ILLINOIS JOURNAL OF MATHEMATICS
Volume 21, Number 4, December 1977

KOLMOGOROV'S LAW OF THE ITERATED LOGARITHM FOR BANACH SPACE VALUED RANDOM VARIABLES[1]

BY

J. KUELBS

1. Introduction

Let B denote a real separable Banach space with norm $\|\cdot\|$, and throughout $X_1, X_2, \ldots$ are independent B-valued random variables such that $EX_k = 0$ and $E\|X_k\|^2 < \infty$ $(k \geq 1)$. As usual $S_n = X_1 + \cdots + X_n$ for $n \geq 1$ and we write Lx to denote $\log x$ for $x \geq e$ and 1 otherwise. The function $L(Lx)$ is written LLx, and B^* denotes the topological dual of B.

In this paper we establish Kolmogorov's version of the *LIL* [6] for B-valued random variables, and this result will have several corollaries dealing with the *LIL* for i.i.d. sequences. In particular, the recent interesting result of G. Pisier [10, Thèoréme 4.3] will be obtained as an easy corollary (see Corollary 4.1).

To motivate Theorem 3.2 we now turn to the *LIL* for i.i.d. sequences in the Banach space setting, but first we need a bit of terminology.

If (M, d) is a metric space and $A \subseteq M$, $x \in M$, we define the distance from x to A by $d(x, A) = \inf_{y \in A} d(x, y)$. If $\{x_n\}$ is a sequence of points in M, then $C(\{x_n\})$ denotes the cluster set of $\{x_n\}$. That is, $C(\{x_n\})$ is all possible limit points of the sequence $\{x_n\}$. We also will use the notation $\{x_n\} \twoheadrightarrow A$ if both $\lim_n d(x_n, A) = 0$ and $C(\{x_n\}) = A$.

Now let $X_1, X_2, \ldots$ be i.i.d. B-valued random variables such that $EX_1 = 0$ and $E\|X_1\|^2 < \infty$. In view of Strassen's formulation of the Hartman-Wintner result [12] and the recent results in [7], [9], [10] we say *X satisfies the* LIL *in B* if for $X_1, X_2, \ldots$ independent copies of X we have a bounded limit set K in B such that

$$P\{\{S_n/a_n \colon n \geq 1\} \twoheadrightarrow K\} = 1 \tag{1.1}$$

where $a_n = \sqrt{2nLLn}$ $(n \geq 1)$.

However, (1.1) is not always true under the classical moment assumptions in the infinite dimensional setting, but necessary and sufficient conditions for (1.1) to hold are known, and another will be established in Theorem 4.1 below.

If $\mu = \mathscr{L}(X_1)$ denotes the distribution of X_1, the limit set K turns out to be the unit ball of a Hilbert space H_μ which is uniquely determined by the covariance function

$$T(f, g) = \int_B f(x)g(x)\, d\mu(x) = E(f(X_1)g(X_1)) \qquad (f, g \in B^*). \tag{1.2}$$

Received March 25, 1976.

[1] Sponsored in part by the United States Army and by a National Science Foundation grant.

The Hilbert space H_μ is carefully defined in Lemma 2.1 of [7], and it is shown that K is necessarily a compact subset of B whenever $E\|X_1\|^2 < \infty$. Hence if $E\|X_1\|^2 < \infty$ and (1.1) holds we must have

$$P(\{S_n/a_n: n \geq 1\} \text{ conditionally compact in } B) = 1. \tag{1.3}$$

In fact, (1.3) is necessary and sufficient for (1.1) with K the unit ball of $H_{\mathscr{L}(X_1)}$, and this is demonstrated in Corollary 3.1 of [7].

The main result of Section 3 is Theorem 3.2 which generalizes (1.1) to the setting of Kolmogorov's LIL [6]. In Theorem 4.1 we consider the LIL for i.i.d sequences of B-valued random variables, and show by applying Theorem 3.2 that (1.1) holds iff $S_n/a_n \to 0$ in probability iff $E\|S_n\| = o(a_n)$.

Several corollaries of Theorem 4.1 are given, and we also obtain a more elementary version of Kolmogorov's theorem in Theorem 3.1. Theorem 4.2 produces some conditions equivalent to $P(\sup_n \|S_n\|/a_n < \infty) = 1$.

2. Some exponential inequalities for B-valued random variables

The inequalities of this section are well known for real-valued random variables, but have proved far more difficult in the vector-valued case. However, recently V. V. Yurinskii obtained some interesting results for Banach valued random variables in [13], and it is his method which we use here. Further references for the vector valued case can also be found in [13].

LEMMA 2.1. *Let $X_1, X_2, \ldots, X_n$ be independent B-valued random variables such that*

$$\|X_j\| \leq cb_n \qquad (j \leq n). \tag{2.1}$$

Then, for $\varepsilon c \leq 1$ we have

$$P\left(\frac{\|S_n\|}{2b_n} \geq \varepsilon\right) \leq \exp\left\{-\varepsilon^2 + (\varepsilon^2/2)\sum_{j=1}^n \frac{E\|X_j\|^2}{b_n^2}\left(1 + \frac{\varepsilon c}{2}\right) + \frac{\varepsilon E\|S_n\|}{2b_n}\right\}. \tag{2.2}$$

Proof. Since

$$P\left(\frac{\|S_n\|}{2b_n} \geq \varepsilon\right) = P\left(\varepsilon\frac{\|S_n\|}{2b_n} \geq \varepsilon^2\right) \leq \exp\{-\varepsilon^2\}E\left(\exp\left(\frac{\varepsilon\|S_n\|}{2b_n}\right)\right) \tag{2.3}$$

it suffices to show that

$$E\left(\exp\left(\frac{\varepsilon\|S_n\|}{2b_n}\right)\right) \leq \exp\left\{(\varepsilon^2/2)\sum_{j=1}^n \frac{E\|X_j\|^2}{b_n^2}\left(1 + \frac{\varepsilon c}{2}\right) + \frac{\varepsilon E\|S_n\|}{2b_n}\right\}. \tag{2.4}$$

To prove (2.4) we set $Y_k = S_n - X_k$, $E_k\eta = E(\eta \mid X_1, \ldots, X_{k-1})$, $E_{n+1}\eta = \eta$, $E_1\eta = E\eta$. Then

$$\begin{aligned} E(\exp(h\|S_n\|)) &= E(E_n(\exp[hE_n\|S_n\| + h\eta_n])) \\ &= E(\exp(hE_n\|S_n\|) \cdot E_n \exp(h\eta_n)) \end{aligned} \tag{2.5}$$

where $\eta_k = E_{k+1}\|S_n\| - E_k\|S_n\|$ $(k = 1, \ldots, n)$.

