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ON A PROBLEM OF STOLZENBERG IN POLYNOMIAL CONVEXITY

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1. Introduction

The following problem has been posed by G. Stolzenberg [3, p. 350, Problem 9]: Let $\mathscr{A}$ be a uniform algebra on the unit circle T which is generated by a finite number of functions of constant unit modulus. Show that $\sigma(\mathscr{A})\backslash T$ has the structure of a (possibly empty) one-dimensional analytic space, where $\sigma(\mathscr{A})$ is the spectrum of the Banach algebra $\mathscr{A}$. By using the finite set of n, say, unimodular generators to imbed T into the torus T^n, one can reformulate the problem in a more geometric setting as that of showing, for a Jordan curve Γ contained in T^n, that $\hat{\Gamma}\backslash\Gamma$ is a (possibly empty) one-dimensional analytic subset of $\mathbf{C}^n\backslash\Gamma$, where $^\wedge$ denotes the polynomially convex hull. Our first result includes a solution to this problem as a special case.

We will say that a compact subset Z of T^n is an AC set (a union of a set of Arcs with a polynomially Convex set) provided that there is a compact polynomially convex set $K \subseteq Z$ such that $Z\backslash K$ has the structure of an arc at each of its points. By the latter we mean that for each $p \in Z\backslash K$ there exists a homeomorphism of a neighborhood of p in $Z\backslash K$ with some open interval on the real axis.

THEOREM 1. *Let X be a compact subset of T^n which is contained in an AC set Z. Then $\hat{X}\backslash X$ is a (possibly empty) analytic subset of pure dimension one in $\mathbf{C}^n\backslash X$. Moreover, $\hat{X}\backslash X$ is algebraic, in the sense that there exists a global algebraic subvariety B of $\mathbf{C}^n$ such that $\hat{X}\backslash X = B \cap (\overline{U^n}\backslash T^n)$, and $(\hat{X}\backslash X)^- \cap T^n$ is a union of real analytic curves contained in X.*

Results of this type were first obtained by Wermer [10], Bishop [4], and Stolzenberg [8], under smoothness restrictions, for real curves. In Theorem 1, no smoothness is assumed, but rather there is the geometric hypothesis that X lies in the torus. It is interesting to note, however, that the boundary curves of the hull are shown, a posteriori, by the application of a reflection principle, to be in fact real analytic.

For X a Jordan curve, we can, for the theorem, take $Z = X$ with K empty. When X is a Jordan arc, take $Z = X$ with K the set of two endpoints, to conclude that $\hat{X}\backslash X$ is either analytic or empty. From the argument principle (cf. [8], [11]), it follows that the latter must be the case; i.e., X is polynomially convex. Thus we recover the following result of Stolzenberg [7].

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COROLLARY 1. *Every Jordan arc Γ in T^n is polynomially convex and satisfies $P(\Gamma) = C(\Gamma)$.*

The last assertion, as Stolzenberg observes, follows from the polynomial convexity because it implies that each z_k is invertible in $P(\Gamma)$ (cf. Lemma 1b below).

Since every compact totally disconnected subset E of T^n is contained in a Jordan arc, Corollary 1 implies that such E is polynomially convex. From this we get:

COROLLARY 2. *If J is a finite union of (not necessarily disjoint) Jordan curves and Jordan arcs contained in T^n, then J is an AC set and so the conclusion of Theorem 1 applies to J.*

To see this, note that if $J = \bigcup \{J_k: 1 \le k \le s\}$ where each J_k is a Jordan curve or arc in T^n, then defining E_{ik} to be the relative boundary of $J_i \cap J_k$ in J_k for $i \neq k$ and E_{ii} to be the set of endpoints (if any) of J_i and putting $K = \bigcup E_{ik}$, a totally disconnected (cf. [2]) and hence polynomially convex set, we conclude, since $J \backslash K$ clearly has the structure of an arc at each point, that J is an AC set.

Theorem 1 and its corollaries suggest the following:

Problem. If $X \subseteq T^n$ and $0 \in \hat{X}$, show that there exists a subvariety V of $\mathbf{C}^n \backslash X$ such that $0 \in V \subseteq \hat{X}$.

A first step might be to show, say for $n = 2$, that a minimal set $X \subseteq T^2$ such that $0 \in \hat{X}$ (adduced from Zorn's lemma) has no interior in T^2.

After giving the proof of Theorem 1 in Section 2, we shall return to the original problem of Stolzenberg in Section 3, where we derive a necessary and sufficient condition that the algebra $\mathscr{A}$ on the circle be a proper subalgebra of $C(T)$.

2. Proof of Theorem 1

We shall employ the theory of uniform algebras, convenient references being the books of Stout [9] and Wermer [11]. Rossi's local maximum modulus principle will be referred to as LMMP. The open unit disc will be denoted by U; and the unit circle by T. Thus U^n is the unit polydisc and T^n the torus in $\mathbf{C}^n$. For X a compact set in $\mathbf{C}^n$, the polynomially convex hull, denoted $\hat{X}$, is $\{z \in \mathbf{C}^n: |p(z)| \le \|p\|_X$ for all polynomials in $z = (z_1, \ldots, z_n)\}$. The maximal ideal space of $P(X)$, the uniform closure in $C(X)$ of the polynomials in z, is (identified with) $\hat{X}$.

