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MINIMAL SETS, RECURRENT POINTS AND DISCRETE ORBITS IN $\beta N\backslash N$

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1. Introduction

Let N be the set of positive integers with the discrete topology and let τ be the mapping on N which sends n to $n+1$. Then τ can be extended to a continuous mapping of βN, the Stone-Čech compactification of N, into itself. The extended mapping, again denoted by τ, is one-one, $\tau(\beta N) = \beta N\backslash\{1\}$ and $\tau(\beta N\backslash N) = \beta N\backslash N$.

A *nonempty* subset K of βN is said to be *τ-invariant* if $\tau K \subset K$. K is said to be *τ-minimal* if K is closed, τ-invariant and is minimal with respect to these two properties. As usual, $\omega \in \beta N$ is said to be *τ-almost periodic* if, for each neighborhood V of ω, the set $\{i \in N: \tau^i\omega \in V\}$ is relatively dense in N. Denote the set of all τ-almost periodic points in βN by A^τ. It is known that A^τ is the union of all the τ-minimal sets of βN (cf. [7]).

$\omega \in \beta N$ is said to be *τ-recurrent* if, for each neighborhood V of ω, the set $\{i \in N: \tau^i\omega \in V\}$ is infinite. Denote the set of all τ-recurrent points by R^τ. The complement of R^τ in $\beta N\backslash N$ is denoted by D^τ. Therefore $\omega \in D^\tau$ if and only if $\omega \in \beta N\backslash N$ and its orbit $o(\omega) = \{\omega, \tau\omega, \tau^2\omega, \ldots\}$ is discrete, and, in this case, we say ω is *τ-discrete*. A^τ is a subset of R^τ and, as pointed out by Nillsen [8], they seem to constitute all the known elements of R^τ. In this paper we shall show that R^τ is much bigger than A^τ. Note that a nonalmost periodic recurrent point was constructed by Gottschalk [6] for a certain discrete *flow* (ϕ, X) where X is *metrizable*. Note also that $(\tau, \beta N)$ and the τ-minimal sets are universal in the sense of Ellis [5, Chapter 7].

Let M^τ be the set of all τ-invariant probability measures on βN. Note that the set M^τ can be identified with the set of Banach limits on N (cf. [10]). It is known that M^τ is ω^*-compact, convex and it contains 2^c points where c is the cardinality of the continuum (cf. [3]). For each $A \subset N$, let $\hat{A} = \mathrm{cl}_{\beta N} A\backslash N$. The set $\hat{A}$ is closed and open in $\hat{N} = \beta N\backslash N$ and sets of the form $\hat{A}$ form a topological basis for $\hat{N}$. (See [11] for these and other basic topological properties of βN.) The *upper τ-density* of a set $A \subset N$ is defined by

$$\bar{d}_\tau(A) = \sup\{\mu(\hat{A}): \mu \in M^\tau\}.$$

The term "upper density" is a proper one, as shown by the following lemma. Its proof involves an application of the Krein-Milman Theorem.

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LEMMA 1.1 (cf. [9]). *For* $A \subset N$,

$$\bar{d}_\tau(A) = \lim\sup_n \sup_k n^{-1} |A \cap \{k, k+1, \ldots, k+n-1\}|.$$

(For a finite set F, $|F|$ stands for the number of elements in F.) As in [10], set

$$K^\tau = \text{cl} \cup \{\text{suppt } \mu: \mu \in M^\tau\}.$$

(For a measure ν, the support of ν is denoted by suppt ν.)

LEMMA 1.2 (cf. [2]). $\omega \in K^\tau$ *if and only if* $\bar{d}_\tau(A) > 0$ *whenever* $\omega \in \hat{A}$.

It is easy to see that the interior of D^τ is dense in $\hat{N}$ (cf. [2]). In [8], Nillsen proved that $D^\tau \cap K^\tau$ is dense in K^τ. Let ex M^τ denote the set of *extreme* points of M^τ. Note that $\mu \in M^\tau$ is extreme if and only if it is ergodic (cf. [1]). In Section 2, we prove the following:

THEOREM. *The set* $D^\tau \cap (\bigcup \{\text{suppt } \mu: \mu \in \text{ex } M^\tau\})$ *is dense in* K^τ.

In particular, the support of an ergodic measure can contain τ-discrete points.

