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COMMUTING SUBNORMAL OPERATORS SIMULTANEOUSLY QUASISIMILAR TO UNILATERAL SHIFTS

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Section 1

Let $\mathbf{S} = (S_1, \ldots, S_n)$ be an n-tuple of pairwise commuting subnormal operators on a separable, infinite dimensional Hilbert space H. An n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ is a normal extension of $\mathbf{S}$ if $T_1, \ldots, T_n$ are normal commuting operators on a Hilbert space $K \supset H$ and T_i is a normal extension of S_i. T. Ito [11, Theorem 1] has given necessary and sufficient conditions in order that $\mathbf{S}$ have a normal extension $\mathbf{T}$. (Recently, A. Lubin [12] and B. Abramhamse [1] have shown (independently) that $\mathbf{S}$ need not have a normal extension.) A normal extension $\mathbf{T}$ of $\mathbf{S}$ is said to be minimal if the smallest subspace of K containing H and reducing $T_1, \ldots, T_n$ is all of K. A minimal extension $\mathbf{T}$ is unique up to unitary equivalence [11, Theorem 2].

A vector $x_0 \in H$ is a joint cyclic vector for $\mathbf{S}$ if the smallest subspace of H containing x_0 and invariant under $S_1, \ldots, S_n$ is all of H. To construct examples, let μ be a measure with compact support in $\mathbf{C}^n$. (By measure is meant a positive, finite measure on the Borel subsets of $\mathbf{C}^n$.) Let $H^2(\mu)$ be the $L^2(\mu)$-closure of $\mathscr{P}_n$, the polynomials in $z = (z_1, \ldots, z_n)$. Define operators $W_{i\mu}$ on $L^2(\mu)$ by $(W_{i\mu} f)(z) = z_i f(z)$ (multiplication by z_i) and let

$$U_{i\mu} = W_{i\mu}|_{H^2(\mu)}.$$

Then $\mathbf{U}_\mu = (U_{1\mu}, \ldots, U_{n\mu})$ is an n-tuple of pairwise commuting subnormal operators and the constant function 1 is a joint cyclic vector. Furthermore $\mathbf{W}_\mu = (W_{1\mu}, \ldots, W_{n\mu})$ is a minimal normal extension.

Theorem 0. *Suppose $\mathbf{S} = (S_1, \ldots, S_n)$ is an n-tuple of pairwise commuting subnormal operators on H with a joint cyclic vector x_0 of norm* 1. *Suppose that* $\mathbf{S}$ *has a commuting normal extension $\mathbf{T} = (T_1, \ldots, T_n)$ on $K \supset H$ and suppose T is minimal. Then there exists a Borel probability measure μ with compact support in $\mathbf{C}^n$ and there exists a unitary operator $V: K \to L^2(\mu)$ such that $Vx_0 = 1$, $VH = H^2(\mu)$ and $T_i = V^* W_{i\mu} V$, $1 \le i \le n$. In particular, $S_i \cong U_{i\mu}$, $1 \le i \le n$.*

Proof. Let $\mathfrak{A}$ be the smallest *-subalgebra of $\mathscr{L}(K)$ which contains $T_1, \ldots, T_n$ and I. The theorem is just the spectral theorem applied to the algebra $\mathfrak{A}$ plus the Gelfand-Naimark theorem.

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Let m_n be Haar measure on the unit torus $\mathbf{T}^n$ in $\mathbf{C}^n$. Then $\mathbf{U} = \mathbf{U}_{m_n}$ is an n-tuple of commuting unilateral shifts of infinite multiplicity. The purpose of this paper is to investigate when an n-tuple of commuting subnormal operators $\mathbf{S}$ is unitarily equivalent or quasisimilar to $\mathbf{U}$. We will write $\mathbf{S} \cong \mathbf{U}$ via V if V is unitary and $V\mathbf{S} = \mathbf{U}V$, i.e., $VS_i = U_i V$, $1 \leq i \leq n$. Similarly, $\mathbf{S} \sim \mathbf{U}$ via X (similar) if X is invertible and $X\mathbf{S} = \mathbf{U}X$, and $\mathbf{S} \sim\sim \mathbf{U}$ via (X, Y) (quasisimilar) if X and Y are quasiinvertible (1-1 and dense range), $X\mathbf{S} = \mathbf{U}X$, and $\mathbf{S}Y = Y\mathbf{U}$.

For $n = 1$, the problem at hand has a nice solution found by S. Clary [3, 4]. Let supp μ denote the closed support of the measure μ, and let

$$\Delta^n = \{z \in \mathbf{C}^n \colon |z_i| < 1\}.$$

A measure μ is said to be of type $\mathscr{S}$ if supp $\mu \subset \Delta^-$, $\mu_0 = \mu|_{\mathbf{T}} \ll m$ and $\int \log (d\mu_0/dm)\, dm > -\infty$. Clary showed that $U_\mu \sim\sim U$ if and only if μ is of type $\mathscr{S}$. By a result of Hoover [10], if supp $\mu \subset \mathbf{T}$, then $U_\mu \sim\sim U$ if and only if $U_\mu \cong U$.

Clearly, if $\mathbf{S} \sim\sim \mathbf{U}$, then $\mathbf{S}$ has a joint cyclic vector. We make the additional assumption that $\mathbf{S}$ has a normal extension. In this case there is a measure μ such that $\mathbf{S} \cong \mathbf{U}_\mu$. We will say that $\phi \in H^2(\mu)$ is cyclic if ϕ is a joint cyclic vector for $\mathbf{U}_\mu$, that is, if $\mathscr{P}_n \phi$ is dense in $H^2(\mu)$.