LEMMA 1. (a) *If $X \subseteq T^n$ is not polynomially convex, then $\hat{X}$ meets the compact set $C = \{z \in \bar{U}^n: z_1 z_2 \cdots z_n = 0\}$.*

(b) *If $X \subseteq T^n$ is polynomially convex, then there exists a compact neighborhood Y of X in T^n such that Y is polynomially convex.*

Proof. (a) If $\hat{X} \cap C$ were empty, then each z_k would be invertible in $P(X)$. Consequently $\bar{z}_k = 1/z_k \in P(X)$ and the Stone-Weierstrass theorem implies $P(X) = C(X)$ and hence that X is polynomially convex.

(b) Let $Y_k = \{z \in T^n\colon \operatorname{dist}(z, X) \leq 1/k\}$. Then $Y_k \downarrow X$ implies $\hat{Y}_k \downarrow \hat{X} = X$. Thus $Y_k \cap C$ is empty for large enough k and then Y_k is polynomially convex by (a), Q.E.D.

At this point we state a useful consequence of Theorem 1.

COROLLARY 3. *Let $X \subseteq T^n$ be such that there is a compact polynomially convex set $K \subseteq X$ with $X \backslash K$ totally disconnected. Then X is polynomially convex.*

Proof. By Lemma 1(b), we can choose a polynomially convex set $L \subseteq T^n$ such that $L^0 \supseteq K$, where the interior is taken in T^n. Then $E \equiv X \backslash L^0$ is a compact totally disconnected set and therefore there exists a Jordan arc α such that $E \subseteq \alpha \subseteq T^n$. Set $Z = L \cup \alpha$. Then Z is clearly an AC set containing X and so by Theorem 1, $V = \hat{X} \backslash X$ is an analytic variety and $\partial V \equiv \bar{V} \backslash V$ is either empty or a union of real analytic curves with $\partial V \subseteq X$ but $\partial V \nsubseteq K$, as K is polynomially convex. Since $X \backslash K$ is totally disconnected, the later possibility gives a contradiction, unless ∂V is empty and, consequently, so is V; i.e., $\hat{X} = X$, Q.E.D.

In the next lemma, we identify $\mathbf{C}^{n-1}$ with the hyperplane in $\mathbf{C}^n$ obtained by holding one of the coordinate projections fixed.

LEMMA 2. *If F is polynomially convex in $\mathbf{C}^n$, then $L = F \cap \mathbf{C}^{n-1}$ is polynomially convex in $\mathbf{C}^{n-1}$.*

Proof. Say $\mathbf{C}^{n-1} \subseteq \mathbf{C}^n$ is (identified with) $\{z \in \mathbf{C}^n\colon z_n = \alpha\}$. Take

$$p = (p_1, p_2, \ldots, p_{n-1}) \in \mathbf{C}^{n-1} \quad \text{with } p \notin L.$$

Then $q = (p_1, p_2, \ldots, p_{n-1}, \alpha) \notin F$. Hence there is a polynomial $f(z_1, z_2, \ldots, z_n)$ such that $|f(q)| > |f|_F$. Then, putting $g(z_1, z_2, \ldots, z_{n-1}) = f(z_1, \ldots, z_{n-1}, \alpha)$, we have $|g(p)| = |f(q)| > |f|_F \geq |g|_L$, Q.E.D.

We now begin the proof of Theorem 1 by induction on n, the case $n = 1$ being obvious. Recall, in what will be our fixed notation, that X is contained in the AC set Z with a polynomially convex set $K \subseteq Z$ such that $Z \backslash K = \gamma$ has the structure of an arc at each of its points. Let $\gamma_0 = \{p \in X\colon$ there exists an open subarc ω of γ such that (i) $p \in \omega \subseteq X$ and (ii) some coordinate function z_k is constant on $\omega\}$. Note that γ_0 is a relatively open subset of X and define the compact set $X_1 \equiv X \backslash \gamma_0$.

LEMMA 3. (a) $\hat{X} \cap U^n \subseteq \hat{X}_1$.

(b) *$z_k^{-1}\{\alpha\} \cap (X_1 \backslash K)$ is totally disconnected for $\alpha \in T$ and for each k.*

(c) $\hat{X}_1 \cap \partial(U^n) = X_1$.

Proof. (a) Let $p \in \hat{X} \cap U^n$ and let σ be a (positive) representing measure on X for p for the algebra $P(X)$. Then, for each k, $z_{k*}(\sigma)$ represents $z_k(p) \in U$ for the disc algebra. Since $z_k{}^*(\sigma)$ lives on T, it is harmonic measure and, in particular, it is absolutely continuous with respect to Lebesgue measure on T.

We claim that σ is supported on X_1 (this implies $p \in X_1$, as desired). To see this, let $q \in X \backslash X_1$. Then $q \in \gamma_0$ and so there is an $\omega \subseteq X$ such that $z_k | \omega \equiv \alpha \in T$ for some k. As indicated above, $z_{k*}(\sigma)\{\alpha\} = 0$ and hence $\sigma(z_k^{-1}\{\alpha\} \cap X) = 0$. Thus $\sigma(\omega) = 0$ and so $q \notin \operatorname{spt} \sigma$.

