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ON FORMAL SPACES AND THEIR LOOP SPACE

BY

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1. Introduction

In this paper we describe a method of fibering a simply connected CW-complex X over a certain product B of Eilenberg–MacLane-spaces $K(\mathbf{Q}, n_i)$. B is determined, essentially, by the rational Hurewicz morphism. The construction of this fibration uses the theory of minimal differential graded algebras, as outlined in Section 2. Our main result is Theorem 4.1 in Section 4: For a particular class of formal CW-complexes—including skeletons in products of Eilenberg–MacLane-spaces—we prove that the fiber F of our fibration has the rational homotopy type of a wedge of spheres. Since the projection of our fibration is surjective in rational homotopy it follows that the Poincaré series of the loop space ΩX for X in this class is rational, thus proving Serre's conjecture for this class of spaces.

In Sections 5 and 6 we construct $\mathscr{B}$-free minimal resolutions of certain algebras of type $\mathscr{B}/\mathscr{I}$, where $\mathscr{B}$ is a free graded-commutative algebra. We iterate our method of fibering, i.e., we fibre F over a product B_1 of Eilenberg–MacLane-spaces etc. It turns out that the minimal model of the P.L.-De Rham complex of X, X in our particular class, is the direct limit of the minimal models of the P.L.-De Rham complex of spaces constructed by successively twisting together the spaces $B, B_1, \ldots$ (Theorem 6.2). We also outline how actually to compute the twistings in the corresponding twisted tensor products.

2. Algebraic preliminaries

Let $\mathscr{A}$ be a *differential graded-commutative algebra* (DGA) over a field $\mathbf{k}$. In other words:

(1) $\mathscr{A} = \sum_{n \geq 0} \mathscr{A}^n$ is a graded vectorspace over $\mathbf{k}$ together with a derivation $d: \mathscr{A}^n \to \mathscr{A}^{n+1}$, $d \cdot d = 0$.

(2) If $a \in \mathscr{A}^n$, $a' \in \mathscr{A}^{n'}$, then $a \cdot a' = (-1)^{nn'} a' \cdot a \in \mathscr{A}^{n+n'}$.

All DGA's will be *connected* and *simply connected*, i.e., $\mathscr{A}^0 = \mathbf{k}$ and $\mathscr{A}^1 = 0$. The cohomology groups of $\mathscr{A}$ are denoted by $H^n(\mathscr{A})$.

Let $f: \mathscr{B} \to \mathscr{A}$ be a morphism of DGA's. There are defined *relative cohomology groups* $H^n(\mathscr{A}, \mathscr{B})$ by taking the cohomology of the relative cochain complex $\{C^n(\mathscr{A}, \mathscr{B}), d\}$, where $C^n(\mathscr{A}, \mathscr{B}) = \mathscr{A}^n \oplus \mathscr{B}^{n+1}$ with differential d given by

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$d(a, b) = (da + f(b), -db)$. There is a long exact sequence

$$(2.1) \quad \cdots \longrightarrow H^n(\mathscr{B}) \xrightarrow{f^*} H^n(\mathscr{A}) \xrightarrow{j} H^n(\mathscr{A}, \mathscr{B}) \xrightarrow{\delta} H^{n+1}(\mathscr{B}) \longrightarrow \cdots,$$

j and δ being defined by $j(a) = (a, 0)$ and $\delta(a, b) = b$.

A DGA $\mathscr{M}$ is called *minimal* if it is free as an algebra, i.e., the only relations in $\mathscr{M}$ are those imposed by associativity and graded-commutativity, and if the differential of every element in $\mathscr{M}$ is decomposable, i.e., $d\mathscr{M} \subseteq \mathscr{M}^+ \cdot \mathscr{M}^+$, where $\mathscr{M}^+ = \sum_{n\geq 1} \mathscr{M}^n$. If $V = \sum_{n\geq 1} V^n$ denotes the graded vectorspace spanned by the generators of $\mathscr{M}$ and if $P = P[\sum_{p\geq 1} V^{2p}]$ is the symmetric algebra generated by all even-degree elements and $E = E(\sum_{p\geq 1} V^{2p-1})$ the exterior algebra of all odd-degree elements, then $\mathscr{M}$ can be written as

$$(2.2) \qquad \mathscr{M} = S^*(V) = P \otimes E.$$

Let $\mathscr{M}$ be minimal and denote by $\mathscr{M}(n-1) \subseteq \mathscr{M}$ the subalgebra generated by all elements of degree $\leq n-1$. According to (2.1) there is a long exact sequence (with respect to the inclusion i: $\mathscr{M}(n-1) \to \mathscr{M}$)

$$(2.3) \quad \cdots \longrightarrow H^n(\mathscr{M}(n-1)) \xrightarrow{i^*} H^n(\mathscr{M}) \xrightarrow{j_n} H^n(\mathscr{M}, \mathscr{M}(n-1)) \xrightarrow{\delta} H^{n+1}(\mathscr{M}(n-1)) \longrightarrow \cdots.$$

$H^n(\mathscr{M}, \mathscr{M}(n-1))$ is isomorphic to the vectorspace $\mathscr{M}^n(n)/\mathscr{M}^n(n-1)$ spanned by all generators of $\mathscr{M}$ of dimension n [2].

With this identification in mind, let $\mathscr{B} \subseteq \mathscr{M}$ be the subalgebra generated by $\sum_{n\geq 0} j_n(H^n(\mathscr{M}))$. It follows that each element in $\mathscr{B}$ is closed and that the generators of $\mathscr{M}$ can be chosen in such a way that $\mathscr{B}$ is generated precisely by all closed generators. Let $\mathscr{F} \subseteq \mathscr{M}$ be the subalgebra generated by $\mathbf{k}$ and all nonclosed generators. $\mathscr{F}$ is isomorphic to $\mathbf{k} \otimes_{\mathscr{B}} \mathscr{M}$ and the differential d in $\mathscr{M}$ induces a differential d_0 in $\mathscr{F}$ such that $\mathscr{F}$ is a minimal DGA. It follows that $\mathscr{M}$ can be written as twisted tensor product (over $\mathbf{k}$) with base $\mathscr{B}$ and fibre $\mathscr{F}$:

$$(2.4) \qquad \mathscr{M} = \mathscr{B} \otimes_t \mathscr{F},$$

where the twisting t is given by $d = d_0 + t$. We shall call (2.4) the *natural decomposition* of $\mathscr{M}$. A geometric interpretation of (2.4) will be given in the next section.

