon^{(\beta+1-k)/(k-\alpha)}} e^{-y} y^{(\alpha-\beta)/(\beta+1-\alpha)}\,dy,$$

where C, c are positive constants. Since $\beta + 1 \le k$, from (4.18) we get

$$\text{(4.19)} \qquad \int_0^\rho \left(\exp\int_0^\sigma h\right) d\sigma \ge C\varepsilon^{1/(\beta+1-\alpha)}.$$

Since γ is as close to 1 as we wish, provided we take ρ_0 sufficiently small, and $I_{\alpha.\beta.k}(\varepsilon, \rho)$ is as close to $I_{\alpha.\beta.k}$ as we wish for $\rho_0/2 \le \rho \le \rho_0$, provided ε is sufficiently small, if we had originally worked with $\tilde{b}'$ so that $\tilde{b} < \tilde{b}' < b_0$, it follows from (4.14), (4.15), and (4.19) that

$$\text{(4.20)} \qquad f'(\rho) \le Ce^{-d/\varepsilon^\zeta}.$$

If we now proceed as in the first part of the proof of Section 2 we see that there is a positive constant A so that

$$\text{(4.21)} \qquad L_\varepsilon\Phi + Ae^{-d/\varepsilon^\zeta}\Phi \ge 0, \quad x \in [0, 1].$$

Since the operator L_ε may not be elliptic at the end points of [0, 1] it may not be possible to use (4.21) in order to apply Lemma 2.2. However, we may proceed in the following way. Let $\eta > 0$ and consider the principal eigenvalue to the problem

$$\text{(4.22)} \qquad L_\varepsilon u(x) = -\lambda_\varepsilon^\eta u(x),\ x \in (\eta, 1-\eta), \qquad u(\eta) = u(1-\eta) = 0.$$

From Corollary 3.3 of [1] it follows that

$$\text{(4.23)} \qquad \lambda_\varepsilon \le \lambda_\varepsilon^\eta.$$

If we now let ρ be the distance from a point in $J_\eta = (\eta, 1-\eta)$ to ∂J_η, it follows from our assumptions that for every $\gamma > 1$, there is an η_γ and ρ_γ so that

$$\varepsilon a_0(x) < \gamma\varepsilon\tilde{a}(\rho+\eta)^\alpha, \quad a_1(x) \le a(\rho+\eta)^\alpha, \quad b(x)(d\rho/dx) > \tilde{b}(\rho+\eta)^\beta$$

for $x \in J_\eta$, $0 < \eta \le \eta_\gamma$, $0 \le \rho \le \rho_\gamma$.

We now set

$$h_\eta(\rho) = \frac{-\tilde{b}(\rho+\eta)^\beta}{\gamma\varepsilon\tilde{a}(\rho+\eta)^\alpha + a(\rho+\eta)^k}, \tag{4.24}$$

and solve equation (4.13) with h_η replacing h. Going through the same computations as before, we find that there is a function Φ, positive in J_η, vanishing on the boundary of J_η and satisfying

$$L_\varepsilon\Phi + Ae^{-d(\varepsilon,\eta)/\varepsilon^\zeta}\Phi \ge 0, \tag{4.25}$$

for all $(\rho_0+\eta)/2 \le \rho \le \rho_0 + \eta$, where ρ_0 is sufficiently small and for all sufficiently small η. Here $A > 0$ and is independent of η,

$$d(\varepsilon, \eta) = \inf\left\{\frac{\tilde{b}a^{(\alpha-(\beta+1))/(k-\alpha)}}{\tilde{a}^{(k-(\beta+1))/(k-\alpha)}} I_{\alpha,\beta,k}(\varepsilon, \rho, \eta);\ \frac{\rho_0+\eta}{2} \le \rho \le \rho_0 + \eta\right\},$$

and

$$I_{\alpha,\beta,k}(\varepsilon, \rho, \eta) = \int_{\eta/(\gamma\tilde{a}\varepsilon/a)^{1/(k-\alpha)}}^{(\rho+\eta)/(\gamma\tilde{a}\varepsilon/a)^{1/(k-\alpha)}} \frac{y^{\beta-\alpha}}{1+y^{k-\alpha}}\,dy.$$

We now proceed exactly as in the proof of Theorem 2.1 to obtain the inequality (4.25) for the whole interval J_η. We may now use (4.25) to apply Lemma 2.2 to the eigenvalue problem (4.22). Taking into account (4.23) we have

$$\lambda_\varepsilon \le Ae^{-d(\varepsilon,\eta)/\varepsilon^\zeta}. \tag{4.26}$$

Allowing $\eta \to 0$ in (4.26) we get $\lambda_\varepsilon \le Ae^{-d/\varepsilon^\zeta}$. Thus we have the inequality (4.6) of Theorem 4.1.