The abundance of τ-discrete points in $\hat{N}$ does not prevent the widespread distribution of its complement in $\hat{N}$. We shall prove the following in Section 3.

THEOREM. *Suppose that* $A \subset N$ *and* $d_\tau(A) > 0$. *Then* $\hat{A}$ *contains a* τ*-recurrent point which is not* τ*-almost periodic.*

The above theorem has the following consequence: $K^\tau \cap (R^\tau \backslash A^\tau)$ is dense in K^τ.

As defined in [9], a *motion* is a one-one mapping of N into N under which N has no periodic points. If σ is a motion, then one may define $A^\sigma, M^\sigma, D^\sigma$, etc. as in the case that $\sigma = \tau$. In [10], Raimi proved that if σ, δ are motions such that $M^\sigma = M^\delta$ then the σ-minimal sets and the δ-minimal sets are identical. He asked whether the converse is true. In Section 4, we shall provide a negative answer:

THEOREM. *There exists a motion* σ *such that* (i) $A^\tau = A^\sigma$ *and* $\sigma = \tau$ *on* $A^\tau = A^\sigma$, *and* (ii) $M^\sigma \neq M^\tau$.

In Section 4 we shall also prove the following.

THEOREM. *Let* K_0 *be a fixed* τ*-minimal set. If* σ *is a motion of* N *and if* K *is a* σ*-minimal set in* βN *then there exists a homeomorphism* ϕ *of* K_0 *onto* K *such that* $\phi(\tau\omega) = \sigma\phi(\omega)$ $(\omega \in K_0)$.

2. Slim sets and τ-discrete points in K^τ

Let k be a positive integer. A subset C of N is called a *k-chain* if whenever p and q are two adjacent integers in C, $|p - q| \leq k$. If C is a k-chain and $C \subset A$ then C is called a k-chain in A. The number of elements in a k-chain C is called the length of C. A maximal k-chain in A will be called a *k-component* of A. Each set A is the disjoint union of its k-components.

DEFINITION. A set $S \subset N$ is said to be *τ-slim* if for each $k \in N$, the length of each k-component of S is bounded by a constant depending only on S and k.

One of the reasons that we study τ-slim sets is given in the following result.

LEMMA 2.1 (cf. [2]). *A set S is τ-slim if and only if $\hat{S} \cap A^{\tau} = \emptyset$.*

In [2, Proposition 2.2], we constructed a τ-slim set A with $\bar{d}_\tau(A) > 0$. A similar but somewhat simpler example in the following.

Example 1. Let $S_1 = \{1, 2\}$. Define S_n inductively by setting

$$S_{n+1} = S_n \cup (\sup S_n + n + S_n), \quad n = 1, 2, \ldots.$$

Let $S = \bigcup_{n=1}^{\infty} S_n$. Note that $|S_n| = 2^n$ and $\sup S_n = 2^{n+1} - n - 1$. Therefore, by Lemma 1.1, $\bar{d}_\tau(S) \geq \lim_n |S_n|/\sup S_n = 1/2$. (In fact, it is easy to see that $\bar{d}_\tau(S) = 1/2$.) On the other hand, the length of each k-component of S *equals* $|S_k|$. Therefore, S is τ-slim.

The above example can be applied to construct many other τ-slim subsets of N.

PROPOSITION 2.2. *Suppose that A is a subset of N with $\bar{d}_\tau(A) > 0$. Then there exists a τ-slim set $B \subset A$ with $\bar{d}_\tau(B) > 0$.*

Proof. If A is already τ-slim then there is nothing to be shown. Therefore, assume that A is not τ-slim. Then there exists $k_0 \in N$ such that A contains k_0-chains of any given length. Let $S = S_1 \cup S_2 \cup \cdots$ be the set in Example 1. Set $t_n = |S_n|$ and $p_n = \sup S_n$. Choose k_0-chains $C_1, C_2, \ldots$ in A such that

(1) $|C_n| = p_n$ and
(2) $\sup C_n + n < \inf C_{n+1}$, $n = 1, 2, \ldots$.

Write C_n as $\{c_{n,i}\colon i = 1, 2, \ldots, p_n\}$ where $c_{n,i} < c_{n,j}$ if $i < j$. By (1), the set $B_n = \{c_{n,k}\colon k \in S_n\}$ is contained in C_n. Let $B = \bigcup_{n=1}^{\infty} B_n$.