Remark 2.1. Let f: $\mathscr{M} \to \mathscr{M}'$ be a morphism. In general, there is no morphism g, homotopic to f, which induces a morphism of the corresponding natural decompositions, so that, in general, $g(\mathscr{B}) \nsubseteq \mathscr{B}'$.

Let $\mathscr{A}$ be a DGA. Up to isomorphism, there exists a unique minimal DGA $\mathscr{M} = \mathscr{M}(\mathscr{A})$ and a morphism f: $\mathscr{M} \to \mathscr{A}$, unique up to homotopy, such that f^*: $H^*(\mathscr{M}) \to H^*(\mathscr{A})$ is an isomorphism. $\mathscr{M}(\mathscr{A})$ is called the *minimal model* of $\mathscr{A}$ [2].

Let $\mathscr{M}(\mathscr{A}) = \mathscr{B} \otimes \mathscr{F}$ be the natural decomposition of the minimal model of $\mathscr{A}$. f induces a morphism

$$(2.5) \qquad f^* | \mathscr{B}: H^*(\mathscr{B}) = \mathscr{B} \to H^*(\mathscr{A}).$$

In general, this morphism is not surjective. If $\mathscr{A}$ is *formal*, i.e., if there exists a morphism of DGA's $H^*(\mathscr{A}) \to \mathscr{A}$ inducing an isomorphism in cohomology, it follows that (2.5) is surjective and $\mathscr{M}(\mathscr{A}) \cong \mathscr{M}(\mathscr{B}/\mathscr{I})$, where $\mathscr{I} = \ker f^* \mid \mathscr{B}$ and $\mathscr{B}/\mathscr{I} \cong H^*(\mathscr{A})$.

Remark 2.2. The definition of formality raises the following question: Suppose $\mathscr{A}$ is formal and let $\mathscr{F}$ be the fibre in the natural decomposition of $\mathscr{M}(\mathscr{A})$. Is $\mathscr{F}$ formal too? I conjecture that the answer is affirmative. A special case will be discussed in Sections 4, 5, and 6.

3. Natural fibrations

Let $f\colon X \to B$ be a morphism of CW-complexes and let

$$P(B, X) = \{(\omega, x) \mid \omega \text{ a path in } B \text{ such that } \omega(1) = f(x),\ x \in X\}.$$

The inclusion $x \in X \mapsto (\omega_x, x) \in P(B, X)$ is a homotopy equivalence, ω_x being the constant path at $f(x)$. The map $\pi\colon (\omega, x) \in P(B, X) \mapsto \omega(0) \in B$ is the projection in the fibration

$$\begin{array}{ccc} F & \longrightarrow & P(B, X) \\ & & \downarrow{\scriptstyle \pi} \\ & & B \end{array} \tag{3.1}$$

where the fibre F is the total space of the induced fibration

$$\begin{array}{ccc} F & \longrightarrow & P(B) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & B \end{array}, \tag{3.2}$$

$P(B) \to B$ being the path fibration of B with fibre $\Omega(B)$.

Suppose X is simply connected and $H^*(X, \mathbf{Q})$ is finite dimensional in each degree. The P.L.-De Rham complex $\mathscr{A}(X)$ of X (with respect to a triangulation) is defined as follows [2], [7], [8]: Let σ be an n-simplex with barycentric coordinates $(t_0, \ldots, t_n)$. A rational p-form ω_σ on σ is given by

$$\omega_\sigma = \sum a_{i_1 \ldots i_p}\, dt_{i_1} \wedge \cdots \wedge dt_{i_p}, \quad dt_0 + \cdots + dt_n = 0,$$

the $a_{i_1 \ldots i_p}$'s being polynomials in $t_0, \ldots, t_n$ with $\mathbf{Q}$-coefficients. A rational p-form ω on X is a collection $\omega = \{\omega_\sigma\}$, σ ranging over all simplexes of the triangulation of X, such that the following compatibility condition holds: Let τ be a face of σ and $i\colon \tau \to \sigma$ the inclusion; then $i^*\omega_\sigma$ equals ω_τ as differential forms. Let $\mathscr{A}^p(X)$ be the $\mathbf{Q}$-vectorspace of all such p-forms and put $\mathscr{A}(X) = \bigoplus_{p \geq 0} \mathscr{A}^p(X)$. Exterior multiplication and differentiation turns $\mathscr{A}(X)$ into a DGA, and there is an algebra isomorphism $H^*(\mathscr{A}(X)) \longrightarrow H^*(X, \mathbf{Q})$ [2], [7], [8].

Let $\mathscr{M}$ be the minimal model of $\mathscr{A}(X)$. In the long exact sequence (2.3), $H^n(\mathscr{M}, \mathscr{M}(n-1))$ is isomorphic to the dual of the homotopy group $\pi_n(X)$ and

j_n is the dual of the Hurewicz morphism. Let $\mathcal{M} = \mathcal{B} \otimes \mathcal{F}$ be the natural decomposition of $\mathcal{M}(\mathcal{A})$ and let $B = \prod K(\mathbf{Q}, n_i)$ be a product of Eilenberg-MacLane spaces such that $H^*(B, \mathbf{Q}) \cong \mathcal{B}$. B is simply connected. Let $f\colon X \to B$ be a morphism inducing $f^* \mid \mathcal{B}$ (see (2.5)). The corresponding fibration (3.1) we shall call the *natural fibration* of X.