THEOREM 1. *Let μ be a measure with compact support in $\mathbf{C}^n$. The following statements are equivalent.*

(1) $\mathbf{U}_\mu \cong \mathbf{U}$.
(2) $\mu \ll m_n$,

$$\int \log \frac{d\mu}{dm_n}\, dm_n > -\infty$$

and $U_{j\mu} U_{k\mu}^* = U_{k\mu}^* U_{j\mu}$ *whenever* $j \neq k$, $1 \leq j, k \leq n$.
(3) $d\mu = |\phi|^2\, dm_n$, *where* $\phi \in H^2(m_n)$ *is cyclic.*

THEOREM 2. *Let μ be a measure with compact support in $\mathbf{C}^n$. Then $U_\mu \sim\sim U$ if and only if*

(a) *there is a cyclic function $\phi \in H^2(m_n)$ such that*

$$|\phi|^2 \leq \frac{d\mu}{dm_n} \text{ a.e. } [m_n]$$

and

(b) *there is a cyclic function $\psi \in H^2(\mu)$ such that*

$$\int |p|^2 |\psi|^2\, d\mu \leq \int |p|^2\, dm_n \textit{ for each } p \in \mathscr{P}_n.$$

In the next two sections we take up the proof of these theorems. Along the way we will develop several properties of pairwise commuting subnormal operators. In the last section, examples are given which show that the situation for

$n > 1$ differs substantially from that for $n = 1$. This has implications regarding the possible boundary values of outer and cyclic functions in $H^2(m_n)$.

Section 2

This section begins with a lemma on the joint spectrum. Let

$$\mathbf{A} = (A_1, \ldots, A_n)$$

be an n-tuple of pairwise commuting operators on H and let $\mathfrak{A}$ be an abelian subalgebra of $\mathscr{L}(H)$ containing $A_1, \ldots, A_n$ and I. Denote the joint spectrum of $\mathbf{A}$ with respect to $\mathfrak{A}$ by $\sigma_{\mathfrak{A}}(\mathbf{A})$.

LEMMA 1. *Let $\mathfrak{A}$ be an abelian subalgebra of $\mathscr{L}(H)$ and let*

$$\mathbf{A} = (A_1, \ldots, A_n)$$

be an n-tuple of elements of $\mathfrak{A}$. Suppose $p_1, \ldots, p_k \in \mathscr{P}_n$. Then

$$\sigma_{\mathfrak{A}}(p_1(\mathbf{A}), \ldots, p_k(\mathbf{A})) = \{(p_1(\lambda), \ldots, p_k(\lambda)) \colon \lambda \in \sigma_{\mathfrak{A}}(\mathbf{A})\}.$$

In particular, if $k = 1$ and $\mathfrak{A}$ is inverse closed, then $\sigma(p(\mathbf{A}))$, the usual spectrum of $p(\mathbf{A})$, equals $p(\sigma_{\mathfrak{A}}(\mathbf{A}))$.

In the absence of a joint cyclic vector, a useful representation of commuting subnormal operators seems out of reach. For their normal extensions we have the following theorem. For a set $\mathscr{E}$ of operators on a Hilbert space H, $\mathscr{E}'$ and $\mathscr{E}''$ denote the first and second commutants of $\mathscr{E}$, respectively, where $\mathscr{E}'$ is the set of all operators on H which commute with each element of $\mathscr{E}$ and $\mathscr{E}'' = (\mathscr{E}')'$.

THEOREM 3. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be an n-tuple of pairwise commuting normal operators on K and let $\mathfrak{A} = \{T_1, \ldots, T_n\}''$. Then there exists a projection-valued spectral measure $\mathbf{P}$ with* supp $\mathbf{P} = \sigma_{\mathfrak{A}}(\mathbf{T})$ *such that*

$$T_i = \int z_i \, d\mathbf{P}(z), \quad 1 \le i \le n.$$

Proof. The proof of this theorem closely parallels the proof of the spectral theorem for a single normal operator as in [7]. The necessary spectral mapping theorem is provided by Lemma 1. That supp $\mathbf{P} = \sigma_{\mathfrak{A}}(\mathbf{T})$ is proved using a theorem of Coburn and Schechter [5, Theorem 2], which shows that $\lambda \notin \sigma_{\mathfrak{A}}(\mathbf{T})$ if and only if there exists $\varepsilon > 0$ such that

$$\sum_{j=1}^{n} \|(T_j - \lambda_j)f\| \ge \varepsilon \|f\| \quad \text{for each } f \in K.$$

The relationship between the spectrum of a subnormal operator S and the spectrum of its minimal normal extension T is well understood (cf. [2]). In particular, $\sigma(T) \subset \sigma(S)$ and $\partial\sigma(S) \subset \sigma(T)$. This implies that $\sigma(S)$ is $\sigma(T)$ together with some "holes" of $\sigma(T)$. The statements about spectrum which generalize to the joint spectrum of commuting subnormal operators with commuting normal extensions are $\sigma(T) \subset \sigma(S)$ and $\sigma(S) \subset \sigma(T)^\wedge$, where for a compact subset E of $\mathbf{C}^n$, $E^\wedge$ denotes the polynomially convex hull of E.

THEOREM 4. *Let* $\mathbf{S} = (S_1, \ldots, S_n)$ *be an n-tuple of pairwise commuting subnormal operators on H with minimal normal extension* $\mathbf{T} = (T_1, \ldots, T_n)$ *on K. Let* $\mathfrak{A} = \{T_1, \ldots, T_n\}''$ *and let* $\mathfrak{B}$ *be the subalgebra of* $\mathscr{L}(H)$ *generated by* $S_1, \ldots, S_n$ *and I. Then*

(1) $\sigma_{\mathfrak{A}}(\mathbf{T}) \subset \sigma_{\mathfrak{B}''}(\mathbf{S})$, *and*
(2) $\sigma_{\mathfrak{B}}(\mathbf{S})$ *is the polynomially convex hull of* $\sigma_{\mathfrak{A}}(\mathbf{T})$.