Using the triangle inequality and the independence of $X_1, \ldots, X_n$ we have

$$\begin{aligned}(2.6)\quad E_{k+1}\|S_n\| - E_k\|S_n\| &\leq E_{k+1}\|Y_k\| + E_{k+1}\|X_k\| - E_k\|Y_k\| + E_k\|X_k\| \\ &= \|X_k\| + E\|X_k\| \qquad (k = 1, \ldots, n)\end{aligned}$$

and similarly

$$(2.7)\qquad E_{k+1}\|S_n\| - E_k\|S_n\| \geq -\|X_k\| - E\|X_k\| \qquad (k = 1, \ldots, n)$$

since $E_{k+1}\|Y_k\| = E_k\|Y_k\|$ (because Y_k is independent of X_k). Hence

$$(2.8)\qquad E_k|\eta_k|^j \leq 2^j E\|X_k\|^j \qquad (j \geq 2)$$

and with probability one, $E_k\eta_k = 0$ $(k = 1, \ldots, n)$. Thus for $h = \varepsilon/2b_n$ and by (2.1) and (2.8) we obtain, for $\varepsilon c \leq 1$,

$$\begin{aligned} E_k(\exp(h\eta_k)) &= 1 - \frac{h^2}{2!}E_k\eta_k^2 + \frac{h^3E_k\eta_k^3}{3!} + \cdots \\ &\leq 1 + \frac{2^2h^2E\|X_k\|^2}{2!}\left[1 + \frac{2hcb_n}{3} + \frac{2^2h^2(cb_n)^2}{4.3} + \cdots\right] \\ (2.9)\qquad &= 1 + \frac{\varepsilon^2E\|X_k\|^2}{2b_n^2}\left[1 + \frac{\varepsilon c}{3} + \frac{(\varepsilon c)^2}{4.3} + \cdots\right] \\ &\leq 1 + (\varepsilon^2/2)\frac{E\|X_k\|^2}{b_n^2}(1 + \varepsilon c/2) \\ &\leq \exp\left\{\frac{\varepsilon^2E\|X_k\|^2}{2b_n^2}(1 + \varepsilon c/2)\right\}\end{aligned}$$

since $1 + x \leq e^x$ for all x.

Combining (2.5) and (2.9) we obtain

$$(2.10)\quad E\left(\exp\left(\frac{\varepsilon\|S_n\|}{2b_n}\right)\right) \leq \exp\left(\frac{\varepsilon^2E\|X_n\|^2}{2b_n^2}\left(1 + \frac{\varepsilon c}{2}\right)\right)E\left(\exp\left(\frac{\varepsilon}{2b_n}E_n\|S_n\|\right)\right).$$

Iterating the previous estimates n times yields (2.4), and combining (2.3) and (2.4) we have (2.2) so the lemma is proved.

Remark. If $b_n = \sigma_n$ in (2.1) where $\sigma_n^2 = \sum_{j=1}^n E\|X_j\|^2$, then for $\varepsilon c \leq 1$ we have from (2.2) that

$$(2.11)\qquad P(\|S_n\|/2\sigma_n \geq \varepsilon) \leq \exp\{-(\varepsilon^2/2)(1 - \varepsilon c/2) + \varepsilon E\|S_n\|/2\sigma_n\}.$$

Remark. The validity of Lemma 2.1 does not depend on the fact that the range space B is a separable Banach space. In fact, both (2.2) and (2.11) hold if $X_1, \ldots, X_n$ are measurable from a probability space $(\Omega, \mathscr{F}, P)$ to the measurable space $(B, \mathscr{B})$ provided $\mathscr{B}$ is a sigma-algebra of subsets of B compatible with the linear structure of B, and B is a normed vector space with norm $\|\cdot\|$ such that $\{x \in B: \|X\| \leq t\} \in \mathscr{B}$ for all $t \geq 0$. For example, if $B = D[0, T]$,

$\|x\| = \sup_{0 \leq t \leq T} |x(t)|$, and $\mathscr{B}$ denotes the minimal sigma-algebra making all the maps $x \to x(t)$ $(0 \leq t \leq T)$ measurable, then Lemma 2.1 can be applied in this setting. See, for example, the second remark following Theorem 3.1.

3. Kolmogorov's LIL for Banach space valued random variables

We will obtain several versions of Kolmogorov's LIL for the Banach space setting and then apply them to obtain results for i.i.d. sequences as in the real-valued case.

THEOREM 3.1. *Let $X_1, X_2, \ldots$ be independent B-valued random variables such that $EX_j = 0$ $(j \geq 1)$, and*

(3.1i) $\|X_n\| \leq \Gamma_n \sigma_n / \sqrt{LL\sigma_n^2}$ a.s. $(n \geq 1)$ *where* $\Gamma_n \to 0$, *and* $\sigma_n^2 = \sum_{j=1}^n E\|X_j\|^2 \to \infty$, *and*

(3.1ii) *there exists an L such that* $\sup_n P(\|S_n\| > La_n) \leq 1/24$ *where* $a_n = \sqrt{2\sigma_n^2 LL\sigma_n^2}$.

Then there exists a constant Λ, $0 \leq \Lambda < \infty$, such that

$$P\left(\overline{\lim_n} \frac{\|S_n\|}{a_n} = \Lambda\right) = 1. \tag{3.2}$$

Proof. Using Kolmogorov's zero-one law we have (3.2) if we can show

$$P\left(\overline{\lim_n} \frac{\|S_n\|}{a_n} < \infty\right) = 1. \tag{3.3}$$

Now (3.3) holds if we can choose M sufficiently large so that $\sum_k P(B_k) < \infty$ where

$$B_k = \left\{\sup_{n_k \leq n \leq n_{k+1}} \|S_n\|/a_{n_k} > M\right\}, \tag{3.4}$$

and n_k is the smallest integer n such that $\sigma_n > 2^k$ $(k \geq 1)$. First of all observe that from (3.1i) we have

$$1 \leq \frac{\sigma_{n+1}^2}{\sigma_n^2} = 1 + \frac{E\|X_{n+1}\|^2}{\sigma_n^2} \leq 1 + \frac{\Gamma_{n+1}\sigma_{n+1}^2}{\sqrt{LL\sigma_{n+1}^2\sigma_n^2}},$$

and since $\lim_n \Gamma_n = 0$ and $\lim_n \sigma_n^2 = \infty$ this implies that $\sigma_{n+1}^2/\sigma_n^2 \sim 1$, $\sigma_{n_k} \sim 2^k$, and $\sigma_{n_k}/\sigma_{n_{k+1}} \sim 1/2$.