Morphism f induces the inclusion $\mathcal{M}(\mathcal{B}) = \mathcal{B} \to \mathcal{B} \otimes \mathcal{F}$ and therefore a morphism between the corresponding long exact sequences (see (2.3)). In particular, f maps $H^n(\mathcal{B}, \mathcal{B}(n-1))$ injectively into $H^n(\mathcal{M}, \mathcal{M}(n-1))$. It follows that the projection π in (3.1) is surjective in rational homotopy whence $\pi_n(X, \mathbf{Q}) = \pi_n(F, \mathbf{Q}) \oplus \pi_n(B, \mathbf{Q})$, and therefore

$$\pi_n(\Omega X, \mathbf{Q}) = \pi_n(\Omega F, \mathbf{Q}) \oplus \pi_n(\Omega B, \mathbf{Q}).$$

Let $p(\Omega X) = \sum_{n \geq 0} (\dim H_n(\Omega X, \mathbf{Q})) \cdot t^n$ be the (rational) Poincaré series of ΩX. By a theorem in [5], the Hurewicz morphism induces an isomorphism of Hopf algebras $U(\pi_*(\Omega X, \mathbf{Q})) \cong H_*(\Omega X, \mathbf{Q})$, where $U(\pi_*(\Omega X, \mathbf{Q}))$ denotes the universal enveloping algebra of the Lie algebra $\pi_*(\Omega X, \mathbf{Q})$. This leads to:

PROPOSITION 3.1. *Let $F \to P(B, X) \to B$ be the natural fibration of X. Then $p(\Omega X, \mathbf{Q}) = p(\Omega B, \mathbf{Q}) \cdot p(\Omega F, \mathbf{Q})$. In particular, if $p(\Omega F, \mathbf{Q})$ is a rational function, so is $p(\Omega X, \mathbf{Q})$.*

Remark 3.2. Let $g\colon X \to X'$ be a morphism and let B and B' be the base spaces of the natural fibrations of X and X', respectively, with corresponding maps $f\colon X \to B$ and $f'\colon X' \to B'$. In general, there exists no morphism $B \to B'$ making the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ B & B' \end{tikzcd}$$

homotopy commutative.

4. Formal spaces

We consider now a particular class of formal spaces (i.e., CW-complexes X such that $\mathcal{A}(X)$ is formal). Let $X = X^{(M)}$ be such that $H^*(X, \mathbf{Q}) \cong \mathcal{B}/\mathcal{I}$, where $\mathcal{I} = \mathcal{I}^{(M)}$ is the ideal generated by all elements of degree larger than M, and the isomorphism is induced by the map f in (3.2).

THEOREM 4.1. *The fibre F of the natural fibration of $X^{(M)}$ has the rational homotopy type of a wedge of spheres.*

Since, by [1], the Poincaré series of the loop space of a wedge of spheres is rational, we get the following corollary from Proposition 3.1.

COROLLARY 4.2. *$p(\Omega X^{(M)}, \mathbf{Q})$ is rational.*

Example 4.3. Let $B = BU(n)$ be the classifying space of the unitary group $U(n)$. B is rationally equivalent to $\prod_{k=1}^{n} K(\mathbf{Q}, 2k)$. Let $X = BU^{(2n)}$ be the $2n$-skeleton and let F be the fibre of the natural fibration. F has the rational homotopy type of a wedge of spheres in dimensions between $2n + 1$ and $n^2 + 2n$ and $H^*(F, \mathbf{Q}) \cong H^*(\mathfrak{A}_n)$, $\mathfrak{A}_n$ being the Lie algebra of formal vector fields in n variables, [3], [4]. More generally, Theorem 4.1 applies to M-skeletons $X^{(M)}$ in spaces B which are rationally equivalent to a product of Eilenberg–MacLane spaces $\prod K(\mathbf{Q}, n_i)$, $n_i \geq 2$.

Proof of Theorem 4.1. Let

$$F \longrightarrow E \xrightarrow{\pi} B$$

be a fibration, B as usual connected and simply connected. Denote by $\mathscr{A}(F)$, $\mathscr{A}(E)$ and $\mathscr{A}(B)$ the corresponding P.L.-De Rham complexes. We define a DGA $\bar{\mathscr{A}}(E)$ and a morphism $h\colon \bar{\mathscr{A}}(E) \to \mathscr{A}(E)$ as follows [9]. Let σ be a simplex in B and $\omega_\sigma \in \mathscr{A}^r(\sigma) \otimes \mathscr{A}^s(\pi^{-1}(\sigma))$. $\bar{\mathscr{A}}^{r,s}(E)$ is the $\mathbf{Q}$-vectorspace formed by all collections $\omega = \{\omega_\sigma\}$, σ ranging over all simplexes of B, such that the following is satisfied: If $i\colon \tau \to \sigma$ is a face and $j\colon \pi^{-1}(\tau) \to \pi^{-1}(\sigma)$ is the inclusion then $\omega_\tau = (i^* \otimes j^*)\omega_\sigma$. Let $\bar{\mathscr{A}}^p(E) = \oplus_{r+s=p} \bar{\mathscr{A}}^{r,s}(E)$ and $\bar{\mathscr{A}}(E) = \oplus_{p>0} \bar{\mathscr{A}}^p(E)$. Under exterior multiplication and derivation, $\bar{\mathscr{A}}(E)$ is a DGA. To define h, let $\omega \in \bar{\mathscr{A}}^{r,s}(E)$. Let τ be a simplex in E, $\sigma = \pi(\tau)$ and $\pi_\tau = \pi \mid \tau$. Let $i_\tau\colon \tau \to \pi^{-1}(\sigma)$ be the inclusion. If

$$\omega_\sigma = \sum \alpha_k \otimes \beta_k \in \mathscr{A}^r(\sigma) \otimes \mathscr{A}^s(\pi^{-1}(\sigma)),$$

then $(h\omega)_\tau = \sum \pi_\tau^* \alpha_k \wedge i_\tau^* \beta_k \in \mathscr{A}^{r+s}(\tau)$. The compatibility condition holds, hence $h\omega \in \mathscr{A}^{r+s}(E)$. h is a morphism inducing an isomorphism in cohomology. There is a canonical injection $\mathscr{A}(B) \to \bar{\mathscr{A}}(E)$ defining a (left) $\mathscr{A}(B)$-module structure on $\bar{\mathscr{A}}(E)$.