Proof. Let $\mathbf{P}$ be the spectral measure such that $T_i = \int z_i \, d\mathbf{P}(z)$, $1 \le i \le n$. By Theorem 3, supp $\mathbf{P} = \sigma_{\mathfrak{A}}(\mathbf{T})$. Suppose $\lambda \notin \sigma_{\mathfrak{B}''}(\mathbf{S})$. To prove (1) we must show that $\lambda \notin$ supp $\mathbf{P}$. Without loss of generality, we will assume that $\lambda = 0$. There exist bounded operators $B_1, \ldots, B_n \in \mathfrak{B}''$ such that $B_1 S_1 + \cdots + B_n S_n = I$. Choose ε such that

$$0 < \varepsilon < (\|B_1\| + \cdots + \|B_n\|)^{-1}.$$

Let $E_\varepsilon = \{z \in \mathbf{C}^n\colon |z_i| \le \varepsilon\}$ and let $K_\varepsilon = \mathbf{P}(E_\varepsilon)K$. Then K_ε reduces $T_1, \ldots, T_n$. We will show that K_ε is orthogonal to H and, consequently, that $K_\varepsilon = 0$. For $f \in H$ and $g \in K_\varepsilon$

$$\begin{aligned}(f, g) &= ((B_1 S_1 + \cdots + B_n S_n)^k f, g) \\ &= \sum_{\substack{j=(j_1, \ldots, j_n) \\ j_1 + \cdots + j_n = k \\ j_i \ge 0}} c_j (S_1^{j_1} \cdots S_n^{j_n} B_1^{j_1} \cdots B_n^{j_n} f, g) \\ &= \sum_j c_j (B_1^{j_1} \cdots B_n^{j_n} f, T_1^{*j_1} \cdots T_n^{*j_n} g).\end{aligned}$$

Therefore,

$$\begin{aligned}|(f, g)| &\le \|f\| \sum_j c_j \|B_1\|^{j_1} \cdots \|B_n\|^{j_n} \|T_1^{*j_1} \cdots T_n^{*j_n} g\| \\ &\le \varepsilon^k \|f\| \|g\| \sum_j c_j \|B_1\|^{j_1} \cdots \|B_n\|^{j_n} \\ &= \varepsilon^k \|f\| \|g\| (\|B_1\| + \cdots + \|B_n\|)^k \\ &\to 0 \text{ as } k \to \infty.\end{aligned}$$

That is, $(f, g) = 0$, and so $K_\varepsilon = 0$. But this is possible only if $0 \notin$ supp $\mathbf{P}$.

To prove (2) note that $\sigma_{\mathfrak{B}}(\mathbf{S})$ is polynomially convex [6, Theorem III.1.4]. In view of (1), supp $\mathbf{P} \subset \sigma_{\mathfrak{B}}(\mathbf{S})$. Suppose $\lambda \notin (\text{supp } \mathbf{P})^\wedge$. We must show that $\lambda \notin \sigma_{\mathfrak{B}}(\mathbf{S})$. Without loss of generality, $\lambda = 0$. Let $\mathfrak{C}$ be the Banach algebra generated by $\mathscr{P}_n$ with norm $\|f\| = \sup\{|f(z)|\colon z \in \text{supp } \mathbf{P}\}$. The maximal ideal space of $\mathfrak{C}$ may be identified with $(\text{supp } \mathbf{P})^\wedge$ [6, Theorem III.1.2]. Since $0 \notin (\text{supp } \mathbf{P})^\wedge$, there exist functions $f_1, \ldots, f_n$ in $\mathfrak{C}$ such that $z_1 f_1 + \cdots +$

$z_n f_n = 1$ on supp $\mathbf{P}$. It is easily seen that if

$$B_j = \left(\int f_j(z)\, d\mathbf{P}(z) \right) \Big|_H ,$$

then $B_j \in \mathfrak{B}$ and $B_1 S_1 + \cdots + B_n S_n = I$. That is, $0 \notin \sigma_{\mathfrak{B}}(\mathbf{S})$.

S. Clary has shown [3] that quasisimilar subnormal operators have equal spectra. While it is not known whether the corresponding statement for joint spectrum is true, we do have the following result.

PROPOSITION 1. *Let μ and ν be measures on $\mathbf{C}^n$ with compact supports. If $\mathbf{U}_\mu \sim\sim \mathbf{U}_\nu$, then* $(\operatorname{supp} \mu)^\wedge = (\operatorname{supp} \nu)^\wedge$.

Proof. Suppose $\mathbf{U}_\mu \sim\sim \mathbf{U}_\nu$ via (X, Y). We will show that $\operatorname{supp} \mu \subset (\operatorname{supp} \nu)^\wedge$. Suppose $\lambda \in \operatorname{supp} \mu$ but $\lambda \notin (\operatorname{supp} \nu)^\wedge$. Then there is a polynomial $p \in \mathscr{P}_n$ such that $|p(\lambda)| > d = \sup \{|p(z)|: z \in \operatorname{supp} \nu\}$. We may assume that $|p(\lambda)| = 1$. Choose d' such that $d < d' < 1$ and let O be an open set such that $\lambda \in O$ and $|p(z)| > d'$ for each $z \in O$. Let $\psi = Y1$. Then for each $k \geq 0$, $p^k \psi = Y(p^k)$ and, therefore,

$$\int |p|^{2k} |\psi|^2 \, d\mu \leq \|Y\|^2 \int |p|^{2k} \, d\nu \leq d^{2k} \|Y\|^2 \int d\nu.$$

On the other hand

$$\int |p|^{2k} |\psi|^2 \, d\mu \geq (d')^{2k} \int_O |\psi|^2 \, d\mu.$$

Since $\lambda \in \operatorname{supp} \mu$ and Y has dense range, $\int_O |\psi|^2 \, d\mu \neq 0$. Therefore, we have

$$(d')^{2k} \int_O |\psi|^2 \, d\mu \leq d^{2k} \|Y\|^2 \int d\nu$$

for all $k \geq 0$, which is impossible. Hence, our assumption that $\lambda \notin (\operatorname{supp} \nu)^\wedge$ must be false.