Note first that, by Example 1 and (2), the length of a k-chain in B is at most $\sum_{i=1}^{k} t_i$. Therefore B is τ-slim. On the other hand, since C_n is a k_0-chain,

$$C_n \cup (C_n + 1) \cup \cdots \cup (C_n + k_0 - 1) \supset \{c_{n,1}, c_{n,1} + 1, c_{n,1} + 2, \ldots, c_{n,p_n}\}$$

Hence,

(3) $k_0 p_n \geq c_{n,p_n} - c_{n,1}$.

By Lemma 1.1,

$$\begin{aligned} \bar{d}_\tau(B) &\geq \limsup_n |B_n|/(c_{n,p_n} - c_{n,1}) \\ &\geq \limsup_n t_n/k_0 p_n \quad \text{(by (3))} \\ &= 1/2k_0 \quad \text{(by the calculation in Example 1).} \end{aligned}$$

So B is the set we are looking for.

The following proposition is contained in [7, p. 65]. For the convenience of the reader, we like to provide a proof here.

PROPOSITION 2.3. *Let K be a closed τ-invariant subset of $\hat{N}$. If $\omega \in K\backslash A^\tau$ and if U is a closed-open neighborhood of ω then $U \cap K \cap D^\tau \neq \emptyset$.*

Proof. Let $\omega \in K\backslash A^\tau$ and U be a closed open neighborhood of ω. Since $\omega \notin A^\tau$, we may assume that the set $\{i \in N: \tau^i\omega \in U\}$ is not relatively dense, in other words,

$$(1) \quad o(\omega) = \{\omega, \tau\omega, \tau^2\omega, \ldots\} \not\subset \tau^{-1}U \cup \tau^{-2}U \cup \cdots \cup \tau^{-k}U \quad \text{for } k = 1, 2, \ldots.$$

Let $U_k = U\backslash(\tau^{-1}U \cup \tau^{-2}U \cup \cdots \cup \tau^{-k}U)$. We claim that

$$(2) \qquad U_k \cap K \neq \emptyset \quad \text{for } k \in N,$$

$$(3) \qquad \tau^k U_k \cap \tau^j U_j = \emptyset \quad \text{if } i \neq j.$$

If (2) and (3) have been established, then, by (2), there exists $\omega' \in \bigcap_k (U_k \cap K)$, and, by (3), $\{\tau^k U_k\}$ is a sequence of disjoint neighborhoods of $\tau^k\omega'$. Therefore, $\omega' \in U \cap K \cap D^\tau$. It remains to prove (2) and (3).

If there exists k such that $U_k \cap K = \emptyset$ then

$$(4) \qquad K \cap U \subset \tau^{-1}U \cup \tau^{-2}U \cup \cdots \cup \tau^{-k}U.$$

Since $\omega \in K \cap U$ and K is τ-invariant, $\tau\omega \in K$. Hence, by (4),

$$\begin{aligned}\tau\omega &\in \tau(\tau^{-1}U \cup \cdots \cup \tau^{-k}U) \cap K \\ &= (U \cap K) \cup (\tau^{-1}U \cap K) \cup \cdots \cup (\tau^{-k+1}U \cap K) \\ &\subset (\tau^{-1}U \cup \tau^{-2}U \cup \cdots \cup \tau^{-k}U) \cup (\tau^{-1}U \cap K) \cup \cdots \cup (\tau^{-k+1}U \cap K) \\ &\subset \tau^{-1}U \cup \tau^{-2}U \cup \cdots \cup \tau^{-k}U.\end{aligned}$$

By induction, one may conclude that $o(\omega) \subset \tau^{-1}U \cup \cdots \cup \tau^{-k}U$ and it contradicts (1). Therefore, (2) holds.

To see (3), note that if $\tau^k\omega_k = \tau^j\omega_j \in \tau^k U_k \cap \tau^j U_j$, $j > k$ and $\omega_k \in U_k$, $\omega_j \in U_j$, then $\omega_k = \tau^{j-k}\omega_j \in U$. So $\omega_j \in \tau^{k-j}U$ and it contradicts the definition of U_j. So $\tau^k U_k \cap \tau^j U_j = \emptyset$ as we have claimed.