Next we use (3.1) to prove that $\sup_n E\|S_n\|/a_n < \infty$. To do this we first assume that $X_1, X_2, \ldots$ are symmetric. Then we know by [3] or [5, Lemma 5.4] that

$$(1/3)\int_0^\infty P(\|S_n\| > t)\,dt \leq \int_0^\infty P(N_n > t)\,dt + 4\int_0^\infty [P(\|S_n\| > t)]^2\,dt$$

where $N_n = \sup_{1 \le j \le n} \|X_j\|$. By (3.1ii) we choose $L > 1$ so that

$$\sup_n P(\|S_n\| > La_n) < 1/24.$$

Then we have

$$(1/3) \int_{La_n}^{\infty} P(\|S_n\| > t)\, dt \le 5La_n + \int_{La_n}^{\infty} P(N_n > t)\, dt + (1/6) \int_{La_n}^{\infty} P(\|S_n\| > t)\, dt$$

and hence

$$(1/6) \int_{La_n}^{\infty} P(\|S_n\| > t)\, dt \le 5La_n + \int_{La_n}^{\infty} P(N_n > t)\, dt.$$

Since for n large, $\sup_{1 \le j \le n} \|X_j\| \le \sup_{1 \le j \le n} \Gamma_j \sigma_j / \sqrt{LL\sigma_j^2} \le \sigma_n$, we have $P(N_n > t) = 0$ for $t > a_n$. Thus $\int_{La_n}^{\infty} P(\|S_n\| > t)\, dt \le 30La_n$ and hence for all n sufficiently large $E\|S_n\| = \int_0^{\infty} P(\|S_n\| > t)\, dt \le 31La_n$. This gives $\sup_n E\|S_n\|/a_n < \infty$ as was asserted.

If $\{X_n\}$ is not symmetric we introduce an independent identically distributed copy, call it $\{X_n' : n \ge 1\}$. Now the sequence $\{X_n - X_n'\}$ satisfies the conditions in (3.1) and since it is symmetric we have by the previous argument

$$\sup_n \frac{E\|S_n - S_n'\|}{b_n} < \infty$$

where $S_n' = \sum_{j=1}^{n} X_i'$, $b_n = \sqrt{2\gamma_n^2 LL\gamma_n^2}$, and $\gamma_n^2 \equiv \sum_{j=1}^{n} E\|X_j - X_j'\|^2$. Since

$$\gamma_n^2 \le \sum_{j=1}^{n} 4E\|X_j\|^2 = 4\sigma_n^2$$

we have $\sup_n E\|S_n - S_n'\|/a_n < \infty$ and since $E\|S_n\| \le E\|S_n - S_n'\|$ we thus have $\sup_n E\|S_n\|/a_n < \infty$.

Now by the same proof as for real random variables we have for all $\lambda > 0$,

$$(3.5) \quad P\left(\max_{n_k \le n \le n_{k+1}} \|S_n\| \ge 2\lambda\right) \le \frac{P(\|S_{n_{k+1}}\| \ge \lambda)}{1 - \max_{n_k \le n \le n_{k+1}} P(\|S_{n_{k+1}} - S_n\| > \lambda)}.$$

Further, for all sufficiently large k,

$$(3.6) \quad \max_{n_k \le n \le n_{k+1}} P(\|S_{n_{k+1}} - S_n\| > Ma_{n_k}/2) \le 2 \max_{n_k \le n \le n_{k+1}} \frac{E\|S_{n_{k+1}} - S_n\|}{Ma_{n_k}},$$

so by (3.1ii) the left-hand member of (3.6) is dominated by 1/2 provided

$$M \ge 16 \sup_n \frac{E\|S_n\|}{a_n}.$$

Applying (3.5) to (3.4) we thus have for $M \geq 16 \sup_n E\|S_n\|/a_n$ that

$$P(B_k) \leq 2P(\|S_{n_{k+1}}\| \geq Ma_{n_k}/2). \tag{3.7}$$

Applying (2.11) and (3.1ii) to (3.7) we have

$$\begin{aligned} P(B_k) &\leq 2P\left(\frac{\|S_{n_{k+1}}\|}{2\sigma_{n_{k+1}}} \geq \frac{M}{4}\sqrt{2LL\sigma_{n_k}^2}\,\frac{\sigma_{n_k}}{\sigma_{n_{k+1}}}\right) \\ &\leq 2\exp\left\{\frac{-\varepsilon_k^2}{2}(1-\varepsilon_k c_k/2) + \frac{\varepsilon_k}{2\sigma_{n_{k+1}}}E\|S_{n_{k+1}}\|\right\} \end{aligned} \tag{3.8}$$

where $\varepsilon_k = (M/4)\sqrt{2LL\sigma_{n_k}^2}\cdot\sigma_{n_k}/\sigma_{n_{k+1}}$ and $c_k = \Gamma_{n_{k+1}}/\sqrt{LL\sigma_{n_{k+1}}^2}$. Since $\sigma_{n_k}/\sigma_{n_{k+1}} \sim 1/2$ and $\sigma_{n_k} \sim 2^k$ we have $\varepsilon_k c_k \to 0$. Now fix $\delta > 0$ and choose $M_0(\delta)$ such that $M \geq M_0(\delta)$ implies

$$\frac{M^2}{64(1+\delta)} - (M/8)\sup_n \frac{E\|S_n\|}{a_n} > 2.$$

Since $\beta \equiv \sup_n E\|S_n\|/a_n < \infty$, (3.8) and $M \geq M_0(\delta)$ implies

$$P(B_k) \leq 2\exp\left\{-\left(\frac{M^2}{64(1+\delta)} - \frac{M\beta}{8}\right)LL2^{2k}\right\} \leq 2\exp\{-2LL2^{2k}\} \tag{3.9}$$

for all sufficiently large k. Hence for $M \geq \max(M_0(\delta), 16\sup_n E\|S_n\|/a_n)$, (3.9) implies $\sum_k P(B_k) < \infty$, and hence

$$P(B_k \text{ i.o.}) = 0. \tag{3.10}$$

Thus (3.3) holds and the theorem is proved.

Remark. The proof actually shows that $\Lambda \leq \max(M_0(\delta), 16\sup_n E\|S_n\|/a_n)$ since for any M which dominates the right-hand side we have proved that $P(B_k \text{ i.o.}) = 0$.

Remark. If $E\|S_n\| = o(a_n)$ or $S_n/a_n \to 0$ in probability in Theorem 3.1, then the method of proof used implies that (3.2) holds with $\Lambda \leq 8$. In fact, if we define n_k as the smallest integer n such that $\sigma_n > \beta^k$ $(\beta > 1)$ then we have $\Lambda \leq 4\beta$, and since $\beta > 1$ is arbitrary we have $\Lambda \leq 4$.