The Poisson kernel representation shows that each point of Δ^n is a bounded point evaluation for $H^2(m_n)$. In general, if μ is a compactly supported measure on $\mathbf{C}^n$, then $\lambda \in \mathbf{C}^n$ is a bounded point evaluation (b.p.e.) for $H^2(\mu)$ if there exists a constant $C > 0$ such that $|p(\lambda)| \leq C \|p\|_\mu$ for every polynomial $p \in \mathscr{P}_n$.

PROPOSITION 2. *A point $\lambda \in \mathbf{C}^n$ is a b.p.e. for $H^2(\mu)$ if and only if*

$$\bigcap_{j=1}^n \ker (U_{j\mu}^* - \bar{\lambda}_j) \neq \{0\}.$$

COROLLARY 1. *Let $\mathfrak{A}$ be a closed abelian subalgebra of $\mathscr{L}(H^2(\mu))$ which contains $U_{1\mu}, \ldots, U_{n\mu}$ and I. If λ is a b.p.e. for $H^2(\mu)$, then $\lambda \in \sigma_{\mathfrak{A}}(\mathbf{U}_\mu)$.*

COROLLARY 2. *Let μ and ν be measures with compact support in $\mathbf{C}^n$. Suppose there exists a bounded operator $X\colon H^2(\mu) \to H^2(\nu)$ with dense range satisfying $X\mathbf{U}_\mu = \mathbf{U}_\nu X$. If λ is a b.p.e. for $H^2(\nu)$, then λ is a b.p.e. for $H^2(\mu)$.*

COROLLARY 3. *If $\mathbf{U}_\mu \sim\sim \mathbf{U}_\nu$, then $H^2(\mu)$ and $H^2(\nu)$ have the same bounded point evaluations.*

COROLLARY 4. *If $\mathbf{U}_\mu \sim\sim \mathbf{U}$, then $\sigma_{\mathfrak{A}}(\mathbf{U}_\mu) = (\Delta^n)^-$, where $\mathfrak{A}$ is any closed, abelian subalgebra of $\mathscr{L}(H^2(\mu))$ containing $U_{1\mu}, \ldots, U_{n\mu}$ and I.*

Proof. In view of Proposition 1 and Theorem 4, $\sigma_{\mathfrak{A}}(\mathbf{U}_\mu) \subset (\Delta^n)^-$. The result follows from Corollary 1 and Corollary 3.

Section 3

We now turn to the problem of characterizing those measures μ for which $\mathbf{U}_\mu \sim\sim \mathbf{U}$. To begin, we examine the measure μ restricted to the boundary of Δ^n. For $n = 1$ the quasisimilarity of U_μ and U_m implies that $\mu|_{\mathbf{T}} \ll m$. For $n > 1$, we must have $\mu|_{\mathbf{T}^n} \ll m_n$, but what can be said about μ restricted to the indistinguished boundary of Δ^n (that part of the boundary disjoint from $\mathbf{T}^n$)? The answer in the following proposition allows us to construct a measure ν with $\operatorname{supp} \nu \subset \mathbf{T}^n$ such that

$$\int |p|^2 \, d\mu \leq \int |p|^2 \, d\nu \quad \text{for each } p \in \mathscr{P}_n.$$

Let σ be a permutation of the first n integers and define

$$\pi_{\sigma k}\colon \mathbf{C}^n \to \mathbf{C}^k \quad (k \leq n)$$

by $\pi_{\sigma k}(z) = (z_{\sigma(1)}, \ldots, z_{\sigma(k)})$.

PROPOSITION 3. *Let μ be a measure with* $\operatorname{supp} \mu \subset (\Delta^n)^-$. *Suppose there exists an operator $Y\colon H^2(m_n) \to H^2(\mu)$ with dense range satisfying $Y\mathbf{U} = \mathbf{U}_\mu Y$. Then for each k and each σ, $\mu(\pi_{\sigma k}^{-1}(E)) = 0$ for every set $E \subset \mathbf{T}^k$ of m_k-measure zero. In particular, $\mu|_{\mathbf{T}^n} \ll m_n$. Furthermore, there is a cyclic function $\phi \in H^2(\mu)$ such that*

$$\int |p|^2 |\phi|^2 \, d\mu \leq \int |p|^2 \, dm_n \quad \textit{for every } p \in \mathscr{P}_n.$$

Proof. Suppose $\|Y\| = 1$ and let $\phi = Y1$. We assume that σ is the identity permutation; the general case follows by a similar argument. Define a measure

ν on $\mathbf{T}^k$ by $\nu(E) = \int_{\pi_{\sigma k}^{-1}(E)} |\phi|^2 \, d\mu$. For any polynomial $p \in \mathscr{P}_k$,

$$\begin{aligned}\int |p|^2 \, d\nu &= \int |p(z_1, \ldots, z_k)|^2 |\phi(z)|^2 \, d\mu(z) \\ &\leq \int |p(z_1, \ldots, z_k)|^2 \, dm_n(z) \\ &= \int |p|^2 \, dm_k.\end{aligned}$$