In [2] we showed that there exists an ergodic $\mu \in M^\tau$ such that its support contains a non-τ-almost periodic point. By the above proposition, we know that suppt μ also contains τ-discrete points. In fact, more can be said:

PROPOSITION 2.4. *The set $D^\tau \cap (\bigcup \{\text{suppt } \mu: \mu \in \text{ex } M^\tau\})$ is dense in K^τ.*

Proof. Let $\omega \in K^\tau$ and let $\hat{A}$ be a closed-open neighborhood of ω in $\hat{N}$. Then, by Lemma 1.2, $\bar{d}_\tau(A) > 0$ and hence, by Proposition 2.2, there exists a τ-slim set $B \subset A$ with $\bar{d}_\tau(B) > 0$. Since $\bar{d}_\tau(B) > 0$, by the Krein-Milman Theorem, there exists $\mu \in \text{ex } M^\tau$ such that $\hat{B} \cap \text{suppt } \mu \neq \emptyset$. Since B is τ-slim, by Lemma 2.1, $\hat{B}$ is disjoint from A^τ. Therefore, by Proposition 2.3, there exists

$$\omega_1 \in \hat{B} \cap \text{suppt } \mu \cap D^\tau \subset \hat{A} \cap \text{suppt } \mu \cap D^\tau.$$

The proof is completed.

We are going to show, in the next section, that if a closed τ-invariant set K is not contained in A^τ then $K \backslash (A^\tau \cup D^\tau) \neq \emptyset$, which perhaps makes the above two propositions more interesting.

Remark. In [8], Nillsen showed that if σ is a motion then $D^\sigma \cap K^\sigma$ is dense in K^σ. When $\sigma = \tau$, the above proposition is stronger than his result. A brief description on how to generalize the results in this section from τ to σ is in order. A set $S \subset N$ is said to be σ-slim if for each $k \in N$,

$$\bar{d}_\sigma(S \cup \sigma S \cup \cdots \cup \sigma^{k-1}S) < 1,$$

or, equivalently, there exists $n \in N$ such that $\{m, \sigma m, \ldots, \sigma^{n-1}m\} \not\subset S \cup \sigma S \cup \cdots \cup \sigma^{k-1}S$, for each $m \in N$. With this definition, one sees right away that Lemma 2.1 and Propositions 2.2–2.4 still hold when τ is changed to σ. (In the proof of Proposition 2.3, if $V \subset \hat{N}$, $\sigma^{-k}V$ should be understood as the preimage of V under σ^k.)

3. Nonalmost periodic recurrent points

The only known method to find τ-recurrent points is to apply Zorn's Lemma to find a τ-minimal set K then show that each $\omega \in K$ is τ-almost periodic and therefore τ-recurrent. In this section we are going to produce many other τ-recurrent points. First of all we need the following.

PROPOSITION 3.1. *Let ϕ be a homeomorphism of a compact Hausdorff space X onto itself. Suppose that $T_1 \supset T_2 \supset \cdots$ is a sequence of nonempty closed subsets of X such that a sequence of positive integers $k_1 < k_2 < \cdots$ can be found to satisfy $\phi^{k_n}T_{n+1} \subset T_n$. Then $\bigcap_{n=1}^\infty T_n$ contains a ϕ-recurrent point.*

Proof. Let $\mathfrak{F}$ be the family of sequences of closed subsets of X defined as follows: A sequence of closed subsets $\{F_n\}_{n=1}^\infty$ of X belongs to $\mathfrak{F}$ if, for each $n \in N$, (i) $F_n \subset T_n$, (ii) $F_{n+1} \subset F_n$, (iii) $\phi^{k_n}F_{n+1} \subset F_n$ and (iv) $F_n \neq \emptyset$.

Note first that $\mathfrak{F} \neq \emptyset$, since $\{T_n\} \in \mathfrak{F}$. $\mathfrak{F}$ can be ordered in a natural way: $\{F_n\} \leq \{G_n\}$ if and only if $F_n \subset G_n$ for each $n \in N$. It is easy to check that each chain in $\mathfrak{F}$ has a lower bound. Therefore, by Zorn's Lemma, $\mathfrak{F}$ has a minimal element $\{K_n\}$.