(b) By construction, $z_k^{-1}\{\alpha\} \cap \gamma \cap X_1$ contains no subarc; i.e., this set is totally disconnected.

(c) As each point of T^n is a peak point for $P(T^n)$, it follows that $\hat{X}_1 \cap T^n = X_1$. To show that $\hat{X}_1 \cap (\partial U^n \backslash T^n)$ is empty, we argue by contradiction and suppose that there is a point $p \in \hat{X}_1$ such that $z_k(p) = \alpha \in T$ for some k and $p \notin T^n$. Let $F = z_k^{-1}\{\alpha\} \cap X_1$. Since $p \in \hat{X}_1 \cap z_k^{-1}\{\alpha\}$, which is a peak set for $P(X_1)$ (with peaking function $f(z) = (1 + \bar{\alpha} z_k)/2$), it follows that $p \in \hat{F}$. On the other hand, $F \cap K$ is polynomially convex in $\mathbf{C}^{n-1}$ $(= z_k^{-1}\{\alpha\})$ by Lemma 2 and $F \backslash (F \cap K)$ is totally disconnected by part (b). Using the induction hypothesis, we can invoke Corollary 3 (in $\mathbf{C}^{n-1}$) to conclude that F is polynomially convex. This is a contradiction, Q.E.D.

Define $X_2 = X_1 \cap (\hat{X} \cap U^n)^-$. It follows from the LMMP and Lemma 3 that $\hat{X}_2 \backslash T^n = \hat{X} \cap U^n$.

LEMMA 4. *The mapping z_n: $X_2 \backslash K \to T$ is an open mapping.*

Proof. Fix $p = (p', \alpha) \in X_2 \backslash K$ where $p' \in T^{n-1}$ and $\alpha \in T$. We shall identify $\mathbf{C}^{n-1}$ with $\{z \in \mathbf{C}^n\colon z_n = \alpha\}$. Let $\mathcal{U}$ be a neighborhood of p in $\mathbf{C}^n$ which is disjoint from K. We shall show that $z_n(\mathcal{U} \cap X_2)$ contains a neighborhood of α in T. From Lemma 3(b) and (c),

$$z_n^{-1}\{\alpha\} \cap (\hat{X}_2 \backslash K) = z_n^{-1}\{\alpha\} \cap (X_2 \backslash K)$$

is a totally disconnected set; call it A and view it as a subset of $\mathbf{C}^{n-1}$. Take a neighborhood V of $p' \in A$ in $\mathbf{C}^{n-1}$ such that $\partial V \cap A = \emptyset$ and $\bar{V} \subseteq \mathcal{U} \cap \mathbf{C}^{n-1}$ with $\bar{V}$ compact. Hence $\partial V \cap \hat{X}_2 = \emptyset$. Therefore, there is a $\delta > 0$ such that

(i) $\partial V \times \{\lambda \in \mathbf{C}\colon |\lambda - \alpha| < \delta\}$ is disjoint from $\hat{X}_2$ and
(ii) $W \equiv V \times \{\lambda \in \mathbf{C}\colon |\lambda - \alpha| < \delta\}$ is a relatively compact subset of $\mathcal{U}$.

Let $Y = \hat{X}_2 \cap \bar{W}$ and observe that the topological boundary of Y in X_2 is contained in $\{z \in X_2\colon |z_n - \alpha| = \delta\}$. Let $Q = z_n(Y)$ and note that Q is contained in the intersection S of $\{\lambda \in \mathbf{C}\colon |\lambda - \alpha| \leq \delta\}$ with $\bar{U}$; S is a closed planar set bounded by two circular arcs, one of which, Γ, is a neighborhood of α in T. By definitio of X_2, p is a limit of points in $\hat{X} \cap U^n$ and so there is $q \in Y$ such that $z_n(q)$ lies in the interior of S. Apply the LMMP to $P(Y)$ to see that the Shilov boundary of Y is contained in $z_n^{-1}(\partial S) \cap Y$. Let σ be a representing measure for $P(Y)$ supported on the Shilov boundary of Y. It follows that $z_{n*}(\sigma)$

has support contained in ∂S and represents $z_n(q)$ for $P(Q)$. We conclude that $z_{n*}(\sigma)$ is harmonic measure on ∂S and, in particular, its support is all of ∂S and so contains Γ. Therefore $z_n(Y \cap X_2) \supseteq \Gamma$ and so $z_n(\mathcal{U} \cap X_2) \supseteq z_n(\bar{W} \cap X_2) \supseteq \Gamma$, Q.E.D.