But on $\mathbf{T}^k$ every trigonometric polynomial (in $z_1, \ldots, z_k$ and $\bar{z}_1, \ldots, \bar{z}_k$) agrees in modulus with a polynomial in $\mathscr{P}_k$. Furthermore, the positive trigonometric polynomials are uniformly dense in the positive continuous functions on $\mathbf{T}^k$. Therefore, $\int h \, d\nu \leq \int h \, dm_k$ for every positive continuous function on $\mathbf{T}^k$. This is possible only if $\nu \ll m_k$. In other words, $\int_{\pi_{\sigma k}^{-1}(E)} |\phi|^2 \, d\mu = 0$ for every Borel set $E \subset \mathbf{T}^k$ with $m_k(E) = 0$. Since Y has dense range, $|\phi| > 0$ a.e. $[\mu]$. Therefore, $\mu(\pi_{\sigma k}^{-1}(E)) = 0$ whenever $E \subset \mathbf{T}^k$ and $m_k(E) = 0$. The last two statements of the proposition follow by taking $k = n$ in the above argument.

Suppose μ is a measure carried by $(\Delta^n)^-$ for which $\mu(\pi_{\sigma k}^{-1}(E)) = 0$ for each set $E \subset \mathbf{T}^k$ of m_k-measure zero. Then we will say that $\mu|_{\partial\Delta^n}$, the restriction of μ to the boundary of Δ^n, is absolutely continuous (with respect to Lebesgue measure). A continuous function ϕ on Δ^n is said to be *n-harmonic* if it is harmonic in each variable separately (cf. [13, page 16]).

PROPOSITION 4. *Let μ be a measure carried by $(\Delta^n)^-$ and suppose $\mu|_{\partial\Delta^n}$ is absolutely continuous. Then there exists a function $R_\mu \in L^1(m_n)$ such that $R_\mu \geq 0$* a.e. $[m_n]$ *and*

$$\int_{(\Delta^n)^-} \phi \, d\mu = \int_{\mathbf{T}^n} \phi \left(R_\mu + \frac{d\mu}{dm_n} \right) dm_n$$

for every function ϕ which is continuous on $(\Delta^n)^-$ and n-harmonic in Δ^n. Furthermore, if $\mu(\Delta^n) > 0$, then there is a constant $c > 0$ such that $R_\mu \geq c$ a.e. $[m_n]$.

Proof. Let

$$E_{\sigma k} = \{z\colon\ |z_{\sigma(j)}| = 1,\ 1 \leq j \leq k, \text{ and } |z_{\sigma(j)}| < 1,\ k < j \leq n\}.$$

Then $\bigcup_{k=1}^{n-1} \bigcup_\sigma E_{\sigma k} = \partial(\Delta^n)^- \backslash \mathbf{T}^n$. For each σ and for each k we will construct a function $R_{\sigma k} \in L^1(m_n)$ such that $R_{\sigma k} \geq 0$ and $\int_{E_{\sigma k}} \phi \, d\mu = \int \phi R_{\sigma k} \, dm_n$ for every function ϕ which is continuous on $(\Delta^n)^-$ and n-harmonic in Δ^n. Assume for now that $\phi \geq 0$. As before, we assume that σ is the identity permutation. Write $z = (z', z'')$, where

$$z' = (z_1, \ldots, z_k) \quad \text{and} \quad z'' = (z_{k+1}, \ldots, z_n).$$

We have

$$\int_{E_{\sigma k}} \phi \, d\mu = \int_{E_{\sigma k}} \phi(z', z'') \, d\mu(z', z'')$$

$$= \int_{E_{\sigma k}} \int_{\mathbf{T}^{n-k}} \phi(z', w) P_{z''}(w) \, dm_{n-k}(w) \, d\mu(z', z'')$$

$$= \int_{\mathbf{T}^{n-k}} \int_{E_{\sigma k}} \phi(z', w) P_{z''}(w) \, d\mu(z', z'') \, dm_{n-k}(w).$$

Define a measure ν_w on $\mathbf{T}^k$ by

$$\nu_w(E) = \int_{E \times \Delta^{n-k}} P_{z''}(w) \, d\mu(z', z'').$$

Then ν_w is positive and finite for almost every (with respect to m_{n-k}) w. Furthermore, $\nu_w \ll m_k$ by hypothesis. If

$$R_{\sigma k}(z', w) = \frac{d\nu_w}{dm_k}(z'),$$

then

$$\int_{\mathbf{T}^{n-k}} \int_{E_{\sigma k}} \phi(z', w) P_{z''}(w) \, d\mu(z', z'') \, dm_{n-k}(w)$$

$$= \int_{\mathbf{T}^{n-k}} \int_{\mathbf{T}^k} \phi(z', w) R_{\sigma k}(z', w) \, dm_k(z') \, dm_{n-k}(w)$$

$$= \int_{\mathbf{T}^n} \phi(z) R_{\sigma k}(z) \, dm_n(z).$$

Next, define R_0 on $\mathbf{T}^n$ by $R_0(w) = \int_{\Delta^n} P_z(w) \, d\mu(z)$. Then

$$\int_{\Delta^n} \phi(z) \, d\mu(z) = \int_{\Delta^n} \int_{\mathbf{T}^n} \phi(w) P_z(w) \, dm_n(w) \, d\mu(z)$$

$$= \int_{\mathbf{T}^n} \phi(w) \int_{\Delta^n} P_z(w) \, d\mu(z) \, dm_n(w)$$

$$= \int_{\mathbf{T}^n} \phi(w) R_0(w) \, dm_n(w).$$

Notice that the above computations show that $R_{\sigma k}$ and R_0 are in $L^1(m_n)$. Therefore, we may drop the requirement that ϕ be positive.