Let $x \in \bigcap_{n=1}^\infty K_n$. We want to show that x is ϕ-recurrent. Indeed, let U be an open neighborhood of x. Let $V = \bigcup_{n=-\infty}^\infty \phi^n U$. Consider the sequence $\{K_n \backslash V\}$. It clearly satisfies conditions (i) and (ii). Using the fact that $\phi V = V$, one sees that $\{K_n \backslash V\}$ satisfies (iii). Since $K_n \backslash V \subsetneq K_n$ and $\{K_n\}$ is minimal in $\mathfrak{F}$, $\{K_n \backslash V\} \notin \mathfrak{F}$. Therefore $\{K_n \backslash V\}$ does not satisfy (iv), i.e., there exists n_0 such that $K_{n_0} \backslash V = \emptyset$, or, equivalently, $K_{n_0} \subset V = \bigcup_{n=-\infty}^\infty \phi^n U$. Since K_{n_0} is compact, there exists $l \in N$ such that

$$K_{n_0} \subset \bigcup_{s=-l}^{l} \phi^s U. \tag{1}$$

If $n \geq n_0$, then $\phi^{k_n} x \in \phi^{k_n} K_{n+1} \subset K_n \subset K_{n_0}$. Hence, by (1), for each $n \geq n_0$ there exists an integer s_n, $-l \leq s_n \leq l$, such that $\phi^{k_n - s_n} x \in U$. Therefore, x is ϕ-recurrent, as we have claimed.

We shall only apply the above proposition to the case that $\phi = \tau$ and $X = \hat{N}$.

LEMMA 3.2. *Suppose that $A \subset N$, $\bar{d}_\tau(A) > 0$ and $n \in N$. Then there exist $B \subset A$, $s \in N$, $s \geq n$, such that $\bar{d}_\tau(B) > 0$ and $B + s \subset A$.*

Proof.[2] By the definition of upper τ-density, there exists $\mu \in M^\tau$ such that $\mu(\hat{A}) > 0$. If for each $s \geq n$, $\mu(\hat{A} \cap \tau^{-s}\hat{A}) = 0$, then

$$\sum_{i=0}^{\infty} \mu(\tau^{-in}\hat{A}) = \mu\left(\bigcup_{i=0}^{\infty} \tau^{-in}\hat{A}\right) \leq 1.$$

This contradicts the fact that μ is τ-invariant. Therefore there exists $s \geq n$ such that $\mu(\hat{A} \cap \tau^{-s}\hat{A}) > 0$. Let $B = A \cap (A - s) \cap N$. Then $\mu(\hat{B}) > 0$ and $B + s \subset A$.

We are now ready to prove the main result of this section.

PROPOSITION 3.3. *Suppose that $A \subset N$, $\bar{d}_\tau(A) > 0$. Then $A \cap (R^\tau \backslash A^\tau) \neq \emptyset$.*

Proof. By Proposition 2.2, we may assume that A is τ-slim and hence, by Lemma 2.1, $\hat{A} \cap A^\tau = \emptyset$. Therefore, it remains to produce a τ-recurrent point in $\hat{A}$.

By Lemma 3.2, it is easy to construct two sequences $s_1 < s_2 < \cdots$ and $A = A_1 \supset A_2 \supset \cdots$, inductively, such that $\bar{d}_\tau(A_i) > 0$ and $s_{i-1} + A_i \subset A_{i-1}$, $i = 2, 3, \ldots$. Therefore, $\hat{A}$ contains a τ-recurrent point, by applying Proposition 3.1 to the case that $\phi = \tau$, $X = \hat{N}$ and $T_n = \hat{A}_n$.

The above proposition tells us that $A^\tau \cup D^\tau \neq \hat{N}$. This answers a question raised in [8].

COROLLARY 3.4. *If K is a closed τ-invariant subset of $\hat{N}$ and $K \not\subset A^\tau$ then $K \cap (R^\tau \backslash A^\tau) \neq \emptyset$.*

Proof. By Proposition 2.3, there exists $\omega \in K \cap D^\tau$. $(\tau, \beta N)$ and $(\tau, \bar{o}(\omega))$ are isomorphic in the obvious sense. ($\bar{o}(\omega)$ is the closure of $o(\omega)$.) Therefore, by the above proposition, there exists

$$\omega_1 \in \bar{o}(\omega) \cap (R^\tau \backslash A^\tau) \subset K \cap (R^\tau \backslash A^\tau).$$

The set $K^\tau \cap D^\tau$ is dense in K^τ (see Section 2). Its complement in K^τ is also dense in K^τ:

[2] This simple proof was provided by the referee. Our original proof was much longer.