Now let

$$\begin{array}{ccc} E' & \xrightarrow{f'} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

be a fibre square. We define a morphism $g\colon \mathscr{A}(B') \otimes \bar{\mathscr{A}}(E) \to \bar{\mathscr{A}}(E')$ as follows: Let σ' be a simplex in B', $\sigma = f(\sigma')$, $f_\sigma\colon \sigma' \to \sigma$ and $f'_\sigma\colon \pi'^{-1}(\sigma') \to \pi^{-1}(\sigma)$ the corresponding restrictions of f and f'. Let $\omega' \in \mathscr{A}^q(B')$, $\omega \in \bar{\mathscr{A}}^{r,s}(E)$. If

$$\omega_\sigma = \sum \alpha_k \otimes \beta_k \in \mathscr{A}^r(\sigma) \otimes \mathscr{A}^s(\pi^{-1}(\sigma)),$$

then

$$(g(\omega' \otimes \omega))_{\sigma'} = \sum \omega'_{\sigma'} \wedge f_\sigma^* \alpha_k \otimes f_\sigma'^* \beta_k \in \mathscr{A}^{q+r}(\sigma') \otimes \mathscr{A}^s(\pi'^{-1}(\sigma')).$$

Morphism g induces a morphism $\mathscr{A}(B') \otimes_{\mathscr{A}(B)} \bar{\mathscr{A}}(E) \to \bar{\mathscr{A}}(E')$, the right $\mathscr{A}(B)$-module structure on $\mathscr{A}(B')$ being induced by f, and we have

$$\operatorname{Tor}_{\mathscr{A}(B)}(\mathscr{A}(B'), \bar{\mathscr{A}}(E)) \longrightarrow H^*(E').$$

(This is shown in [9] for the real case but the proof also works for the rational case.)

Consider now (3.2). Let $\mathscr{B}$ be generated by b_l, $l \in L$, and let $\sum^{-1} \mathscr{B}$, the (-1)-suspension, be generated by u_l, $l \in L$, with $\deg u_l = \deg b_l - 1$, i.e., $u_l = \sum^{-1} b_l$. $\sum^{-1} \mathscr{B}$ corresponds to the cohomology of ΩB. In $\mathscr{A} = \mathscr{B} \otimes \sum^{-1} \mathscr{B}$ define a derivation d by $du_l = b_l$, $db_l = 0$. Then $H^*(\mathscr{A}) \cong \mathbf{Q}$. Since $\mathscr{B}$ is the minimal model of $\mathscr{A}(B)$ and the latter injects into $\bar{\mathscr{A}}(E)$ it follows that there is a morphism $\mathscr{B} \otimes \sum^{-1} \mathscr{B} = \mathscr{A} \to \bar{\mathscr{A}}(E)$ commuting with the $\mathscr{B}$-action and inducing an isomorphism in cohomology. By general properties of Tor [6] it follows that

$$\operatorname{Tor}_{\mathscr{B}} (\mathscr{A}(B'), \mathscr{A}) \xrightarrow{\cong} H^*(F);$$

moreover, since the projection $\mathscr{A} \to \mathbf{Q}$ commutes with the $\mathscr{B}$-action, we get

$$\operatorname{Tor}_{\mathscr{B}} (\mathscr{A}(B'), \mathbf{Q}) \cong H^*(F).$$

By assumption, there is a morphism $\mathscr{B}/\mathscr{I} \to \mathscr{A}(B')$ inducing an isomorphism in cohomology. This morphism commutes with the action of $\mathscr{B}$, hence

$$\operatorname{Tor}_{\mathscr{B}} (\mathscr{B}/\mathscr{I}, \mathscr{A}) \xrightarrow{\cong} H^*(F). \tag{4.1}$$

Let $P(\mathscr{B}/\mathscr{I})$ be the bar-resolution of the (right) $\mathscr{B}$-module $\mathscr{B}/\mathscr{I}$. $P(\mathscr{B}/\mathscr{I})$ has the structure of a DGA [9] and the isomorphism (4.1) is induced by the morphism ϕ defined by

$$\begin{tikzcd} P(\mathscr{B}/\mathscr{I}) \otimes_{\mathscr{B}} \mathscr{A} \arrow[r, "\phi"] \arrow[d, "\varepsilon \otimes 1"] & \bar{\mathscr{A}}(F) \\ \mathscr{B}/\mathscr{I} \otimes_{\mathscr{B}} \mathscr{A} \cong \mathscr{B}/\mathscr{I} & \otimes \sum^{-1} \mathscr{B}. \arrow[u, "g"] \end{tikzcd}$$

Both complexes $P(\mathscr{B}/\mathscr{I}) \otimes_{\mathscr{B}} \mathscr{A}$ and $\mathscr{B}/\mathscr{I} \otimes \sum^{-1} \mathscr{B}$ compute $\operatorname{Tor}_{\mathscr{B}} (\mathscr{B}/\mathscr{I}, \mathbf{Q})$, the first one using resolutions of $\mathscr{B}/\mathscr{I}$ and $\mathbf{Q}$, the second one using a resolution of $\mathbf{Q}$ only. $\varepsilon \otimes 1$ establishes an isomorphism in cohomology, hence g induces an isomorphism in cohomology. The theorem is therefore proved if there exist cocycles in $\mathscr{A}$, forming a base of $H^*(\mathscr{A})$, such that on the cochain level the product of two of them each is zero.

The construction of this base will be done using a spectral sequence. First we are going to relabel the generators b_l of $\mathscr{B}$ by integers $l' \in L'$ such that, for a certain integer N, $\deg b_{l'_1} \cdot \cdots \cdot b_{l'_p} > M$ iff $l'_1 + \cdots + l'_p > N$. Note that if $\mathscr{B}$ has at most one generator in each dimension we could choose l' to be the degree of the corresponding generator and $N = M$. We will construct the new index set L' using the following lemma.

Lemma 4.4. *Let $V = \sum_{n \geq 2} V^n$ be a finite dimensional graded vector space, let $S^*(V)$ be defined as in (2.2) and let $\mathscr{I}^{(M)} \subseteq S^*(V)$ be the ideal generated by all elements of degree larger than M. Then there exists a graded vector space $V' = \sum_{n' \geq 2} V'^{n'}$, a linear isomorphism $\psi\colon V \to V'$ (not respecting the degrees) and an integer N such that:*

(1) $\dim V'^{n'} \leq 1$, *for all* n'.

(2) *The induced maps*

$$S^*(V)\to S^*(V'),\quad \mathscr{I}^{(M)}\to\mathscr{I}^{(N)},\quad S^*(V)/\mathscr{I}^{(M)}\to S^*(V')/\mathscr{I}^{(N)}$$

are algebra isomorphisms (not respecting degrees), mapping even- and odd-degree elements onto even- and odd-degree elements, respectively.