Finally, let $R_\mu = R_0 + \sum_{k=1}^{n-1} \sum_\sigma R_{\sigma k}$, where the inner sum is over those permutations of the first n integers for which $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(n)$. (This restriction on σ is merely a device to avoid duplication; different choices of σ give rise to the same set $E_{\sigma k}$ whenever $k < n - 1$.)

If $\mu(\Delta^n) > 0$, then the function R_0 is positive and lower semicontinuous (cf. [9, pp. 454–455]).

COROLLARY 5. *If μ is a measure which satisfies the hypotheses of the proposition, then*

$$\int |p|^2\, d\mu \leq \int |p|^2 \left(R_\mu + \frac{d\mu}{dm_n}\right) dm_n$$

for every polynomial $p \in \mathscr{P}_n$.

Proof. Let ϕ be the function which is continuous on $\Delta^n \cup \mathbf{T}^n$, n-harmonic in Δ^n and agrees with $|p|^2$ on $\mathbf{T}^n$. Then ϕ is a trigonometric polynomial, hence continuous in $(\Delta^n)^-$. Since $|p|^2$ is n-subharmonic, $|p|^2 \leq \phi$ in Δ^n; by continuity, this inequality holds throughout $(\Delta^n)^-$. Applying the proposition to ϕ, the proof is complete.

COROLLARY 6. *If μ is a measure which satisfies the hypotheses of the proposition, then $H^\infty(m_n) \subset H^2(\mu)$.*

COROLLARY 7. *Suppose the measure μ satisfies the hypotheses of the proposition and suppose $m_n \ll \mu$. Define a measure v by*

$$dv = \left(R_\mu + \frac{d\mu}{dm_n}\right) dm_n.$$

If $\psi \in H^2(v)$ is cyclic, then there exists a cyclic function $\tilde{\psi} \in H^2(\mu)$ such that $\tilde{\psi} = \psi$ a.e. $[m_n]$ and such that $\int |p|^2 |\tilde{\psi}|^2\, d\mu \leq \int |p|^2 |\psi|^2\, dv$ for each $p \in \mathscr{P}_n$.

Proof. Suppose $\psi \in H^2(v)$ is cyclic. Choose polynomials p_k such that $p_k \to \psi$ in $H^2(v)$. The sequence $\{p_k\}$ is Cauchy in $H^2(\mu)$ by Corollary 5 and hence has a limit $\tilde{\psi} \in H^2(\mu)$. Since m_n is absolutely continuous with respect to μ and v, we have $\tilde{\psi} = \psi$ a.e. $[m_n]$. Again by Corollary 5, for $p \in \mathscr{P}_n$, $\int |pp_k|^2\, d\mu \leq \int |pp_k|^2\, dv$. Hence,

$$\int |p|^2 |\tilde{\psi}|^2\, d\mu \leq \int |p|^2 |\psi|^2\, dv.$$

Finally, let q_k by polynomials such that $q_k \psi \to 1$ in $H^2(v)$. Then

$$\begin{aligned}\int |1 - q_k \tilde{\psi}|^2\, d\mu &= \lim_{j\to\infty} \int |1 - q_k p_j|^2\, d\mu \\ &\leq \lim_{j\to\infty} \int |1 - q_k p_j|^2\, dv \\ &= \int |1 - q_k \psi|^2\, dv.\end{aligned}$$

Therefore, $\tilde{\psi}$ is cyclic.

LEMMA 2. *Let μ be a measure carried by $(\Delta^n)^-$. Suppose there is an operator $X: H^2(\mu) \to H^2(m_n)$ such that X has dense range and $X\mathbf{U}_\mu = \mathbf{U}X$. Then there exists a cyclic function $\phi \in H^2(m_n)$ such that*

$$|\phi|^2 \leq \frac{d\mu}{dm_n} \text{ a.e. } [m_n].$$

Proof. We may assume that $\|X\| = 1$. If $\phi = X1$, then $Xp = p\phi$ for each polynomial $p \in \mathscr{P}_n$. Since X has dense range, ϕ is cyclic.

For $f \in H^2(\mu)$,

$$\begin{aligned} \|Xf\|_{m_n} &= \lim_{k\to\infty} \|(z_1 \cdots z_n)^k Xf\|_{m_n} \\ &= \lim_{k\to\infty} \|X(z_1 \cdots z_n)^k f\|_{m_n} \\ &\leq \lim_{k\to\infty} \|(z_1 \cdots z_n)^k f\|_\mu \\ &= \|f\|_{\mu_0}. \end{aligned}$$

In particular, for $p \in \mathscr{P}_n$, $\int |p|^2 |\phi|^2 \, dm_n \leq \int |p|^2 \, d\mu_0$. Arguing as in the proof of Proposition 3, we may conclude that $\int h|\phi|^2 \, dm_n \leq \int h \, d\mu_0$ for every positive continuous function h. This is possible only if

$$|\phi|^2 \leq \frac{d\mu}{dm_n} \text{ a.e. } [m_n].$$

Proof of Theorem 2. Suppose $\mathbf{U}_\mu \sim\sim \mathbf{U}$. By Proposition 1, supp $\mu \subset (\Delta^n)^-$. Statement (a) follows from Lemma 2 and (b) from Proposition 3.