LEMMA 5. *The mapping z_n: $X_2 \backslash K \to T$ is locally one-to-one and $X_2 \backslash K$ is a relatively open subset of γ.*

Proof. To verify the first statement we argue by contradiction and suppose that there is some $p \in X_2 \backslash K$ such that $z_n | X_2$ is not locally 1-1 in any neighborhood of p. Fix a small neighborhood $\mathcal{U}$ of p in $\mathbf{C}^n$ such that (i) $Z \cap \mathcal{U}$ is an open Jordan arc $\gamma_1 \subseteq \gamma$ and (ii) z_n maps γ_1 into a subarc of T of arclength less than π. By our assumption, there are $q_1 \neq q_2$ in $\gamma_1 \cap X_2$ such that $z_n(q_1) = z_n(q_2) = \alpha$. By the construction of $X_1 \supseteq X_2$, $\gamma_1 \cap z_n^{-1}\{\alpha\} \cap X_2$ is totally disconnected and, since it contains at least the two points q_1 and q_2, there is a nonempty open subarc γ_2 of γ_1 such that z_n takes the endpoints of γ_2 to α and $z_n \neq \alpha$ on $\gamma_2 \cap X_2$. This implies that the map

$$z_n\colon \gamma_2 \cap X_2 \to T \backslash \{\alpha\}$$

is a proper mapping. As a consequence, the image is closed in $T\backslash\{\alpha\}$. By Lemma 4, the map is also open. Thus, by connectedness, $z_n(\gamma_2 \cap X_2)$ is either all of $T\backslash\{\alpha\}$ or is empty. Because of (ii) above, the former possibility is ruled out and we conclude that $\gamma_2 \cap X_2$ is empty.

Let the endpoints of γ_2 be p_1 and p_2. Choose any $q \in \gamma_1 \cap X_2$ such that $z_n(q) \neq \alpha$ $(= z_n(p_i))$. Then for one of the points p_1 and p_2, say p_1, the open subarc γ_3 of γ_1 joining q to p_i $(i = 1, 2)$ is disjoint from γ_2. By the last paragraph, $(X_2 \cap \gamma_3) \cup \{p_1\}$ is a neighborhood of p_1 in X_2 and therefore its image under z_n is a neighborhood of α in T. Let $\beta = z_n(q)$ $(\neq \alpha)$ and put $E = X_2 \cap \gamma_3 \backslash z_n^{-1}(\beta)$. Consider the map z_n: $E \cup \{p_1\} \to T\backslash\{\beta\}$ and note that $E \cup \{p_1\}$ contains, at least, the point p_1. By the argument of the last paragraph, since this map is proper, we conclude, this time, that $E \cup \{p_1\}$ is empty. This is a contradiction.

Now we know that z_n: $X_2 \cap \gamma \to T$ is an open, continuous map which is locally 1-1. It follows that this map is a local homeomorphism. Therefore $X_2 \cap \gamma$ has the structure of an arc at each of its points. It follows that $X_2 \cap \gamma$ is relatively open in γ, Q.E.D.

LEMMA 6. *For each point $p \in X_2 \backslash K$ there is a neighborhood W of p in $\mathbf{C}^n$ such that $W \cap \hat{X}_2 \cap U^n$ is a pure one-dimensional analytic subvariety of $W\backslash T^n$.*

Proof. We shall employ the notation and information contained in the proof of Lemma 4. By Lemma 5, we first choose a neighborhood $\mathcal{U}$ of p such that z_n is 1-1 on $\mathcal{U} \cap X_2 \subseteq \gamma$. Repeating the construction of Lemma 4, we get W, Y, Q, S and Γ. By an argument of Bishop [4, p. 496], the topological boundary of Q is contained in the Shilov boundary of $P(Q)$ which is ∂S. This implies that $Q = S$. Also, because z_n is a homeomorphism of $Y \cap z_n^{-1}(T)$ to $\Gamma \subseteq T$, by a further argument of Bishop [4, p. 497], $W \cap \hat{X}_2 = Y \cap z_n^{-1}(S^0)$ is an analytic subvariety of $z_n^{-1}(S^0)$ of dimension one, Q.E.D.

LEMMA 7. *The set $X_2 \backslash K$ is locally a real analytic arc.*

Proof. For $p \in X_2 \backslash K$ choose a neighborhood W given by Lemma 6 with $W = \rho(W)$ where $\rho(z_1, z_2, \ldots, z_n) = (1/\bar{z}_1, 1/\bar{z}_2, \ldots, 1/\bar{z}_n)$. Let $V = W \cap \hat{X}_2 \cap U^n$ and observe that $V_1 \equiv V \cup \rho(V)$ is a subvariety of pure dimension one of $W \backslash T^n$ satisfying $\rho(V_1) = V_1$. By the extension result of [1], $W \cap \bar{V}_1$ is an analytic subvariety V_2 of W. Hence $X_2 \cap W = T^n \cap V_2$ is a real analytic arc, Q.E.D.

Remark 1. The result of [1] actually extends varieties through $\mathbf{R}^n$. To apply it in Lemma 7, use, locally, the map

$$(z_1, z_2, \ldots, z_n) \to (e^{iz_1}, e^{iz_2}, \ldots, e^{iz_n})$$

which takes $\mathbf{R}^n$ to T^n. The global version of this was obtained by Shiffman [5], [6] who applied it to prove the following result of Tornehave. If A is a subvariety of U^n of pure dimension one, such that $\bar{A} \backslash A \subseteq T^n$, then there is an algebraic subvariety B in $\mathbf{C}^n$ of pure dimension one with $A = B \cap U^n$. We shall use this below.