Remark. Assume $X_1, X_2, \ldots$ are independent $(D[0, T], \mathscr{B})$ valued random variables (see the remarks following Lemma 2.1) such that each is a martingale in t $(0 \leq t \leq T)$ and satisfying $\sup_{\omega\in\Omega}\sup_{0\leq t\leq T}|X_j(t,\omega)| \leq M$ $(j \geq 1)$. Then if $\sigma_n^2 \to \infty$ and $EX_j(t) = 0$ $(j \geq 1, 0 \leq t \leq T)$ we have

$$P\left(\overline{\lim_n}\frac{\|S_n\|}{a_n} \leq 4\right) = 1.$$

In view of the previous remark, to prove $P(\overline{\lim}_n \|S_n\|/a_n \leq 4) = 1$ it suffices to show that $S_n/a_n \to 0$ in probability where $a_n \equiv \sqrt{2\sigma_n^2 LL\sigma_n^2}$. By the martingale property we have

$$P\left(\frac{\|S_n\|}{\sigma_n} \geq \lambda\right) \leq \int_\Omega \frac{|S_n(T)|}{\sigma_n}\, dP/\lambda$$
$$\leq \left(\sum_{j=1}^n E|X_j(T)|^2\right)^{1/2} \Big/ \lambda\sigma_n$$
$$\leq \left(\sum_{j=1}^n E\|X_j\|^2\right)^{1/2} \Big/ \lambda\sigma_n = 1/\lambda.$$

Hence $\{S_n/\sigma_n \colon n \geq 1\}$ is bounded in probability in $(D[0, T], \mathscr{B})$, and therefore we have $S_n/a_n \to 0$ in probability so the proof is complete.

Remark. The previous remark can easily be applied to the empirical distribution function, and we refer the reader to Corollary 4.3 of [7] for details as well as further references.

In order to formulate a Kolmogorov type LIL for B-valued random variable involving clustering and convergence to the unit ball of some Hilbert space inside B (as in [7] for i.i.d. sequences) we require some additional terminology.

We say the sequence of B-valued random variables $\{X_n\}$ *is uniformly approximable in second moment through finite-dimensional random variables* if for each $\delta > 0$ there exists a Borel function $\tau_\delta \colon B \to B$ such that τ_δ has finite-dimensional range and for some constant c_δ, $0 < c_\delta < \infty$, we have

$$\text{(3.11)} \qquad \begin{aligned} &\text{(i)} \quad \|\tau_\delta(x)\| \leq c_\delta(\|x\| + 1) \quad (x \in B),\\ &\text{(ii)} \quad E(\tau_\delta(X_j)) = 0 \quad (j \geq 1),\\ &\text{(iii)} \quad E\|X_j - \tau_\delta(X_j)\|^2 \leq \delta \quad (j \geq 1). \end{aligned}$$

Of course, if $X_1, X_2, \ldots$ are identically distributed with $EX_j = 0$, $E\|X_1\|^2 < \infty$, and common distribution μ, then $\{X_n\}$ is uniformly approximable in second moment through finite-dimensional random variables. To see this let $\mathscr{F}$ denote a finite sigma-algebra of Borel subsets of B such that

$$\text{(3.12)} \qquad \int_B \left\| \sum_{A_k \in I} \lambda_k 1_{A_k}(x) - x \right\|^2 d\mu(x) \leq \delta$$

where $\lambda_k = \int_{A_k} y\, d\mu(y)/\mu(A_k)$ if $\mu(A_k) \neq 0$ and zero otherwise, and I is the disjoint partition of B determined by $\mathscr{F}$. Let $\tau_\delta(x) = \sum_{A_k \in I} \lambda_k 1_{A_k}(x)$ for $x \in B$. Then $\tau_\delta(X_j) = E(X_j \mid \mathscr{F}_j)$ where $\mathscr{F}_j = X_j^{-1}(\mathscr{F})$ and (3.11i, ii, iii) are easily seen to be satisfied.

It might also be worthwhile to point out that if the X_j's are symmetric then (3.11ii) is always true if the function τ_δ is symmetric.

THEOREM 3.2. *Let $X_1, X_2, \ldots$ be independent B-valued random variables satisfying* (3.1i) *and such that $EX_j = 0$ $(j \geq 1)$. If $\{X_n\}$ is uniformly approximable in second moment through finite-dimensional random variables, $S_n/a_n \to 0$*

in probability, and $\lim_n E\|X_n\|^2 = 1$, *then*

$$P(\{S_n/a_n: n \geq 1\} \text{ conditionally compact in } B) = 1 \tag{3.13}$$

where, of course, $a_n = \sqrt{2\sigma_n^2 LL\sigma_n^2}$ *and* $\sigma_n^2 = \sum_{j=1}^n E\|X_j\|^2$. *Furthermore, if* X *is a* B*-valued random variable such that* $EX = 0$, $E\|X\|^2 < \infty$, *and*

$$\lim_n E(f(X_n)g(X_n)) = E(f(X)g(X)) \quad (f, g \in B^*), \tag{3.14}$$

then

$$P(\{S_n/a_n: n \geq 1\} \twoheadrightarrow K) = 1 \tag{3.15}$$

where K *is the unit ball of the Hilbert space* $H_{\mathscr{L}(X)}$.

Remark. If $\lim_n E\|X_n\|^2 = \Lambda$ $(0 < \Lambda < \infty)$ in the above theorem, then (3.13) still holds and (3.15) holds with K replaced by $\Lambda^{-1/2}K$.

Remark. If $X_1, X_2, \ldots$ are independent B-valued random variables such that $EX_j = 0$ $(j \geq 1)$, $\lim_n E\|X_n\|^2 = 1$, and satisfying (3.1i), then the proof of Theorem 3.2 will show that (3.13) and (3.14) imply (3.15). The point to be emphasized is that we use the condition "$\{X_n\}$ is uniformly approximable through finite-dimensional random variables" only to establish (3.13). Hence if (3.13) can be established in some other manner, then (3.14) will imply (3.15). For example, if (3.13) holds and $X \equiv 0$, i.e., $E(f^2(X)) = 0$ for all $f \in B^*$, then we have $K = \{0\}$ and $P(\overline{\lim}_n \|S_n\|/a_n = 0) = 1$.

Remark. If B is a type-2 space then $X_1, X_2, \ldots$ independent, $EX_j = 0$ $(j \geq 1)$, and $E\|X_j\|^2 < \infty$ $(j \geq 1)$ imply

$$E\|S_n\| \leq (E\|S_n\|^2)^{1/2} \leq \left(A \sum_{j=1}^n E\|X_1\|^2\right)^{1/2} = (A\sigma_n^2)^{1/2} = O(\sigma_n).$$

Hence $E\|S_n\| = o(a_n)$ whenever $\sigma_n \nearrow \infty$, and thus we always have $S_n/a_n \to 0$ in probability in this setting.

Proof. Let B_0 denote a countable dense subset of B. To establish (3.13) it suffices to show that for every $\varepsilon > 0$,

$$P(\{S_n/a_n: n \geq 1\} \text{ is covered by finitely many } \varepsilon\text{-balls centered at points in } B_0) = 1. \tag{3.16}$$

For any $\delta > 0$ let τ_δ be a Borel function from B into B with finite dimensional range such that (3.11) holds and define $S_n' = \sum_{j=1}^n \tau_\delta(X_j)$ $(n \geq 1)$. Then the range of $\{S_n'\}$ is a fixed finite-dimensional subspace of B and since bounded subsets of a finite-dimensional Banach space are conditionally compact we have (3.16) if there exists a $\delta > 0$ such that

$$P\left(\overline{\lim_n} \frac{\|S_n'\|}{a_n} < \infty\right) = 1 \tag{3.17}$$

and

$$P\left(\overline{\lim_n} \frac{\|S_n - S_n'\|}{a_n} > \varepsilon\right) = 0. \tag{3.18}$$

First of all observe that since $\lim_n E\|X_n\|^2 = 1$ we have $\sigma_n^2 \sim n$ as $n \to \infty$, and that by the argument employed in Theorem 3.1, $S_n/a_n \to 0$ in probability implies $E\|S_n\| = o(a_n)$ in this setting.