COROLLARY 3.5. *The set $K^\tau \cap (R^\tau \backslash \text{cl } A^\tau)$ is dense in K^τ.*

Proof. If $\omega \in K^\tau$ and if $\hat{B}$ is a closed-open neighborhood of ω then $\bar{d}_\tau(B) > 0$. Choose a τ-slim set $A \subset B$ such that $\bar{d}_\tau(A) > 0$. From the set A, construct A_n and s_n as in the proof of Proposition 3.3. The result follows by applying Proposition 3.1 to the case that $\phi = \tau$ and $T_n = \hat{A}_n \cap K^\tau$.

To conclude this section, we would like to provide an example to show that $R^\tau \not\subset K^\tau$.

Example 2. Let $F_1 = \{1\}$. Define F_n inductively by the relation $F_{n+1} = F_n \cup (F_n + \sup F_n + 2^n)$. Set $F = \bigcup_{n=1}^{\infty} F_n$. It is easily checked that $\bar{d}_\tau(F) = 0$.

On the other hand, for each $k \in N$, there are infinitely many 2^k-components of F. Let

$$C_k = \{n \in N: n \text{ is the smallest element of a } 2^k\text{-component}\}.$$

From the definition of F, one sees that $C_k + \sup F_{k-1} + 2^{k-1} \subset C_{k-1}$, $k = 2, 3, \ldots$. Therefore, it follows from Proposition 3.1, with $T_k = \hat{C}_k$, $s_k = \sup F_k + 2^k$, that there exists $\omega \in R^\tau \cap \hat{F}$, $\omega \notin K^\tau$.

4. Minimal sets for motions of N

Recall that a motion is a one-one mapping of N into N under which N has no periodic points. Raimi [9] provided a necessary and sufficient condition for two motions σ and δ to satisfy $M^\sigma = M^\delta$. In [10] he showed that if $M^\sigma = M^\delta$ then the σ-minimal sets and the δ-minimal sets are identical. He asked whether the converse holds. In this section we shall provide a negative answer.

LEMMA 4.1. *Suppose that σ and δ are two motions of N. Suppose that $S \subset N$ is both σ-slim and δ-slim and $\sigma = \delta$ on $N \backslash S$. Then $A^\sigma = A^\delta$ and if $\omega \in A^\sigma = A^\delta$ then $\sigma\omega = \delta\omega$.*

Proof. Since $\hat{S} \cap A^\sigma = \emptyset$ and $\hat{S} \cap A^\delta = \emptyset$, if $\omega \in A^\sigma \cup A^\delta$ then $\omega \in (N\backslash S)^\wedge$ and, by assumption, $\sigma\omega = \tau\omega$. The fact that $A^\sigma = A^\delta$ follows easily from this observation.

PROPOSITION 4.2. *There exists a motion σ such that:*

(i) $A^\sigma = A^\tau$ *and* $\sigma = \tau$ *on* $A^\sigma = A^\tau$;
(ii) $M^\sigma \neq M^\tau$.

Proof. Let $A = \{a_1, a_2, \ldots\}$, $a_1 < a_2 < \cdots$, be a τ-slim subset of N with $\bar{d}_\tau(A) > 0$, $1 \notin A$. Let $B = N \backslash A = \{b_1, b_2, \ldots\}$, $b_1 < b_2 < \cdots$. Let σ be the motion defined by the following listing of N:

$$b_1, b_2, a_1; b_3, b_4, a_2; \cdots; b_{2^{n-1}+1}, b_{2^{n-1}+2}, \ldots, b_{2^n}, a_n; \ldots.$$

It means that if c_k denotes the kth element in the above listing then $\sigma c_k = c_{k+1}$. We claim that σ satisfies (i) and (ii).