Remark 2. An alternate proof of Lemma 7 can be obtained from the Schwarz reflection principle. In the notation of Lemma 6, Bishop's argument shows that z_n: $W \cap X_2 \to S^0$ is a homeomorphism and

$$f_k = z_k \circ (z_n | W \cap \hat{X}_2)^{-1}$$

is analytic on S^0 with $|f_k(\lambda)| \to 1$ as $\lambda \to \partial S \cap T$ for $1 \leq k \leq n$. Thus f_k reflects to be analytic across $\partial S \cap T$ and then

$$\partial S \cap T \ni \lambda \to (f_1(\lambda), \ldots, f_n(\lambda))$$

gives a real analytic parameterization of a neighborhood of p in X_2.

LEMMA 8. *Let V be an analytic subvariety of $\mathbf{C}^n \backslash T^n$ of pure dimension one which is contained in $\overline{U^n}$. Put $\partial V \equiv \bar{V} \backslash V (\subseteq T^n)$.*

(a) *The variety V has a finite number of irreducible analytic components.*

(b) *Each analytic component lies either in U^n or in a polydisc U^k contained in ∂U^n.*

(c) *If $\omega \subseteq T^n$ is an open Jordan arc such that $\omega \cap \partial V$ is nonempty and relatively open in ∂V, then ω is a real analytic subarc of ∂V.*

Proof. (a) Apply Lemma 1(a) to conclude that each component of V meets the compact set C defined there.

(b) Let W be a component of V. For each coordinate function z_j, either $|z_j| < 1$ on W or $|z_j|$ attains the value 1 on W and so, by the maximum principle, $z_j | W$ is a constant of modulus one. The first alternative must occur for at least one j; say for $j_1, j_2, \ldots, j_k$. Then

$$V \subseteq U^k \equiv \prod \{|z_j| < 1: j = j_1, j_2, \ldots, j_n\}$$

with a certain abuse of notation.

(c) We claim that $\omega \cap \partial V$ is relatively open in ω. In fact, if $x \in \omega \cap \partial V$, since $\omega \cap \partial V$ is relatively open in ∂V which is a real analytic curve by the reflection argument of Lemma 7, it follows that the germs at x of the sets ω and ∂V are identical. Consequently, $\omega \cap \partial V$ is relatively open in ω. As $\omega \cap \partial V$ is relatively closed in ω, it follows by connectedness that $\omega \subseteq \partial V$. Thus ω is an open subarc of ∂V and hence is real analytic, Q.E.D.

LEMMA 9. *Let $q \in (\hat{X} \backslash X) \cap \partial U^n$. Then there exists an analytic subvariety V of $\mathbf{C}^n \backslash T^n$ of pure dimension one such that $q \in V \subseteq \hat{X}$ and $\partial V \equiv \bar{V} \backslash V$ is a real analytic curve contained in X.*

Proof. One of the coordinates of q, say the nth, has unit modulus; i.e., $z_n(q) = \alpha \in T$. Then $q \in \hat{L}$ where $L = X \cap z_n^{-1}\{\alpha\}$ (because $\hat{X} \cap z_n^{-1}\{\alpha\}$ is a peak set; cf. the proof of Lemma 3(c)). Let σ be the relative interior of $L \cap \gamma$ in γ. Let $K_1 = K \cap z_n^{-1}\{\alpha\}$, a polynomially convex set by Lemma 2. Then $L \backslash \sigma$ is the union of K_1 and a totally disconnected set and so is polynomially convex by Corollary 3 and induction. This shows that L is an AC set in $\mathbf{C}^{n-1}$ $(= z_n^{-1}\{\alpha\})$. By induction, $\hat{L} \backslash L$ is an analytic subvariety of $\mathbf{C}^n \backslash L$. We can take V to be this set. The reflection argument of Lemma 7 implies that ∂V is a real analytic curve, Q.E.D.

Now put $X_3 = (\hat{X} \backslash X)^- \cap X$. Then $X_3 \supseteq X_2$ and X_3 locally has the structure of a real analytic arc at points of $X_2 \backslash K$ from Lemma 7. Also $\hat{X}_3 \supseteq \hat{X} \backslash X$ by the LMMP.

LEMMA 10. *The set X_3 locally has the structure of a real analytic arc at points of $X_3 \backslash K$.*

Proof. By what we have just observed, we need to verify this only for points $p \in X_3 \backslash (X_2 \cup K)$. Such a point p is in the closure of $(\hat{X} \backslash X) \cap \partial U^n$. Let f be a polynomial which peaks on $\bar{U}^n$ at p. Hence there is an open Jordan subarc ω of γ containing p and $0 < b < 1$ such that $|f| < b$ on $X \backslash \omega$. Choose $q \in (\hat{X} \backslash X) \cap \partial U^n$ so close to p that $|f(q)| > b$. Let V be the variety produced in Lemma 9 for this point q. Since $|f(q)| \leq |f|_{\partial V}$ and $|f|_{X \backslash \omega} < |f(q)|$, it follows that $\partial V \cap \omega$ is nonempty. Also since $\partial V \subseteq X$ and ω is relatively open in X, $\omega \cap \partial V$ is relatively open in ∂V. By Lemma 8(c), ω is real analytic, Q.E.D.