To verify (3.17) for $0 < \delta < 1$ we apply Theorem 3.1 to the random variables $\{\tau_\delta(X_j): j \geq 1\}$. Since these random variables are independent it suffices to show

(3.19i) $\|\tau_\delta(X_n)\| \leq \Gamma_n'\sigma_n'/\sqrt{LL(\sigma_n')^2}$ where $\Gamma_n' \to 0$ and $(\sigma_n')^2 = \sum_{j=1}^n E\|\tau_\delta(X_j)\|^2$,

(3.19ii) $E\|S_n'\| = o(a_n')$ where $a_n' = \sqrt{2(\sigma_n')^2 LL(\sigma_n')^2}$,

(3.19iii) $\overline{\lim}_n a_n'/a_n < \infty$.

To verify (3.19) note that (3.11i) and $\sigma_n^2 \sim n$ imply

$$(\sigma_n')^2 \leq 2C_\delta^2(\sigma_n^2 + n) = O(\sigma_n^2). \tag{3.20}$$

Then we immediately have (3.19iii), and (3.19i) follows since

$$\begin{aligned} \|\tau_\delta(X_n)\| &\leq c_\delta(\|X_n\| + 1) \\ &\leq c_\delta\left(\frac{\Gamma_n\sigma_n}{\sqrt{LL\sigma_n^2}} + 1\right) \\ &\leq \Gamma_n' \frac{\sigma_n'}{\sqrt{LL(\sigma_n')^2}} \end{aligned} \tag{3.21}$$

where $\Gamma_n' \to 0$. The last inequality in (3.21) holds since $\sigma_n^2 \sim n$ and $0 < \delta < 1$ and (3.11iii) imply $\overline{\lim}_n n/(\sigma_n')^2 < \infty$.

Thus for (3.17) to hold it remains to verify (3.19ii) and this follows immediately since $ES_n' = 0$, $\{S_n': n \geq 1\}$ takes values in a fixed finite-dimensional subspace of B,

$$\overline{\lim_n} \sqrt{\frac{2nLLn}{2(\sigma_n')^2 LL(\sigma_n')^2}} < \infty \quad \text{and} \quad \lim_n \frac{E\|S_n'\|}{\sqrt{2nLLn}} = 0.$$

Hence (3.17) holds for $0 < \delta < 1$ and (3.16) (and hence (3.13)) will be established provided we can choose $\delta > 0$ sufficiently small so that (3.18) holds

To verify (3.18) let $Y_j = X_j - \tau_\delta(X_j)$ $(j \geq 1)$, and $T_n = \sum_{j=1}^n y_j$ $(n \geq 1)$. The Y_j's are independent and by (3.19) and (3.20) we have

$$\begin{aligned} &\text{(i)} \quad \|Y_n\| \leq \frac{\Gamma_n''\sigma_n}{\sqrt{LL\sigma_n^2}} \quad \text{where } \Gamma_n'' \to 0, \\ &\text{(ii)} \quad E\|T_n\| \leq E\|S_n\| + E\|S_n'\| = o(a_n). \end{aligned} \tag{3.22}$$

Let $n_k = 2^k$ $(k \geq 1)$ and note that

$$(3.23) \quad \max_{n_k \leq n \leq n_{k+1}} P(\|T_{n_{k+1}} - T_n\| > a_{n_k}\varepsilon/2) \leq (2/\varepsilon) \max_{n_k \leq n \leq n_{k+1}} \frac{E\|T_{n_{k+1}} - T_n\|}{a_{n_k}}.$$

Now the right-hand side of (3.23) tends to zero as $k \to \infty$ by using (3.22ii) and the fact that $a_n \sim \sqrt{2nLLn}$, and hence by (3.5) for all k sufficiently large we have

$$(3.24) \qquad P\left(\max_{n_k \leq n \leq n_{k+1}} \|T_n\| > a_{n_k}\varepsilon\right) \leq 2P(\|T_{n_{k+1}}\| \geq \varepsilon a_{n_k}/2).$$

Now $a_{n_k}/a_{n_{k+1}} \sim 1/2$ so (3.18) holds if

$$(3.25) \qquad \sum_k P(\|T_{n_{k+1}}\| \geq \varepsilon a_{n_{k+1}}/4) < \infty.$$

Now for $0 < \delta < 1$ and using (3.22i) we have from Lemma 2.1 that

$$(3.26) \quad \begin{aligned} P(\|T_n\| \geq \varepsilon a_n/4) &= P\left(\frac{\|T_n\|}{2\sqrt{n\delta}} \geq \frac{\varepsilon a_n}{8\sqrt{n\delta}}\right) \\ &\leq \exp\left\{-\varepsilon_n^2 + \frac{\varepsilon_n^2}{2}\sum_{j=1}^n \frac{E\|Y_j\|^2}{n\delta}\left(1 + \frac{\varepsilon_n c_n}{2}\right) + \frac{\varepsilon_n E\|T_n\|}{2\sqrt{n\delta}}\right\} \end{aligned}$$

provided $\varepsilon_n c_n \leq 1$, and $\varepsilon_n = \varepsilon a_n/8\sqrt{n\delta}$, $b_n = \sqrt{n\delta}$, and $c_n = \Gamma_n''\sigma_n/\sqrt{n\delta}\sqrt{LL\sigma_n^2}$. Now fix $\delta > 0$, $0 < \delta < 1$, such that $\varepsilon^2/64\delta > 4$. Now $\varepsilon_n c_n \to 0$ as $n \to \infty$ since $\sigma_n^2 \sim n$ and hence (3.26) holds for all n sufficiently large. Further, by (3.11iii) we have $\sum_{j=1}^n E\|Y_j\|^2 \leq n\delta$ so

$$(3.27) \quad P(\|T_n\| \geq \varepsilon a_n/2) \leq \exp\left\{-\varepsilon_n^2 + \frac{\varepsilon_n^2}{2}(1 + c_n\varepsilon_n) + \frac{\varepsilon_n E\|T_n\|}{2\sqrt{n\delta}}\right\}$$

for all n sufficiently large. Since $\sigma_n^2 \sim n$ and (3.22ii) holds we have

$$\frac{\varepsilon_n E\|T_n\|}{2\sqrt{n\delta}} = O\,\frac{(\varepsilon_n^2 E\|T_n\|)}{a_n} = o(\varepsilon_n^2),$$

and hence for all n sufficiently large,

$$(3.28) \qquad P(\|T_n\| \geq \varepsilon a_n/4) < \exp\{-\varepsilon_n^2/4\}.$$

Since $\sigma_n^2 \sim n$ and $\varepsilon^2/64\delta > 4$. We have from (3.28) that for all n sufficiently large,

$$(3.29) \quad P\{\|T_n\| > \varepsilon a_n/4\} \leq \exp\left\{\frac{-\varepsilon^2(2nLLn)}{64(n\delta)4}\right\} \leq \exp\{-2LLn\} = 1/(Ln)^2.$$

Using (3.29) we see (3.25) converges and hence (3.13) holds.