Conversely, suppose (a) and (b) are true. By (b), $\int |z_1^{k_1} \cdots z_n^{k_n}|^2 |\psi|^2 \, d\mu \leq 1$ for every choice of $k_1, \ldots, k_n$, $k_j \geq 0$. This is possible only if $\int_{\mathbf{C}^n \setminus (\Delta^n)^-} |\psi|^2 \, d\mu = 0$. Since ψ is cyclic, this implies that supp $\mu \subset (\Delta^n)^-$. We may define bounded operators

$$X: H^2(\mu) \to H^2(m_n) \quad \text{and} \quad Y: H^2(m_n) \to H^2(\mu)$$

by the requirement that $Xp = \phi p$ and $Yp = \psi p$ for each $p \in \mathscr{P}_n$. The operators X and Y have dense range because ϕ and ψ are cyclic. By Proposition 3 and (a), $\mu|_{\mathbf{T}^n}$ and m_n are mutually absolutely continuous. It follows easily that Y is quasiinvertible. Finally, suppose $Xf = 0$. Clearly, $f = 0$ a.e. $[m_n]$. It is easily verified that there is a cyclic function $h \in H^2(\mu)$ such that $h = \phi\psi$ a.e. $[m_n]$. Combining (a) and (b), we have

$$\int |f|^2 |h|^2 \, d\mu \leq \int_{\mathbf{T}^n} |f|^2 \, d\mu = 0;$$

that is, $f = 0$.

We now turn to the proof of Theorem 1. Let $\mathbf{Z}^n$ be the set of all n-tuples of integers and let $\mathbf{Z}^n_+$ be those elements of $\mathbf{Z}^n$ whose coordinates are all nonnegative. The set $\mathbf{Z}^n$ with the operation of addition is a group. Let Γ be a subsemigroup of $\mathbf{Z}^n$ which contains $\mathbf{Z}^n_+$ and such that

$$\Gamma \cap -\Gamma = \{(0, \ldots, 0)\},$$

while $\Gamma \cup -\Gamma = \mathbf{Z}^n$. Let A be the set of those continuous functions on $\mathbf{T}^n$ whose Fourier coefficients vanish off Γ. Then A is a Dirichlet algebra (cf. [8]) and m_n is a multiplicative measure on A. Let A_0 be those elements $f \in A$ for which $\int f\, dm_n = 0$. Let μ be a measure on $\mathbf{T}^n$. A generalized version of Szegö's Theorem states that

$$\inf_{f \in A_0} \int |1 - f|^2\, d\mu = \exp\left\{\int \log \frac{d\mu}{dm_n}\, dm_n\right\}.$$

LEMMA 3. *Suppose* supp $\mu \subset \overline{\Delta^n}$ *and suppose*

$$\int \log \frac{d\mu}{dm_n}\, dm_n > -\infty.$$

Then 0 *is a b.p.e. for* $H^2(\mu)$.

Proof. We may assume that μ is carried by $\mathbf{T}^n$. Let f_0 be the orthogonal projection of 1 into the closure of A_0 in $L^2(\mu)$. By the generalization of Szego's Theorem mentioned above, $f_0 \neq 1$. Let ϕ be the orthogonal projection of $1 - f_0$ into $H^2(\mu)$. Then ϕ is not zero because $(\phi, 1) = (1 - f_0, 1) \neq 0$. But

$$\phi \in \bigcap_{j=1}^{n} \ker U_j^*,$$

because for $p \in \mathscr{P}_n$, $(U_j^*\phi, p) = (\phi, z_j p) = (1 - f_0, z_j p) = 0$. In view of Proposition 2, the proof is complete.

Proof of Theorem 1. (1)$\Rightarrow$(2). Suppose $\mathbf{U}_\mu \cong \mathbf{U}$. Then $U_{1\mu}, \ldots, U_{n\mu}$ are isometries, which implies that supp $\mu \subset \mathbf{T}^n$. By (a) of Theorem 2 and Proposition 3,

$$\int \log \frac{d\mu}{dm_n}\, dm_n > -\infty$$

and $\mu \ll m_n$. A simple computation using the unitary equivalence shows that $U_{j\mu} U_{k\mu}^* = U_{k\mu}^* U_{j\mu}$ whenever $j \neq k$.

(2)$\Rightarrow$(3). Suppose (2) holds. By Lemma 3 there exists $\psi \in \bigcap_{j=1}^n \ker U_j^*$ such that $\|\psi\|_\mu = 1$. Since $U_{1\mu}, \ldots, U_{n\mu}$ are commuting isometries and $U_{j\mu} U_{k\mu}^* = U_{k\mu}^* U_{j\mu}$ for $j \neq k$, it is easy to verify that $|\psi|^2\, d\mu = dm_n$ or $d\mu = |\psi^{-1}|^2\, dm_n$. Since μ is a finite measure, $\psi^{-1} \in L^2(m_n)$. Furthermore, for $j \in \mathbf{Z}^n$,

$$(z^j, \psi^{-1})_{m_n} = (z^j \psi, 1)_\mu = 0$$

whenever j_k is negative for some k, $1 \leq k \leq n$. That is, $\phi = \psi^{-1} \in H^2(m_n)$. That ϕ is cyclic follows from the fact that $\phi^{-1} \in H^2(\mu)$.

(3) $\Rightarrow$ (1). Trivial.

Section 4

If $\mathbf{U}_\mu \sim\sim \mathbf{U}$, several properties of μ follow immediately.

(1) $\operatorname{supp} \mu \subset (\Delta^n)^-$.
(2) The restriction of μ to the boundary of Δ^n is absolutely continuous.
(3) $\int \log (d\mu/dm_n)\, dm_n > -\infty$.
(4) If μ_0 is the restriction of μ to $\mathbf{T}^n$, then $\mathbf{U}_{\mu_0} \sim\sim \mathbf{U}$.

For $n = 1$, properties (1), (2), and (3) imply that $\mathbf{U}_\mu \sim\sim \mathbf{U}$. If $n > 1$, however, properties (1)–(4) do not imply that $\mathbf{U}_\mu \sim\sim \mathbf{U}$. Even if $\operatorname{supp} \mu \subset \mathbf{T}^n$, (1)–(3) do not imply that $\mathbf{U}_\mu \sim\sim \mathbf{U}$.