Proof of Theorem 1. We know that $\hat{X} \backslash X = \hat{X}_3 \backslash X_3$ and that $X_3 \backslash K$ is locally a real analytic arc. By a basic theorem of Stolzenberg [8], $\hat{X}_3 \backslash X_3 \equiv A$ is an analytic subvariety of $\mathbf{C}^n \backslash X_3$ of pure dimension one. By Lemma 8, A ($\subseteq \overline{U^n}$) has a finite number of components, each contained in some U^k. Applying the Tornehave-Shiffman result (see Remark 1 after Lemma 7) to each analytic component, we conclude that A extends to be a global algebraic subvariety B in $\mathbf{C}^n$. This implies that $\bar{A} \cap T^n = B \cap T^n$ is a set of real analytic curves, Q.E.D.

3. Stolzenberg's problem

We shall now return to the original problem of Stolzenberg. Let $\phi_1, \phi_2, \ldots, \phi_n$ be continuous, unimodular, nonconstant functions on T which separate the points. Let $\Gamma = \{(\phi_1(z), \phi_2(z), \ldots, \phi_n(z)): z \in T\} \subseteq T^n$. We know that either Γ is polynomially convex or $\hat{\Gamma}$ is a nonempty algebraic variety. Assuming that the latter is the case, we shall derive some necessary conditions on the ϕ_i.

Because the ϕ_i are nonconstant, it follows from our previous work that the coordinate projections restricted to Γ are locally one-to-one maps. This implies that each ϕ_i is a covering projection of the circle to itself with winding number $m_i \neq 0$.

LEMMA 11. *The m_i are all of the same sign.*

Proof. Suppose not; say, without loss of generality, that $m_1 > 0$ and $m_2 < 0$. It follows that $f \equiv z_1^{-m_2} z_2^{m_1}$ has a continuous logarithm on Γ. By the argument principle, f has no zeros on $\hat{\Gamma}$ and, in particular, z_1 is invertible in $P(\hat{\Gamma})$. Therefore $|z_1|_{\hat{\Gamma}} \leq |z_1|_{\Gamma} = 1$ and $|z_1^{-1}|_{\hat{\Gamma}} \leq |z_1^{-1}|_{\Gamma} = 1$. Consequently, z_1 is of constant modulus one on $\hat{\Gamma}$ and, as $\hat{\Gamma} \backslash \Gamma$ is analytic, z_1 is constant on $\hat{\Gamma}$ and so on Γ—a contradiction, Q.E.D.

Because ϕ_1 is a covering projection, we can make a change of variable on T so that $\phi_1(\zeta)$ becomes ζ^{m_1}. More precisely, there is a homeomorphism $h: T \to T$ so that $\phi_1(h(\zeta)) \equiv \zeta^{m_1}$ for $\zeta \in T$. Now we replace each of the functions ϕ_i by $\phi_i \circ h$ and refer to this change of variable as "normalizing in the first variable". Define $\Phi: T \to \mathbf{C}^n$ by

$$\Phi(\zeta) = (\phi_1(\zeta) \equiv \zeta^{m_1}, \phi_2(\zeta), \ldots, \phi_n(\zeta)),$$

where, here and below, normalization in the first variable is in force.

For any polynomial f in $\mathbf{C}^n$, $\lambda \in T$ and an indeterminant X define

$$Q_1(f, \lambda, X) = \prod_{k=1}^{m_1} [X - f(\Phi(\lambda_k))],$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{m_1}$ are the m_1 values of λ^{1/m_1}. The subscript 1 of Q refers to the fact that we have normalized in the *first* variable. Then $Q_1(f, \lambda, X)$ is, for f fixed, a polynomial in X with coefficients in $C(T)$ whose roots, for a given $\lambda \in T$, are the values of f on $(z_1 | \Gamma)^{-1}(\lambda)$.

LEMMA 12. *The coefficients of $Q_1(f, \lambda, X)$ are the restrictions to T of rational functions having no poles in U.*

Proof. We know that $\hat{\Gamma}$ is the intersection of $\bar{U}^n$ with an algebraic subvariety B of $\mathbf{C}^n$ and that the coordinate projection $z_1: B \to \mathbf{C}$ is m_1-to-1 over T and is a (branched) m_1-to-1 cover over U. Let

$$(z_1 \mid B)^{-1}(\lambda) = \{p_1, p_2, \ldots, p_{m_1}\} \subseteq \mathbf{C}^n \quad \text{for } \lambda \in U$$

and form

$$P_1(f, \lambda, X) = \prod_{k=1}^{m_1} [X - f(p_k)]$$

for any polynomial f and $\lambda \in U$. Then P_1 is a polynomial in the indeterminant X with analytic functions in U as coefficients. Moreover, as $B \cap \{z \in \mathbf{C}^n: |z_1| > 1\}$ and the coefficients of P_1 for $|\lambda| > 1$ are obtained by reflection in T^n and T respectively (cf. [5]), it follows that (i) $H \equiv \bar{B} \cap \bar{\mathbf{C}}^n$, the closure of B in $\bar{\mathbf{C}}^n$ (where $\bar{\mathbf{C}}$ is the Riemann sphere), is such that the coordinate projection $z_1: H \to \bar{\mathbf{C}}$ is a branched cover of order m_1 and (ii) $P_1(f, \lambda, X)$ has coefficients which are meromorphic on $\bar{\mathbf{C}}$; i.e., P_1 has rational coefficients. For $|\lambda| = 1$, $P_1(f, \lambda, X)$ and $Q_1(f, \lambda, X)$ have the same roots and therefore these monic polynomials have the same coefficients. This shows that $Q_1(f, \lambda, X)$ has rational coefficients which are analytic on $\bar{U}$, Q.E.D.