Now we turn to (3.15). To establish (3.15) we first show

$$(3.30) \qquad P(C(\{S_n/a_n : n \geq 1\}) \subseteq K) = 1$$

and since (3.13) holds (3.30) implies

$$P\left(\lim_n d(S_n/a_n, K) = 0\right) = 1. \tag{3.31}$$

Now (3.30) follows immediately from Theorem 3.1(I) of [7] (note that $\dim H_{\mathscr{L}(X)} < \infty$ is not required to apply part (I)). To see this assertion notice that for $g \in B^*$ we have $g(S_n) = \sum_{j=1}^n g(X_j)$, and applying Kolmogorov's LIL [6] to the random variables $\{g(X_j): j \geq 1\}$ we have from (3.14) and $\sigma_n^2 \sim n$ that

$$P\left(\overline{\lim_n}\, g(S_n/a_n) = [E(g^2(X))]^{1/2}\right) = 1. \tag{3.32}$$

Now by Lemma 2.1 of [7] we have

$$[E(g^2(X))]^{1/2} = \left[\int_B g^2(x)\, d\mu(x)\right]^{1/2} = \sup_{x \in K} g(x),$$

and hence Theorem 3.1(I) applies (with $Y_n = S_n$) to give (3.30) (and hence (3.31) since (3.13) holds).

Thus to show (3.15) holds it suffices to prove that

$$P(C(\{\Pi_N(S_n/a_n): n \geq 1\}) = \Pi_N K) = 1 \quad (N \geq 1) \tag{3.33}$$

where Π_N denotes the mapping defined in [7], equation (2.4). Note that if $\dim H_{\mathscr{L}(X)} < \infty$, then only finitely many Π_N are defined. Now (3.33) holds by using the argument in Lemma 2 of [2] or by adapting the proof of Proposition 2.7 in [11] to N dimensions.

Thus (3.15) holds and the proof is complete.

4. The LIL for i.i.d. sequences and some corollaries

Using Theorem 3.2 we can establish a LIL for i.i.d. sequences using a generalization of the approach due to Hartman and Wintner [4].

THEOREM 4.1. *Let $X, X_1, X_2, \ldots$ be* i.i.d. *B-valued random variables such that $EX = 0$, $E\|X\|^2 < \infty$, and assume $a_n = \sqrt{2nLLn}$ $(n \geq 1)$. If K denotes the unit ball of $H_{\mathscr{L}(X)}$ then the following are equivalent:*

$$\begin{aligned} &\text{(i)} \quad P(\{S_n/a_n: n \geq 1\} \twoheadrightarrow K) = 1, \\ &\text{(ii)} \quad E\|S_n\| = o(a_n), \textit{ and} \\ &\text{(iii)} \quad S_n/a_n \to 0 \textit{ in probability}. \end{aligned} \tag{4.1}$$

Before proving Theorem 4.1 we will state and prove some immediate corollaries. Corollary 4.1 includes G. Pisier's recent result [10] that if X is a B-valued random variable such that $EX = 0$, $E\|X\|^2 < \infty$, and X satisfies the central limit theorem in B, then X satisfies the LIL in B. Here, of course, by saying X

satisfies the CLT on B we mean that if $X_1, X_2, \ldots$ are independent copies of X then $S_n/\sqrt{n}$ converges in distribution to a mean-zero Gaussian measure on B. Corollary 4.2 is Pisier's LIL for type-2 Banach spaces [9], and Corollary 4.3 is a special case of Corollary 4.1 of [7].

COROLLARY 4.1 (Pisier). *Let $X, X_1, X_2, \ldots$ be* i.i.d. *B-valued random variables such that $EX = 0$ and $E\|X\|^2 < \infty$. If $a_n = \sqrt{2nLLn}$ $(n \geq 1)$ and $\{S_n/\sqrt{n}: n \geq 1\}$ is stochastically bounded, then $P(\{S_n/a_n: n \geq 1\} \twoheadrightarrow K) = 1$ where K denotes the unit ball of $H_{\mathscr{L}(X)}$.*

Proof. $\{S_n/\sqrt{n}: n \geq 1\}$ stochastically bounded means that for each $\varepsilon > 0$ there exists an M such that $\sup_n P(\|S_n/\sqrt{n}\| \geq M) \leq \varepsilon$. Hence this condition implies $S_n/a_n \to 0$ in probability and the result now follows immediately from Theorem 4.1.

COROLLARY 4.2 (Pisier). *If B is a type-2 Banach space and $X, X_1, X_2, \ldots$ are* i.i.d. *B-valued random variables such that $EX = 0$, $E\|X\|^2 < \infty$, then*

$$P(\{S_n/a_n: n \geq 1\} \twoheadrightarrow K) = 1$$

where K denotes the unit ball of $H_{\mathscr{L}(X)}$.

Proof. If B is a type-2 space, there exists a uniform constant $A < \infty$ such that

$$E\|S_n\| \leq (E\|S_n\|^2)^{1/2} \leq \left(A \sum_{j=1}^{n} E\|X_j\|^2\right)^{1/2} = (AnE\|X\|^2)^{1/2}.$$

Hence $E\|S_n\| = o(a_n)$ and Theorem 4.1 gives us the result.

COROLLARY 4.3. *If $X, X_1, X_2, \ldots$ are* i.i.d. *$C[0, 1]$-valued random variables such that $EX = 0$, $E\|X\|^2 < \infty$, and $\{X(t): 0 \leq t \leq 1\}$ is a martingale on $[0, 1]$, then $P(\{S_n/a_n: n \geq 1\} \twoheadrightarrow K) = 1$ where K denotes the unit ball of $H_{\mathscr{L}(X)}$.*

Proof. By the martingale property we have

$$P\left(\frac{\|S_n\|}{\sqrt{n}} > \lambda\right) = \frac{E|S_n(1)|}{\sqrt{n}\,\lambda} \leq (E(X^2(1)))^{1/2}/\lambda.$$

Hence $\{S_n/\sqrt{n}: n \geq 1\}$ is stochastically bounded and the result follows immediately from Corollary 4.1.

Proof of Theorem 4.1. First assume (4.1i) holds and fix $\varepsilon > 0$. Using [10, Théorème 3.1] we choose a B-valued random variable Y with finite range, $EY = 0$, and such that

$$N(X - Y) < \varepsilon. \tag{4.2}$$

Here, for any B-valued random variable U,

$$N(U) = E\left(\sup_n \frac{\|U_1 + \cdots + U_n\|}{a_n}\right) \tag{4.3}$$

where $U_1, U_2, \ldots$ are independent copies of U.