Example 1. Define the measure μ on $\mathbf{T}^2$ by

$$d\mu(z, w) = |z - w|^2\, dm_2(z, w).$$

Then there does not exist an operator $X\colon H^2(\mu) \to H^2(m_2)$ such that $X\mathbf{U}_\mu = \mathbf{U}X$ and such that $(X1)(0) \neq 0$. In other words, there is no outer function $\phi \in H^2(m_2)$ such that

$$|\phi(z, w)|^2 \leq |z - w|^2 \text{ a.e. } [m_2].$$

Proof. Suppose $X\colon H^2(\mu) \to H^2(m_2)$ and $XU_{i\mu} = U_i X$, $i = 1, 2$. Let $\phi = X1$, and write $\phi = a + \phi_1$, where $\phi_1(0) = 0$. Let $p_n(z, w) = \sum_{k=0}^n z^k w^{n-k}$. Then we must have

$$\int |p_n|^2 |\phi|^2\, dm_2 \leq \|X\|^2 \int |p_n|^2 |z - w|^2\, dm_2 = 2\|X\|^2.$$

Since $(p_n, p_n \phi_1) = 0$,

$$\int |p_n \phi|^2\, dm_2 \geq |a|^2 \int |p_n|^2\, dm_2 = |a|^2(n + 1).$$

This is possible only if $a = 0$.

Example 2. Define $h \in L^2(m_2)$ by $h(z, w) = \sum_{k=0}^\infty (k + 1)^{-2/3} \bar{z}^k w^k$, and define a measure μ by $d\mu = |h|^2\, dm_2$. If $Y\colon H^2(m_2) \to H^2(\mu)$ such that $YU_i = U_{i\mu} Y$, $i = 1, 2$, then $(Y1)(0) = 0$. In particular, Y cannot be quasiinvertible.

Proof. This proof is similar to the previous one. Let $\phi = Y1$, and write $\phi = a + \phi_1$ where $\phi_1(0) = 0$. (Since $d\mu/dm_2 \geq c > 0$, 0 is a b.p.e., and so $\phi(0)$ is well-defined.) For the polynomial p_n defined in the last proof

$$\int |p_n\phi h|^2\, dm_2 \le \|Y\|^2 \int |p_n|^2\, dm_2 = \|Y\|^2(n+1).$$

Since $(p_n, p_n\phi_1)_\mu = 0$, we must have $|a|^2 \int |p_n h|^2\, dm_2 \le \|Y\|^2(n+1)$. One may check that

$$\int |p_n h|^2\, dm_2 \ge c(n+1)^{5/3}$$

Thus, we must have $a = 0$.

Example 3. Let μ be the measure defined in Example 2. Let

$$E = \{(z_1, z_2, z_3) \in \mathbf{C}^3 \colon |z_1| = |z_2| = 1 \text{ and } z_3 = 0\},$$

and define a measure ν carried by $\mathbf{T}^3 \cup E$ by

$$\nu|_{\mathbf{T}^3} = m_3 \quad \text{and} \quad d\nu(z_1, z_2, 0) = d\mu(z_1, z_2).$$

Suppose there exists an operator $Y\colon H^2(m_3) \to H^2(\mu)$ such that $Y\mathbf{U} = \mathbf{U}_\mu Y$. Let $\phi = Y1$ and let

$$\tilde{\phi}(z_1, z_2) = \phi(z_1, z_2, 0).$$

Then $\tilde{\phi} \in H^2(\mu)$ and the operator $\tilde{Y}\colon H^2(m_2) \to H^2(\mu)$ defined by $\tilde{Y}f = \tilde{\phi}f$ is bounded. As shown in Example 2, this implies that $\phi(0) = \tilde{\phi}(0) = 0$. Therefore, Y cannot have dense range.

Example 4. Suppose $h = \sum_{m=0}^{\infty} a_m \bar{z}^m w^m \in L^2(m_2)$ with $\|h\|_{m_2} = 1$. Then there exists a cyclic function $f \in H^2(m_2)$ such that $|h| = |f|$ a.e. $[m_2]$ if, and only if, $|h| = 1$ a.e. $[m_2]$.

Proof. Suppose there is a cyclic function $f \in H^2(m_2)$ such that $|f| = |h|$ a.e. $[m_2]$. Define a measure μ by $d\mu = |h|^2\, dm_2$. By Theorem 2, there is a unitary operator $X\colon H^2(\mu) \to H^2(m_n)$ such that $\mathbf{U}_\mu \cong \mathbf{U}$ via X. If $\phi = X^*1$, then $\phi \in \ker U_{1\mu}^* \cap \ker U_{2\mu}^*$ and $|\phi|^2\, d\mu = dm_2$; that is,

$$|\phi|^2 |f|^2 = 1 \text{ a.e. } [m_2].$$

But, it is easy to show that $\ker U_{1\mu}^* \cap \ker U_{2\mu}^*$ is spanned by the constant function 1.

To see a consequence of Example 4, suppose $f(z, w) = 2z - w$. Then there is no outer function $g \in H^2(m_2)$ such that $|f|^2 = |g|^2$ a.e. $[m_2]$. The reason is that $|2z - w| = |2 - \bar{z}w|$ on $\mathbf{T}^2$, and consequently, there is no cyclic function $g \in H^2(m_2)$ such that

$$|g| = |2z - w| \text{ a.e. } [m_2].$$

Furthermore, g cannot be outer because an outer function bounded away from zero is easily seen to be cyclic. This example also shows that unlike the case $n = 1$, we can have $\mathbf{U}_\mu \sim \mathbf{U}$ but $\mathbf{U}_\mu \not\cong \mathbf{U}$, where $U_{1\mu}, \ldots, U_{n\mu}$ are isometries.

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