Proof. Let $a^1, \ldots, a^m$ span V^{n_0} and define

$$p_0 = 2[M/n_0]\cdot(m-1)+1,\quad q_0 = p_0\cdot n_0,\quad M_0 = p_0(M+1)-1.$$

Let $V_0 = \sum_{q\geq 2} V_0^q$ be the graded vector space defined as follows: $V_0^{q_0+2i}$ is spanned by b_{q_0+2i}, $i = 0, \ldots, m-1$, $V_0^q \cong V^n$ if $q = p_0\cdot n$, $n \neq n_0$, and $V_0^q = 0$ in the other cases. The elements in V_0^q have now the degree q. The map $\psi_0\colon V\to V_0$ is defined to be the identity on V^n, $n\neq n_0$, only changing the degrees, and

$$\psi_0\colon a^i\in V^{n_0}\mapsto b_{q_0+2i}\in V_0^{q_0+2i},\quad i = 0,\ldots,m-1.$$

It is not hard to verify that V_0, ψ_0 and M_0 satisfy requirement (2) above. Applying this process stepwise to each V^n leads to V', ψ and N.

We may therefore assume that the generators b_l, $l\in L$, of $\mathscr{B}$ already are labeled in such a way that

$$b_{l_1}\cdot\cdots\cdot b_{l_p}\in\mathscr{I}\quad\text{iff}\quad l_1+\cdots+l_p>N, \tag{4.1}$$

where N is the integer constructed in Lemma 4.4 and l is even iff deg b_l is even, for all $l\in L$. We denote by $J\subseteq L$ the subset of all odd integers.

Let $a_{\ell m} = b_{l_1}\cdot\cdots\cdot b_{l_p}\otimes u_{m_1}\cdot\cdots\cdot u_{m_q}$ such that $l_1\leq\cdots\leq l_p$, $m_1\leq\cdots\leq m_q$ and let $l = l_1+\cdots+l_p$. The ideals $\mathscr{A}_r = (a_{\ell;m}, l>r)\subseteq\mathscr{A}$ define a filtration of $\mathscr{A}$ and for the corresponding spectral sequence $\{E_r, d_r\}$ we have:

LEMMA 4.5. *The sets of elements*

$$A_r^1 = \{a_{\ell;m}\mid m_1<r,\ l+m_1>N,\ l_1\geq m_1,\ m_1\notin J\},$$

$$A_r^2 = \{a_{\ell;m}\mid m_1<r,\ l+m_1>N,\ l_1>m_1,\ m_1\in J\},$$

$$D_r = \{a_{\ell;m}\mid m_1\geq r,\ l_1\geq r\}$$

form a base of E_r.

Proof. Decompose D_r as follows:

$$D_r^1 = \{a_{\ell;m}\mid m_1 = r,\ l+m_1\leq N,\ l_1\geq r,\ r\notin J\},$$

$$D_r^2 = \{a_{\ell;m}\mid m_1 = r,\ l+m_1\leq N,\ l_1> r,\ r\in J\},$$

$$D_r^3 = \{a_{\ell;m}\mid m_1 = r,\ l+m_1> N,\ l_1\geq r,\ r\notin J\},$$

$$D_r^4 = \{a_{\ell;m}\mid m_1 = r,\ l+m_1> N,\ l_1> r,\ r\in J\},$$

$$D_r^5 = \{a_{\ell;m}\mid m_1 > r,\ l_1 = r,\ r\notin J\},$$

$$D_r^6 = \{a_{\ell;m}\mid m_1 \geq r,\ l_1 = r,\ r\in J\},$$

$$D_r^7 = \{a_{\ell;m}\mid m_1 > r,\ l_1 > r\}.$$

From the particular labeling of the b_l's it follows that $d_r(a_{\ell;m})$ is either zero or consists of exactly one nonzero element. d_r maps D_r^1 and D_r^2 isomorphically onto D_r^5 and D_r^6, respectively. The elements in A_r^1, A_r^2, D_r^3, D_r^4, and D_r^7 are closed under d_r and it follows that $A_{r+1}^1 = A_r^1 \cup D_r^3$, $A_{r+1}^2 = A_r^2 \cup D_r^4$, and $D_{r+1} = D_r^7$.

COROLLARY 4.6. *The set* $A = A^1 \cup A^2$ *where*

$$A^1 = \{a_{\ell;m} \mid l + m_1 > N,\ l_1 \geq m_1,\ m_1 \notin J\},$$

$$A^2 = \{a_{\ell;m} \mid l + m_1 > N,\ l_1 > m_1,\ m_1 \in J\}$$

forms a base for $H^*(\mathscr{A})$ *and* $a_{\ell;m} \cdot a_{\ell;m'} = 0$ *in* $\mathscr{A}$ *for all* $a_{\ell;m}, a_{\ell;m'} \in A$.

This proves Theorem 4.1.

Remark 4.7. We actually did compute a $\mathscr{B}$-free minimal resolution of $\mathscr{B}/\mathscr{I}^{(M)}$ (see Section 5).

Remark 4.8. If $\mathscr{B}$ has odd-dimensional generators, then $H^*(\mathscr{A})$ is infinite dimensional, although finite dimensional in each degree.

Remark 4.9. It is clear how Theorem 4.1 generalizes to other formal spaces, where there is a labeling of the generators of $\mathscr{B}$ such that (4.1) holds for a certain N. For instance, this is possible for $\mathscr{I} = (\mathscr{B}^+)^M$, i.e., $\mathscr{I}$ consists of all products of exactly M elements. On the other hand, not every ideal can be obtained via (4.1) even if the corresponding F has the rational homotopy type of a wedge of spheres.

5. Resolutions

Let $\{b_l,\ l \in L\}$ be a set of indeterminants having degrees ≥ 2 and let $\mathscr{S}$ be the *free, skew* algebra generated by this set, i.e., the underlying graded vector space of $\mathscr{S}$ is isomorphic to the underlying graded vector space of the polynomial algebra generated by the b_l's and the multiplication is given by $b_l \cdot b_{l'} = (-1)^{nn'} b_{l'} \cdot b_l$, where $l \neq l'$, $n = \deg b_l$, $n' = \deg b_{l'}$. Note that $(b_l)^2 \neq 0$ for all $l \in L$.