We now consider the case $1 + \beta = k$. We again take h_η as in (4.24) and again solve (4.13) with h_η replacing h. Going through the computations in this case we find there is a function Φ, positive in J_η, vanishing on ∂J_η so that

$$L_\varepsilon\Phi + AI_{\alpha,k}(\varepsilon, \eta)^{-1}\Phi \ge 0, \tag{4.27}$$

for all sufficiently small η, where

$$I_{\alpha,k}(\varepsilon, \eta) = \inf\left\{\int_0^\rho \frac{d\sigma}{[\gamma\varepsilon\tilde{a} + a(\sigma+\eta)^{k-\alpha}]^{\tilde{b}/\tilde{a}(k-\alpha)}};\ \frac{\rho_0+\eta}{2} \le \rho \le \rho_0 + \eta\right\}.$$

We may now use (4.27) to apply Lemma 2.2 to the eigenvalue problem (4.22). Taking into account (4.23) we have $\lambda_\varepsilon \le AI_{\alpha,k}(\varepsilon, \eta)^{-1}$. Allowing $\eta \to 0$ we get (4.7). Finally, a similar procedure will give (4.8) for $\alpha < k < 1 + \beta$, so that the proof of Theorem 4.1 is complete.

We shall now state and prove a theorem on lower estimates for λ_ε. In addition to the hypothesis (4.2) we now make the assumption that there is a $\rho_1 > 0$ so that

$$a_1(x) \geq a\rho^k, \quad 0 \leq \rho \leq \rho_1, \quad a > 0, k > \alpha. \tag{4.27}$$

THEOREM 4.2. *Under the hypothesis* (4.2) *and* (4.27) *there exists a positive constant A so that*

$$\lambda_\varepsilon \geq Ae^{-d/\varepsilon^\zeta} \quad \textit{for } 1 + \beta < k, \tag{4.28}$$

$$\lambda_\varepsilon \geq \begin{matrix} A\varepsilon^\theta & \textit{if } \tilde{b}/\tilde{a} > 1 \\ A & \textit{if } \tilde{b}/\tilde{a} < 1 \end{matrix} \quad \textit{for } 1 + \beta = k. \tag{4.29}$$

$$\lambda_\varepsilon \geq A \quad \textit{for } \alpha < k < 1 + \beta. \tag{4.30}$$

In this theorem the constant $\tilde{b}$ which appears in (4.5) *and* (4.29) *is to be taken so that $\tilde{b} > b_0$.*

Proof. The proof proceeds entirely parallel to the proof of Theorem 3.1, so that we shall only discuss in detail those points which are different. The reader can easily fill in the missing details. For this proof we take

$$h_\tau = h - \tau = -\left[\frac{\tilde{b}\rho^\beta}{\gamma\varepsilon\tilde{a}\rho^\alpha + a\rho^k} + \tau\right], \quad \tilde{b} > b_0,$$

where τ is taken as a sufficiently large constant. Hence, working as in Section 3, we find that if $1 + \beta < k$, there is a function $\Phi \in C^2[0, 1]$ so that

$$L_\varepsilon\Phi + Ae^{-d/\varepsilon^\zeta}\Phi < 0 \quad \text{in } [0, 1]. \tag{4.31}$$

Now, we cannot immediately use (4.31) in the application of Lemma 3.2 since L_ε may not be elliptic at the end points of [0, 1]. In order to get around this difficulty we let $\eta > 0$ and let $L_{\varepsilon,\eta} \equiv L_\varepsilon + \eta(d^2/dx^2)$. Since $\Phi \in C^2[0, 1]$, if η is sufficiently small we get (4.31) with $L_{\varepsilon,\eta}$ instead of L_ε. Let λ_ε^η be the principal eigenvalue of the problem (4.1) with $L_{\varepsilon,\eta}$ replacing L_ε. Since $L_{\varepsilon,\eta}$ is elliptic in [0, 1] we may apply Lemma 3.2 to get

$$\lambda_\varepsilon^\eta \geq Ae^{-d/\varepsilon^\zeta}. \tag{4.32}$$

From Corollary 3.2 of [1] we have $\lambda_\varepsilon^\eta \to \lambda_\varepsilon$ as $\eta \to 0$. Thus we have (4.32) with λ_ε replacing λ_ε^η. This is of course the inequality (4.28).

If $\beta + 1 = k$ or if $\alpha < k < 1 + \beta$ we obtain the other inequalities of Theorem 4.2 by similar reasoning. Thus the proof of Theorem 4.2 is complete.

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Northwestern University
Evanston, Illinois