Note that $\bar{d}_\sigma(A) = 0$, while by assumption $\bar{d}_\tau(A) > 0$. Therefore $M^\sigma \neq M^\tau$, i.e., (ii) holds. Let

$$S = A \cup (A - 1) \cup \{b_{2^n}: n = 1, 2, \ldots\}.$$

Note that $\sigma = \tau$ on $N \backslash S$, since if $p \in N \backslash S$ then $p = b_m$ for some $m \in N$, $m \neq 2^n$ ($n \in N$) and $b_m + 1 = b_{m+1}$. To prove (i), by Lemma 4.1, we only have to show that S is both τ-slim and σ-slim.

By assumption, A is τ-slim and, hence, $A - 1$ is also τ-slim. Since $b_{2^n} - b_{2^n-1} \geq 2^{n-1}$, $\{b_{2^n}, n = 1, 2, \ldots\}$ is τ-slim. Therefore, S being a union of three τ-slim sets, is τ-slim.

It is easy to see that A and $\{b_{2^n}: n = 1, 2, \ldots\}$ are σ-slim. Therefore, S will be σ-slim if $(A - 1) \cap (N \backslash A) = (A - 1) \cap B$ is. Let the 1-components of B be $B_1, B_2, \ldots$ where $\sup B_i < \inf B_{i+1}$. Denote the largest element in B_i by t_i. Note that

$$(A - 1) \cap B = \{t_i: i = 1, 2, \ldots\}.$$

Since S is τ-slim, there exists $c \in N$ such that the length of each 1-component of S is bounded by c. Let $\{t_i, t_{i+1}, \ldots, t_{i+l-1}\}$ be a (σ)-k-chain in $(A - 1) \cap B$ of length l, i.e., for each j, $i \leq j \leq i + l - 2$, there exists $p \in N$, $p \leq k$, such that $\sigma^p t_j = t_{j+1}$. We claim that

(iii) $\{t_i + 1, t_{i+1} + 1, \ldots, t_{i+l-1} + 1\}$ is a $(k + c)$-chain in A.

Let the maximal length of a $(k + c)$-chain in A be q. If (iii) holds, then l is bounded by q. In other words, each (σ)-k-chain in $(A - 1) \cap B$ is bounded by the constant q which depends only on k. So $(A - 1) \cap B$ is σ-slim as we have claimed. To see (iii), note first that if $|B_{j+1}| > k$ then $\sigma^p t_j \neq t_{j+1}$ for $p = 1, 2, \ldots, k$. Therefore, $|B_{j+1}| \leq k$ if $i \leq j \leq i + l - 2$. Also note that between t_j and the smallest element of B_{j+1} there is exactly one 1-component of A which, as we have pointed out earlier, is of length $\leq c$. So $t_{j+1} - t_j \leq c + k$, if $i \leq j \leq i + l - 2$. This finishes the proof of (iii) and hence of the proposition.

In [8, Proposition 4.3], Nillsen showed that if σ_1 and σ_2 are motions then each σ_1-minimal set is homeomorphic to each set in an uncountable family of σ_2-minimal sets. He asked whether there exist two nonhomeomorphic σ-minimal sets. The answer is negative:

PROPOSITION 4.3. *Let K_0 be a fixed τ-minimal set. If σ is a motion of N and if K is a σ-minimal set in βN then there exists a homeomorphism ϕ of K_0 onto K such that $\phi(\tau\omega) = \sigma\phi(\omega)$, $\omega \in K_0$.*

Before proving the above proposition, let us look at the general motions more closely. If σ is a motion of N then N can be written as a disjoint union of *infinite cycles* and *infinite half cycles* (cf. [4, Section 4]). Dean and Raimi [4] showed that if σ is a motion then there exists a motion δ such that δ is defined by a single infinite half cycle and $M^\sigma = M^\delta$. Note that $M^\sigma = M^\delta$ implies that the σ-minimal sets and the δ-minimal sets are identical (cf. [10]) but it does not imply that $\sigma = \delta$ on $A^\sigma = A^\delta$. We need the following modification of their result.