Let $\{\mathscr{z}_k,\ k \in K\} = \mathrm{Mon}_0\,(\mathscr{K})$ be a set of not necessarily independent monomials of $\mathscr{S}$ and let $\mathscr{K} \subseteq \mathscr{S}$ be the ideal generated by $\mathrm{Mon}_0\,(\mathscr{K})$. An $\mathscr{S}$-free resolution of $\mathscr{S}/\mathscr{K}$ is obtained as follows:

Let $\mathscr{W}^0 = \mathbf{k}$ and let $\mathscr{W}^n$ be the $\mathbf{k}$-vector space spanned by the n-tuples $\mathscr{k} = (k_1, \ldots, k_n)$, $k_1, \ldots, k_n \in K$, where $(k_{\pi(1)}, \ldots, k_{\pi(n)}) = (-1)^\pi (k_1, \ldots, k_n)$, π being a permutation. Let $|\mathscr{k}| \in \mathscr{S}$ be the smallest common multiple of the monomials $\mathscr{z}_{k_1}, \ldots, \mathscr{z}_{k_n}$. In $\mathscr{S} \otimes \mathscr{W}$ define a differential d by

$$d\mathscr{k} = \sum_{\kappa=1}^{n} (-1)^\kappa |\mathscr{k}| / |\mathscr{k}_\kappa| \otimes \mathscr{k}_\kappa, \tag{5.1}$$

where $\mathscr{k}_\kappa = (k_1, \ldots, \hat{k}_\kappa, \ldots, k_n)$, ($d$ is zero on $\mathscr{S}$) and define a multiplication by

$$\mathscr{k} \cdot \mathscr{k}' = |\mathscr{k}| \cdot |\mathscr{k}'| / |(\mathscr{k}, \mathscr{k}')| \otimes (\mathscr{k}, \mathscr{k}'),$$

where $(\mathscr{k}, \mathscr{k}') = (k_1, \ldots, k_n, k'_1, \ldots, k'_{n'})$. With respect to this product, d is a derivation.

PROPOSITION 5.1. *$\mathscr{S} \otimes \mathscr{W}$ is a resolution of $\mathscr{S}/\mathscr{K}$.*

Proof. Consider the complex

$$0 \underset{h_0}{\overset{d_0}{\rightleftarrows}} \mathscr{K} \underset{h_1}{\overset{d_1}{\rightleftarrows}} \mathscr{S} \otimes \mathscr{W}^1 \qquad \cdots.$$

Let Mon $(\mathscr{K})$ be the set of all monomials in $\mathscr{K}$. Define a map σ: Mon $(\mathscr{K}) \to K$ such that

$$(5.2) \qquad \mathscr{s}_k \mid \mathscr{s}, \; k = \sigma(\mathscr{s}).$$

The contracting homotopy is defined by

$$h_n(\mathscr{s} \otimes \mathscr{k}) = \mathscr{s} \cdot |\mathscr{k}| / |(k, \mathscr{k})| \otimes (k, \mathscr{k}),$$

where $\mathscr{k} \in \mathscr{W}^n$, $\mathscr{s} \in$ Mon $(\mathscr{S})$ and $k = \sigma(\mathscr{s} \cdot |\mathscr{k}|)$.

Now let $\mathscr{B}$ be a free, graded-commutative algebra generated by b_l, $l \in L$, and suppose L has an ordering which is consistent with the degrees of the b_l's. Let $J = \{j \in L \mid \deg b_j \text{ is odd}\}$. Suppose $\mathscr{I} = (c_i, i \in I) \subseteq \mathscr{B}$ is an ideal, generated by monomials and let σ': Mon $(\mathscr{I}) \to I$ be a map satisfying (5.2). In order to construct a $\mathscr{B}$-free resolution of $\mathscr{B}/\mathscr{I}$, let $\mathscr{S} = \mathscr{S}(\mathscr{B})$ be the free, skew algebra as constructed above and let $\mathscr{J} = ((b_j)^2, j \in J)$. Clearly, $\mathscr{S}/\mathscr{J} \cong \mathscr{B}$ as graded-commutative algebras. Let $\mathscr{K} \subseteq \mathscr{S}$ be the ideal with generators

$$\text{Mon}_0\,(\mathscr{K}) = \{c_i, i \in I\} \cup \{(b_j)^2, (b_j)^3, \ldots, j \in J\}.$$

The corresponding index set is $K = I \cup K'$, $K' = J \times \mathbf{N}'$, where $\mathbf{N}'$ is the set of integers larger than 1.

We decompose $\mathscr{s} \in$ Mon $(\mathscr{S})$ into $\mathscr{s} = \mathscr{s}' \cdot \mathscr{s}''$, where $\mathscr{s}' = b_{l_1} \cdot \cdots \cdot b_{l_r}$ $(l_1 < \cdots < l_r)$ is the linear part of $\mathscr{s}$ and $\mathscr{s}'' = (b_{j_1})^{\eta_1 - 1} \cdot \cdots \cdot (b_{j_s})^{\eta_s - 1}$, $j_1 < \cdots < j_s, j_\nu \in J, \eta_\nu \geq 2$ and $(b_{j_\nu})^{\eta_\nu} \mid \mathscr{s}$. Define $\sigma(\mathscr{s})$ as follows: If $\mathscr{s} \in$ Mon $(\mathscr{I})$, then $\sigma(\mathscr{s}) = \sigma'(\mathscr{s})$. If $\mathscr{s} \notin$ Mon $(\mathscr{I})$, i.e., if $s \geq 1$, let ν be maximal such that $\eta_\nu = \max\{\eta_1, \ldots, \eta_s\}$ and define $\sigma(\mathscr{s}) = k = (j_\nu, \eta_\nu)$, i.e., $\mathscr{s}_k = (b_{j_\nu})^{\eta_\nu}$. σ satisfies (5.2).