Let $Y_1, Y_2, \ldots$ be independent copies of Y on the same probability space as $X_1, X_2, \ldots$ and define $T_n = \sum_{j=1}^{n} Y_j$ $(n \geq 1)$. Then, by (4.2) and (4.3),

$$E\|S_n\| \leq E\|S_n - Y_n\| + E\|T_n\| \leq \varepsilon a_n + E\|T_n\|. \tag{4.4}$$

Since $Y_1, Y_2, \ldots$ are i.i.d., $EY_i = 0$, and each has finite range we have

$$E\|T_n\| = O(\sqrt{n}). \tag{4.5}$$

Combining (4.4) and (4.5) we thus have $\overline{\lim}_n E\|S_n\|/a_n \leq \varepsilon$. Now $\varepsilon > 0$ arbitrary implies (4.1ii).

By Chebyshev's inequality (4.1ii) immediately gives (4.1 iii), and hence it remains only to show (4.1iii) implies (4.1i).

Our proof that X satisfies the LIL applies Theorem 3.2 in a manner which is due to Hartman and Wintner [4].

Let $F(t) = P(\|X\| \leq t)$ $(-\infty < t < \infty)$. Then, $E\|X\|^2 < \infty$ implies $\int_0^\infty t\, dF(t) < \infty$ and $\int_0^\infty t^2\, dF(t) < \infty$, and hence we can choose a distribution function $\tau(x)$ such that τ increases only on $[0, \infty)$, $\tau(0) = 0$, $\lim_{x\to\infty} \tau(x) = 1$, and such that as $r \to \infty$,

$$\int_r^\infty t\, dF(t) = o\left(\int_r^\infty t\, d\tau(t)\right), \quad \int_r^\infty t^2\, dF(t) = o\left(\int_r^\infty t^2\, d\tau(t)\right) \tag{4.6}$$
$$\int_0^\infty t^2\, d\tau(t) < \infty.$$

Hence there exists a strictly positive decreasing function $\Phi \equiv \Phi(r)$, $0 < r < \infty$, such that $\lim_{r\to\infty} \Phi(r) = 0$ and

$$E(\|X_j\| 1_{\{\|X_j\| \geq r\}}(X_j)) = \int_{t \geq r} t\, dF(t) \leq \Phi(r) \int_{t \geq r} t\, d\tau(t) \quad (j \geq 1, r \geq 0). \tag{4.7}$$

We now can choose a strictly positive decreasing function $\varepsilon(r)$, $0 < r < \infty$, such that

(4.8)
(i) $\varepsilon(r) > \Phi(r^{1/q})$,
(ii) $\varepsilon(r) > (LLr)^{1/2}/r^{1/6}$,
(iii) $\lim_{r\to\infty} \varepsilon(r) = 0$,
(iv) $\lambda(r) \equiv (r/LLr)^{1/2}\varepsilon(r)$ is monotone increasing on $0 < r < \infty$.

Then (4.8ii, iv) imply $\lambda(r) > r^{1/3}$ $(0 < r < \infty)$.

By [1] it suffices to prove the theorem under the assumption that X is symmetric, so we do this. Next we truncate as follows:

$$X_j' = \begin{cases} X_j & \text{if } \|X_j\| \le \lambda(j) \equiv (j/LLj)^{1/2}\varepsilon(j)\ (j \ge 1) \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

The remainder of the proof now consists of two major steps. The first involves showing that

$$P\left(\overline{\lim_n}\left\|\sum_{j=1}^n \frac{(X_j - X_j')}{a_n}\right\| = 0\right) = 1, \tag{4.10}$$

and the second is the application of Theorem 3.2 to the truncated random variables $\{X_j': j \ge 1\}$.

In view of the truncation made and (4.6), (4.7), and (4.8) the proof of (4.10) follows exactly as in [4, pp. 174–176] and hence it only remains to show that Theorem 3.2 applies to the truncated variables $\{X_j': j \ge 1\}$.

First observe that $X_1', X_2', \ldots$ are independent symmetric, B-valued random variables and by (4.9) we have (3.1i) holding for the sequence $\{X_j': j \ge 1\}$. Furthermore, symmetry implies $EX_j' = 0$ $(j \ge 1)$ and since $\lim_n \lambda(n) = +\infty$ we have $\lim_n E\|X_n'\|^2 = E\|X\|^2$ and

$$\lim_n E(f(X_n')g(X_n')) = E(f(X)g(X)) \quad (f, g \in B^*).$$

Next we establish $\{X_n': n \ge 1\}$ is uniformly approximable through finite-dimensional random variables. To see this fix $\delta > 0$ and take $\mathscr{F}$ to be a finite sigma-algebra of Borel subsets of B such that $\{0\} \in \mathscr{F}$, (3.12) holds with μ equal to the common distribution of the X_n and I the disjoint partition of B determined by $\mathscr{F}$, and $\mathscr{F}$ is symmetric in the sense that $A \in \mathscr{F} \Rightarrow -A \in \mathscr{F}$. As in Section 3 we define $\mathscr{F}_j = X_j^{-1}(\mathscr{F})$ for $j \ge 1$, and setting $\tau_\delta(x) = \sum_{A_k \in I} \lambda_k 1_{A_k}(x)$ as in (3.12) we have $\tau_\delta(X_j) = E(X_j \mid \mathscr{F}_j)$. Further, since $\mathscr{F}$ and $\mu = \mathscr{L}(X_j)$ are both symmetric, we have $\tau_\delta(-x) = \tau_\delta(x)$. Thus X_j' symmetric implies

$$E(\tau_\delta(X_j')) = 0 \quad (j \ge 1). \tag{4.11}$$

Next we note that since $\{0\} \in \mathscr{F}$, we have

$$\begin{aligned} E\|\tau_\delta(X_j') - X_j'\|^2 &= E\left\|\sum_{A_k \in I} \lambda_k 1_{A_k}(X_j') - X_j 1_{\Lambda_j}(X_j)\right\|^2 \\ &= E\left\|\sum_{A_k \in I} \lambda_k 1_{A_k}(X_j) 1_{\Lambda_j}(X_j) - X_j 1_{\Lambda_j}(X_j)\right\|^2 \\ &\le E\left\|\sum_{A_k \in I} \lambda_k 1_{A_k}(X_j) - X_j\right\|^2 \\ &\le \delta \quad (j \ge 1) \end{aligned} \tag{4.12}$$

where $\Lambda_j = \{\|X_j\| \le \lambda(j)\}$. Hence $\{X_j': j \ge 1\}$ is uniformly approximable through finite-dimensional random variables as asserted.