PROPOSITION 4.4. *Let σ be a motion of N. Then there exists a motion δ such that:*

(i) *δ is defined by a single infinite half cycle, i.e., there is $c \in N$ such that $N = \{c, \delta c, \delta^2 c, \ldots\}$,*
(ii) *$A^\sigma = A^\delta$ and $\sigma = \delta$ on $A^\sigma = A^\delta$.*

Proof. The proof is similar to that of Lemma 4.3 and Lemma 4.7 of [4]. Therefore, we shall skip some of the details here. Let B_i, $i \in I$, be the infinite cycles of σ, say, $B_i = \{b_{i,n}, n = 0, \pm 1, \pm 2, \ldots\}$ where $\sigma b_{i,n} = b_{i,n+1}$. B_i can be rearranged as follows:

$$B_i = \{b_{i,0}; b_{i,1}, b_{i,2}, b_{i,-2}, b_{i,-1}; \ldots;$$
$$b_{i,t_n}, b_{i,t_n+1}, \ldots, b_{i,t_{n+1}-1}, b_{i,-t_{n+1}+1}, b_{i,-t_{n+1}+2}, \ldots, b_{i,-t_n}; \ldots\}$$
$$\equiv \{b_1^i, b_2^i, \ldots\}$$

where $t_n = n(n+1)/2$. Define a motion γ as follows: $\gamma(k) = \sigma(k)$ if $k \notin \bigcup_{i \in I} B_i$ and $\gamma(k) = b_{j+1}^i$ if $k = b_j^i$. Let

$$S = \bigcup_{i \in I} \{b_{i,2}, b_{i,-1}; \ldots; b_{i,t_{n+1}-1}, b_{i,-t_n}; \ldots\}.$$

Note that S is both σ-slim and γ-slim and that $\sigma = \gamma$ on $N \backslash S$. Therefore, by Lemma 4.1,

$$(1) \qquad A^\sigma = A^\gamma \text{ and } \sigma = \gamma \text{ on } A^\sigma = A^\gamma.$$

Now γ only has infinite half cycles. For convenience, we assume that there are infinitely many of them, say, A_i, $i = 1, 2, \ldots$. (The finite case is easier.) Assume that $A_i = \{a_{i,1}, a_{i,2}, \ldots\}$ where $\gamma a_{i,k} = a_{i,k+1}$. Let δ be defined by the following single half cycle:

$$\{a_{1,1}; a_{1,2}a_{1,3}, a_{2,1}a_{2,2}a_{2,3}; \ldots; a_{1,s_n+1}a_{1,s_n+2} \cdots a_{1,s_{n+1}},$$
$$a_{2,s_n+1}a_{2,s_n+2} \cdots a_{2,s_{n+1}}, \ldots, a_{n-1,s_n+1}a_{n-1,s_n+2} \cdots a_{n-1,s_{n+1}},$$
$$a_{n,1}a_{n,2} \cdots a_{n,s_{n+1}}; \ldots\}$$

where $s_n = n(n-1)/2$, $n = 2, 3, \ldots$. Let $E = \{a_{n,s_m}: m, n \in N, m \geq n+1\}$. As before, note that E is both γ-slim and δ-slim and that $\gamma = \delta$ on $N \backslash E$. Again, by Lemma 4.1,

$$(2) \qquad A^\gamma = A^\delta \text{ and } \gamma = \delta \text{ on } A^\gamma = A^\delta.$$

Combining (1) and (2), it follows that δ is the motion we are looking for.

Suppose K is a σ-minimal set in βN. Choose δ as in Proposition 4.4. Then $\sigma = \delta$ on K. Let ψ be the homeomorphism of βN onto itself given by $\psi(\delta^n c) = n+1$, $n = 0, 1, \ldots$. Then, clearly, $\psi(\sigma\omega) = \psi(\delta\omega) = \tau\psi(\omega)$, $\omega \in K$ and $\psi(K)$ is a τ-minimal subset of βN. Therefore, Proposition 4.3 follows from the following result.

LEMMA 4.5 [5, p. 62]. *If K_1 and K_2 are two τ-minimal sets of βN then there exists a homeomorphism ϕ of K_1 onto K_2 such that $\phi(\tau\omega) = \tau\phi(\omega)$, $\omega \in K_1$.*

Proof. If T is a discrete group, let T act on βT in the usual way. In [5], Ellis showed that if K_1 and K_2 are two T-minimal sets of βT then there exists a homeomorphism ϕ of K_1 onto K_2 such that $\phi(t \cdot \omega) = t \cdot \phi(\omega)$, $\omega \in K_1$, $t \in T$. It is easily checked that his result also holds for the additive semigroup N. Translating into our language, it means that the lemma holds.

Finally, we like to point out that βN has exactly 2^c τ-minimal sets (cf. [3]).

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