Let $\mathscr{V}^0 = \mathbf{k}$ and let $\mathscr{V}^n \subseteq \mathscr{W}^n$ be spanned by the elements

$$\mathscr{k} = (i_1, \ldots, i_r, j_1^{\eta_1}, \ldots, j_s^{\eta_s}),$$

where $n = r + \eta_1 + \cdots + \eta_s - 1$, $\eta_\nu \geq 2$, and j^η, $j \in J$, denotes the sequence $(j, \eta), \ldots, (j, 2)$ in K. $\mathscr{B} \otimes \mathscr{V}$ is closed under multiplication and the projection $\mathscr{W}^n \to \mathscr{V}^n$ induces a differential d and a contracting homotopy in $\mathscr{B} \otimes \mathscr{V}$, whence:

PROPOSITION 5.2. *$\mathscr{B} \otimes \mathscr{V}$ is a resolution of $\mathscr{B}/\mathscr{I}$.*

Note that, by construction of the induced differential in $\mathscr{B} \otimes \mathscr{V}$, $|\mathscr{k}|/|\mathscr{k}_\kappa| \otimes \mathscr{k}_\kappa = 0$ in (5.1), if $\mathscr{k}_\kappa \notin \mathscr{V}$.

We define the degree of $\mathscr{k} \in \mathscr{V}^n$ by deg $\mathscr{k}$ = deg $|\mathscr{k}| - n$, where deg $|\mathscr{k}|$ is the degree in $\mathscr{S}$. With respect to this degree, d becomes a coboundary operator. On the other hand, $\mathscr{M} = \mathscr{M}(\mathscr{B}/\mathscr{I}) = \mathscr{B} \otimes \mathscr{F}$ produces a $\mathscr{B}$-free resolution of $\mathscr{B}/\mathscr{I}$. Hence, there is a chain morphism ϕ: $\mathscr{B} \otimes \mathscr{V} \to \mathscr{B} \otimes \mathscr{F}$ such that $H^*(\mathscr{V}) \cong \operatorname{Tor}^{\mathscr{B}}(\mathscr{B}/\mathscr{I}, \mathbf{k}) \cong H^*(\mathscr{F})$ as graded vector spaces. Now $\phi\,|\,\mathscr{V}$: $\mathscr{V} \to \mathscr{F}$ is not an algebra morphism, however the induced map ϕ^*: $H^*(\mathscr{V}) \to H^*(\mathscr{F})$ is actually a ring isomorphism.

Consider now the case of a truncated algebra $\mathscr{B}/\mathscr{I}^{(M)}$ more in detail. We assume that the generators b_l, $l \in L$, of $\mathscr{B}$ are labeled as in the proof of Theorem 4.1. In order to express the base in Corollary 4.6 in terms of the complex $\mathscr{V}$, we define a mapping $|\quad|$: Mon $(\mathscr{A}) \to \mathscr{S}$ by $|a_{\ell;m}| = |a_\ell| \cdot |a_m|$, where $|a_\ell| = b_{l_1} \cdot \cdots \cdot b_{l_p}$, $|a_m| = b_{m_1} \cdot \cdots \cdot b_{m_q}$, $\sum^{-1} b_{m_\lambda} = u_{m_\lambda}$ (see Section 4). Let $\mathscr{s}'$ be the linear part of $|a_m|$ and let $b_{n_1} \cdot \cdots \cdot b_{n_r}$ be the largest divisor of $\mathscr{s}'$ such that $b_{n_\mu} \cdot |a_\ell| \neq 0$ in $\mathscr{B}$, $\mu = 1, \ldots, r$. It follows that

$$|a_m| = b_{n_1} \cdot \cdots \cdot b_{n_r} \cdot (b_{j_1})^{\eta_1 - 1} \cdot \cdots \cdot (b_{j_s})^{\eta_s - 1}, \quad j_\nu \in J,\ \eta_\nu \geq 2,$$

and $(b_{j_\nu})^{\eta_\nu} \,|\, |a_{\ell;m}|$, $\nu = 1, \ldots, s$. Now let $a_{\ell;m} \in A$ and define $c_{i_1}, \ldots, c_{i_r} \in$ Mon$_0$ $(\mathscr{I}^{(M)})$ by

$$c_{i_1} = b_{n_1} \cdot |a_\ell|,$$

$$c_{i_\mu} = b_{n_\mu} \cdot \mathscr{s}_\mu, \quad i_\mu = \sigma(b_{n_\mu} \cdot \mathscr{s}_{\mu-1}),\ \mathscr{s}_1 = |a_\ell|,\ \mu = 2, \ldots, r.$$

PROPOSITION 5.3. *The elements* $\mathscr{k} = (i_1, \ldots, i_r, j_1^{\eta_1}, \ldots, j_s^{\eta_s}) \in \mathscr{V}^n$, $n = r + \eta_1 + \cdots + \eta_s - s$, *such that*

(1) $$|(i_1, \ldots, i_r)| = b_{n_1} \cdot \cdots \cdot b_{n_r} \cdot \mathscr{s}_1,\ n_1 < \cdots < n_r,\ c_{i_\mu} = b_{n_\mu} \cdot \mathscr{s}_\mu,$$
$$i_\mu = \sigma(b_{n_\mu} \cdot \mathscr{s}_{\mu-1}),\ \mu = 1, \ldots, r,$$

(2) $b_{j_\nu} \,|\, |(i_1, \ldots, i_r)|$, $\eta_\nu \geq 2$, $\nu = 1, \ldots, s$, *and*

(3) *either* $n_1 \leq l_0$, *if* $n_1 \notin J$ *or* $n_1 < l_0$, *if* $n_1 \in J$, *where* b_{l_0} *is the minimal degree element dividing* $\mathscr{s}_1$, *form a base of* $H^*(\mathscr{V})$.