We shall summarize this state of affairs by saying that "normalizing in the first variable leads to polynomials $Q_1(f, \lambda, X)$ with rational coefficients".

LEMMA 13. *Each $Q_1(f, \lambda, X)$ is a power of an irreducible polynomial in $\mathscr{R}[X]$ where $\mathscr{R}$ is the field of rational functions in λ.*

Proof. In fact, let $g(\lambda, X)$ and $h(\lambda, X)$ be any two irreducible factors of Q_1 and let $\tilde{g}$ and $\tilde{h}$ be the polynomials in two variables obtained from g and h, respectively, by rationalizing the denominators of the coefficients in λ.

Let $\Gamma_1(f) = \{(\phi_1(\zeta), f(\Phi(\zeta)): \zeta \in T\} \subseteq \mathbf{C}^2$. Because we have normalized in the first variable, $\phi_j (2 \le j \le n)$ is real analytic; indeed, we can write, locally, $\phi_j(\lambda) = z_j \circ (z_1 \mid \Gamma)^{-1} \circ \lambda^{m_1}$ which is the composition of real analytic homeomorphisms. Hence $\zeta \to (\phi_1(\zeta), f(\Phi(\zeta)))$ is real analytic on T. Therefore $\tilde{g}(\phi_1(\zeta), f(\Phi(\zeta)))$ and $\tilde{h}(\phi_1(\zeta), f(\Phi(\zeta)))$ are real analytic for $\zeta \in T$ and, as each vanishes on an open subset of T, it follows that each vanishes identically on T. Thus the irreducible polynomials $\tilde{g}$ and $\tilde{h}$ both vanish on the set $\Gamma_1(f)$ which is not a discrete set. Consequently, $\tilde{g}$ and $\tilde{h}$ are, up to a unit, identical; i.e., g and h agree up to a unit, Q.E.D.

Of course, we can also normalize with respect to the kth variable. Then, for a polynomial f in $\mathbf{C}^n$, we will obtain $Q_k(f, \lambda, X)$ with the same properties as $Q_1(f, \lambda, X)$. Thus we have verified the necessity of the conditions in the following theorem.

THEOREM 2. *Let $\phi_1, \phi_2, \ldots, \phi_n$ be nonconstant continuous unimodular functions which separate the points of T with $\mathscr{A}$ the uniform algebra on T which they generate. If $\mathscr{A}$ is a proper subalgebra of $C(T)$, then:*

(a) *The ϕ_i are covering projections of the circle.*

(b) *The winding numbers $\{m_i\}$ of the $\{\phi_i\}$ are all of the same sign.*

(c) *For each k, $1 \leq k \leq n$, and for any polynomial f in $\mathbf{C}^n$, normalization in the kth variable leads to polynomials $Q_k(f, \lambda, X)$ which have coefficients rational in λ, without poles in $\bar{U}$, and which are powers of an irreducible polynomial in $\mathscr{R}[X]$, where $\mathscr{R}$ is the field of rational functions in λ.*

Conversely, suppose that (a), (b), *and* (c) *hold. Then $\mathscr{A}$ is a proper subalgebra of $C(T)$.*

Remark. In order that $Q_k(f, \lambda, X)$ in $C(T)[X]$ be defined, it is necessary that normalization in the kth variable be possible; i.e., that (a) holds. The import of (c) is then that the coefficient functions of Q_k are restrictions to the circle of rational functions.

Proof of sufficiency. Write $Q_{kj}(\lambda, X)$ for $Q_k(z_j, \lambda, X)$, where z_j is the jth coordinate polynomial. Let P_{kj} be the polynomial in two variables obtained by rationalizing the λ denominators of Q_{kj}. We shall view $P_{kj}(z_k, z_j)$ as a polynomial in $\mathbf{C}^n$. Let $V_1 = \{z \in \mathbf{C}^n\colon\ P_{1j}(z) = 0,\ 2 \leq j \leq n\}$. Then V_1 is a one-dimensional algebraic subvariety of $\mathbf{C}^n$ containing Γ. Also

$$V_1 \cap \{z \in \mathbf{C}^n\colon\ |z_1| = 1\} \subseteq T^n$$

because, for $|z_1| = 1$, $P_{1j}(z_1, X)$ has m_1 roots, all of unit modulus.