Now let $S_n' = X_1' + \cdots + X_n'$ for $n \ge 1$. Hence $\sigma_n^2 \equiv \sum_{j=1}^n E\|X_j'\|^2 \sim$

$nE\|X\|^2$, and since $S_n/a_n \to 0$ in probability (4.10) immediately implies $S_n'/a_n \to 0$ in probability.

We now can apply Theorem 3.2. That is, (3.14) and the first remark following Theorem 3.2 imply that

$$P\left(\left\{\frac{S_n'}{\sqrt{(2nLLn)E\|X\|^2}} : n \geq 1\right\} \twoheadrightarrow K/(E\|X\|^2)^{1/2}\right) = 1 \tag{4.13}$$

where K denotes the unit ball of $H_{\mathscr{L}(X)}$: Thus

$$P(\{S_n'/a_n : n \geq 1\} \twoheadrightarrow K) = 1 \tag{4.14}$$

and combining (4.9) and (4.14) we obtain (4.1i). Thus the proof is complete.

In [10], G. Pisier produced an example of a bounded symmetric random variable X such that if $X_1, X_2, \ldots$ are independent copies of X, then

$$P\left(\sup_n \|S_n/a_n\| < \infty\right) = 1, \tag{4.15}$$

and yet

$$P(\{S_n/a_n\} \text{ conditionally compact}) = 0. \tag{4.16}$$

This then raises the question of characterizing those random variables X where (4.15) holds. In some sense we can do this, and our result is the next theorem.

THEOREM. 4.2 *Let $X, X_1, X_2, \ldots$ be* i.i.d. *B-valued random variables such that $EX = 0$ and $E\|X\|^2 < \infty$. Then the following are equivalent:*

(1) $P(\sup_n \|S_n\|/a_n < \infty) = P(\overline{\lim}_n \|S_n\|/a_n = \Lambda) = 1$ *for some* Λ, $0 \leq \Lambda < \infty$.
(2) $\sup_n (E\|S_n\|^2)^{1/2}/a_n < \infty$.
(3) *For all* $\varepsilon > 0$ *there exists an* M *such that* $\sup_n P(\|S_n\|/a_n > M) < \varepsilon$, i.e., $\{S_n/a_n\}$ *is stochastically bounded.*

Proof. That (1) implies (2) follows easily from an application of Corollary 3.4 of [3] in a way similar to that at the beginning of the proof of Theorem 3.1, using the fact that

$$E\left(\sup_{i \leq n} \|X_i\|^2\right) \leq \sum_{i=1}^{n} E\|X_i\|^2 = nE\|X\|^2 = o(a_n^2).$$

That (2) implies (3) is obvious so it remains to show that (3) implies (1).

Let $(Y_1, Y_2, \ldots)$ be an independent copy of $(X_1, X_2, \ldots)$ and assume both sequences are defined on the same probability space. Let $S_n = \sum_{j=1}^{n} X_j$,

$T_n = \sum_{j=1}^{n} Y_j$ $(n \geq 1)$, and define $\{X_n' : n \geq 1\}$ and $\{Y_n' : n \geq 1\}$ as follows:

$$(4.17)\quad X_n' = \begin{cases} X_n & \text{if } \|X_n\| \leq \lambda(n) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad Y_n' = \begin{cases} Y_n & \text{if } \|Y_n\| \leq \lambda(n) \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda(n)$ is as in (4.9).

Then, (3) implies $\{(S_n - T_n)/a_n\}$ is stochastically bounded, and if $S_n' = \sum_{j=1}^{n} X_j'$, $T_n' = \sum_{j=1}^{n} Y_j'$ we can prove as in [4, pp. 174–176] that

$$(4.18)\qquad \overline{\lim_n} \frac{\|S_n - S_n'\|}{a_n} = 0 \text{ a.s.} \quad \text{and} \quad \overline{\lim_n} \frac{\|T_n - T_n'\|}{a_n} = 0 \text{ a.s.}$$

Thus $\{(S_n - T_n)/a_n\}$ stochastically bounded and (4.18) imply that $(S_n' - T_n')/a_n$: $n \geq 1$ is stochastically bounded. Now arguing as in Theorem 3.1, we have

$$(4.19)\qquad \sup_n \frac{E\|S_n' - T_n'\|}{a_n} < \infty,$$

and since $ES_n' = ET_n'$ (4.19) easily implies

$$(4.20)\qquad \sup_n \frac{E\|S_n' - ES_n'\|}{a_n} < \infty.$$

Using the argument in [4, pp. 174–176] we have

$$(4.21)\qquad \lim_n ES_n'/a_n = 0,$$

and hence by combining (4.20) and (4.21) we get

$$(4.22)\qquad \sup_n E\|S_n'\|/a_n < \infty.$$

Now (4.22) implies (3.1ii) and since $\lim_n E\|X_n'\|^2 = E\|X\|^2$ we also have (3.1i). Thus (3.2) holds and gives (1), so the proof is complete.

REFERENCES

1. J. CRAWFORD, Ph.D. Thesis, University of Wisconsin, 1976.
2. H. FINKELSTEIN, *The law of the iterated logarithm for empirical distributions*, Ann. Math. Stat., vol. 42 (1971), pp. 607–615.
3. J. HOFFMANN-JØRGENSEN, *Sums of independent Banach space valued random variables*, Studia Math., vol. 52 (1974), pp. 159–186.
4. P. HARTMAN AND A. WINTNER, *On the law of the iterated logarithm*, Amer. J. Math., vol. 63 (1941), pp. 169–176.
5. N. C. JAIN, *The central limit theorem in a Banach space*, Proc. First Conference on Probability in Banach spaces, July 1975 (to appear in Springer Lecture Notes).
6. A. KOLMOGOROV, *Über das Gesetz des iterierten logarithmus*, Math. Ann., vol. 101 (1929), pp. 126–135.
7. J. KUELBS, *A strong convergence theorem for Banach space valued random variables*, Annals of Probability, vol. 4 (1976), pp. 744–771.

8. ———, *The law of the iterated logarithm and related strong convergence theorems for Banach space valued random variables*, Lectures given to Ecole d'été calcul des probabilitiés St. Flour, 1975 (to appear in Springer Lecture Notes).
9. G. Pisier, *Sur la loi du logarithme itéré dans les espaces de Banach*, Proc. First Conference on Probability in Banach Spaces, July 1975 (to appear in Springer Lecture Notes).
10. ———, *Le Théorème de la limite centrale et la loi du logarithme itéré dans les espace de Banach*, Séminaire Maurey-Schwartz 1975–1976, Exposé III, Ecole Polytechnique, Paris.
11. S. S. Sheu, *Some iterated logarithm results for sums of independent two-dimensional random variables*, Ann. of Prob., vol. 2 (1974), pp. 1139–1151.
12. V. Strassen, *An invariance principle for the law of the iterated logarithm*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, vol. 4 (1966), pp. 265–268.
13. V. V. Yurinskii, *Exponential bounds for large deviations*, Theor. Probability Appl., vol. 19 (1974), pp. 154–155.

University of Wisconsin
Madison, Wisconsin