COROLLARY 5.4. *Let* $\mathscr{U}^n \subseteq \mathscr{V}^n$ *be spanned by the elements in Proposition* 5.3. $\phi(\mathscr{B} \otimes \mathscr{U}) \subseteq \mathscr{B} \otimes \mathscr{F}$ *is a subcomplex and in fact is a minimal resolution of* $\mathscr{B}/\mathscr{I}^{(M)}$. $\phi\,|\,\mathscr{B} \otimes \mathscr{U}$ *is injective.*

6. Twistings

Let $\mathscr{B}_0 = \mathscr{B}$, $\mathscr{I}_0 = \mathscr{I}^{(M)}$, $\mathscr{M}(\mathscr{B}_0/\mathscr{I}_0) = \mathscr{B}_0 \otimes_{t_0} \mathscr{F}_0$. The generators of $\mathscr{B}_0$ are denoted by $b_{0,l}$, $l \in L_0$. Let $\mathscr{B}_1 \subseteq \mathscr{F}_0$ be the subalgebra generated by $b_{1,l}$, $l \in L_1$, where $b_{1,l} = \phi(\mathscr{k})$, $\mathscr{k}$ running through the elements defined in Proposition 5.3. It follows that $\mathscr{F}_0 = \mathscr{B}_1 \otimes_{t_1} \mathscr{F}_1$ is the natural decomposition of $\mathscr{F}_0$.

According to (5.1) the twisting t_0 of the elements in $\mathscr{B}_1$ is given by

(6.1) $$t_0(b_{1,l}) = t_0(\phi(\mathscr{k})) = \sum (-1)^\kappa |\mathscr{k}|/|\mathscr{k}_\kappa| \otimes \phi(\mathscr{k}_\kappa).$$

Since $\mathscr{F}_0$ is the minimal model of the cohomology of a wedge of spheres it follows that $H^*(\mathscr{F}_0) = \mathscr{B}_1/\mathscr{I}_1$, where $\mathscr{I}_1$ is the ideal generated by all pairwise products of the generators of $\mathscr{B}_1$, and therefore $\mathscr{M}(\mathscr{B}_1/\mathscr{I}_1) = \mathscr{F}_0 = \mathscr{B}_1 \otimes \mathscr{F}_1$.

Although $\mathscr{B}_1$ is in general generated by infinitely many elements, a direct limit process gives the following description of $H^*(\mathscr{F}_1)$:

PROPOSITION 6.1. *Let $\mathscr{V}_1$ be the complex $\mathscr{V}$ obtained by replacing $\mathscr{B}$ with $\mathscr{B}_1$ and $\mathscr{I}$ with $\mathscr{I}_1$ in Proposition 5.2. Let L_1, the index set of the generators of $\mathscr{B}_1$, be given an ordering which is consistent with the degrees of the $b_{1,l}$'s. The elements*

$$(i_1, \ldots, i_r, j_1^{\eta_1}, \ldots, j_s^{\eta_s}) \in \mathscr{V}_1^n, \quad i_\mu \in I_1, j_\nu \in J_1, n = r + \eta_1 + \cdots + \eta_s - s,$$

such that

(1) $$|(i_1, \ldots, i_r)| = b_{1,n_1} \cdot \cdots \cdot b_{1,n_r} \cdot b_{1,l_0}, n_1 < \cdots < n_r,$$
$$c_{1,i_\mu} = b_{1,n_\mu} \cdot b_{1,l_0} \in \mathscr{I}_1, \mu = 1, \ldots, r,$$

(2) $b_{i,j_\nu} \,|\, |(i_1, \ldots, i_r)|$, $\eta \geq 2$, $\nu = 1, \ldots, s$, *and*

(3) *either $n_1 \leq l_0$ if $n_1 \notin J$ or $n_1 < l_0$ if $n_1 \in J$*

form a base of $H^*(\mathscr{F}_1)$.

We have therefore the following result.

THEOREM 6.2. *Let $\mathscr{B}_0$ be a minimal DGA with trivial differential and let $\mathscr{I}_0 = \mathscr{I}^{(M)}$ be an ideal truncated at* degree *M. Let $\mathscr{M}(\mathscr{B}_0/\mathscr{I}_0) = \mathscr{B}_0 \otimes \mathscr{F}_0$ and $\mathscr{F}_{n-1} = \mathscr{B}_n \otimes_{t_n} \mathscr{F}_n$, $n = 1, \ldots$, be the natural decompositions.*

(1) *$H^*(\mathscr{F}_{n-1}) = \mathscr{B}_n/\mathscr{I}_n$, $n \geq 1$, where $\mathscr{I}_n = (\mathscr{B}_n^+)^2$. The twisting t_{n-1} in $\mathscr{F}_{n-2}$ of the generators of $\mathscr{B}_n$ is given by formulas corresponding to* (6.1).

(2) *$\mathscr{M}_n = \mathscr{B}_0 \otimes \cdots \otimes \mathscr{B}_n$ is a sub-DGA of $\mathscr{M}$ and $\mathscr{M} = \text{inj lim } \mathscr{M}_n$ as DGA.*

Remark 6.3. Let $\mathscr{M}$ be any minimal DGA with natural decomposition $\mathscr{M} = \mathscr{B}_0 \otimes \mathscr{F}_0$. Let $\mathscr{F}_0 = \mathscr{B}_1 \otimes \mathscr{F}_1$ be the natural decomposition of $\mathscr{F}_0$. In general, $\mathscr{B}_0 \otimes \mathscr{B}_1$ is not invariant under the differential of $\mathscr{M}$. There is, however, a certain subalgebra $\mathscr{B}'_1 \subseteq \mathscr{B}_1$, such that $\mathscr{B}_0 \otimes \mathscr{B}'_1$ is a sub-DGA of $\mathscr{M}$. One has then a similar situation as in (2) above.

Remark 6.4. If the conjecture in Remark 2.2 is true, it would follow that (2) of Theorem 6.2 holds for any minimal DGA $\mathscr{B}$ with trivial differential and any ideal $\mathscr{I} \subseteq \mathscr{B}$.

Remark 6.5. The computation of the twisting t_0 of, say, the generators of $\mathscr{B}_2$ is more complicated than in (6.1). For instance, the elements of $\mathscr{B}_2$ hitting the generators $c_{1,i}$ of the ideal $\mathscr{I}_1 \subseteq \mathscr{B}_1$ are of two different types depending on whether or not the $c_{1,i}$'s, which are products, are zero in $\mathscr{V}_0$. If $c_{1,i}$ is nonzero in

$\mathscr{V}_0$, the element in $\mathscr{B}_2$ hitting $c_{1,i}$ is in fact the ϕ-image of a certain $\ell \in \mathscr{V}_0$ and its twisting t_0 can be obtained by (6.1). In the other case the construction of the element hitting $c_{1,i}$ is more complicated; it can be done inductively. We omit details.

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