Fix $\zeta_0 \in T$ such that the discriminant of the unique irreducible factor of each $Q_{1,j}(\zeta_0, X)$ is different from zero for $2 \leq j \leq n$. This means that the map $z_1\colon V_1 \to \mathbf{C}$ is unbranched over a neighborhood of ζ_0. Let

$$V_1 \cap z_1^{-1}\{\zeta_0\} = \{p_1, p_2, \ldots, p_N\} \subseteq T^n$$

with $p_1, p_2, \ldots, p_{m_1} \in \Gamma$ and $p_{m_1+1}, \ldots, p_N \notin \Gamma$. Choose a polynomial f which separates the N points p_j. Form $Q_1(f, \lambda, X)$ and rationalize the λ denominators to get $R_1(f, \lambda, X)$. Define a polynomial h in $\mathbf{C}^n$ by

$$h(z_1, \ldots, z_n) = R_1(f, z_1, f(z_1, \ldots, z_n)).$$

Define $W_1 = V_1 \cap \{z \in \mathbf{C}^n\colon h(z) = 0\}$. Then W_1 is an algebraic subvariety of $\mathbf{C}^n$ for which the map $z_1\colon W_1 \to \mathbf{C}$ is a proper branched cover over $\bar{U}$. As W_1 is m_1-to-1 over a neighborhood of ζ_0, it follows that W_1 is m_1-to-1 (with possible branching) over $\bar{U}$. Since $h \equiv 0$ on Γ, we conclude that $W_1 \cap \Gamma$ contains, and therefore is equal to, Γ. Also, as $z_1\colon \Gamma \to T$ is m_1-to-1, it follows that W_1 is an unbranched covering of some annulus about T; more precisely, there is a neighborhood $\mathscr{U}$ of T^n in $\mathbf{C}^n$ and $\delta > 0$ such that the map $z_1\colon W_1 \cap \mathscr{U} \to \{\lambda\colon 1 - \delta < |\lambda| < 1 + \delta\}$ is a covering projection of order m_1. Then $(\mathscr{U} \cap W_1)\backslash\Gamma$ is a union of two annular regions Ω_1 and Ω_2 with $z_1(\Omega_1) \subseteq U$ and $z_1(\Omega_2) \subseteq U^* \equiv \mathbf{C}\backslash\bar{U}$.

We know that z_j for $2 \leq j \leq n$ maps Γ into T in a locally one-to-one way. Therefore, if $\mathscr{U}$ is a sufficiently small neighborhood of T^n, $z_j(\Omega_s)$, for $s = 1$ or 2, is contained in either U or U^*. Because $z_j|\Gamma$ and $z_1|\Gamma$ have, by (b), winding numbers with the same sign, it follows, from the argument principle, that $z_j(\Omega_1) \subseteq U$ and also that $z_j(\Omega_2) \subseteq U^*$ for $2 \leq j \leq n$. From this we conclude that for some neighborhood $\mathscr{U}$ of T^n in $\mathbf{C}^n$, that $W_1 \cap \mathscr{U}$ is a disjoint union of three nonempty subsets: one in U^n, one in U^{*n}, and Γ in T^n.

In a completely analogous way, for $2 \leq k \leq n$, we obtain varieties $W_2, W_3, \ldots, W_n$ in $\mathbf{C}^n$, containing Γ, and such that (i)

$$W_k \cap \{z \in \mathbf{C}^n\colon |z_k| = 1\} = \Gamma$$

and (ii) near T^n, W_k splits into three nonempty subsets: one in U^n, one in U^{*n}, and Γ in T^n. Now consider the algebraic variety

$$B = W_1 \cap W_2 \cap \cdots \cap W_n$$

in $\mathbf{C}^n$. Observe that $\Gamma \subseteq B$. Furthermore, (i) implies that $B \cap \partial(U^n) = \Gamma$. By (ii), there is a neighborhood $\mathscr{U}$ of T^n such that $B \cap \mathscr{U} \subseteq U^n \cup \Gamma \cup (U^*)^n$. It follows that $B \cap U^n$ is nonempty, because otherwise $1/z_1$ would be analytic on $B \cap \mathscr{U}$ and would attain its maximum modulus at each point of Γ.

Thus $B \cap U^n \equiv A$ is a nonempty subvariety of $\mathbf{C}^n \backslash \Gamma$ with $\partial A \equiv \bar{A} \backslash A = \Gamma$. Hence $\bar{A} = \hat{\Gamma}$ is the maximal ideal space of $\mathscr{A}$, which is consequently a proper subalgebra, Q.E.D.

Remark 1. As the proof shows, a necessary condition that $\mathscr{A}$ be proper is that $\phi_1, \phi_2, \ldots, \phi_n$ can be made real analytic by a change of variable on the circle.

Remark 2. In many examples, the variety B is actually equal to one of the V_k (in the notation of the proof of Theorem 2). It would be interesting to verify whether B is always equal to $V_1 \cap V_2 \cdots \cap V_n$.

Remark 3. In the case $n = 2$, it is easy to see that

$$B = \{z \in \mathbf{C}^2\colon P_{12}(z) = 0\}.$$

Thus, for two generators, (c) can be replaced by the weaker statement that the single polynomial Q_{12} has rational coefficients with poles off $\bar{U}$.

Example. Consider $\phi_1(\lambda) \equiv \lambda$ and $\phi_2(\lambda) \equiv \bar{\lambda}$. Clearly (a) holds and also the $Q_k(f, \lambda, X)$ have rational coefficients, which may, however, have poles in U. Here Γ lies in an algebraic variety; namely, $\{(z_1, z_2)\colon z_1 z_2 = 1\}$. However, Γ does not bound a relatively compact subset of this variety.

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